DCU School of Mathematical Sciences

BASIC SKILLS WORKSHEET 10

Derivatives

The aim of this worksheet is to revise material on differentiation.

Exercise

Draw a rough sketch of the graph of a function which satisfies these properties.

- The graph starts at $x = 0$, with the value $y = -4$.
- The function increases from $x = 0$ up to $x = 4$, where $y = 20$.
- The graph decreases from $x = 4$ up to $4x = 10$, where $y = 0$.
- The graph increase from $x = 10$ onwards.
- The graph is smooth everywhere, with no sharp corners.

This information allows us to draw a very good sketch of what the graph of the function looks like. As we saw in an earlier worksheet, a graph is a very useful tool, so it would be good to have a simple way of drawing graphs. The point of the exercise above is this:

To draw the graph of a function, all we really need to know is where is it increasing, where is it decreasing, and where does it change from one to the other?

(We might add some frills to get a more accurate picture: How quickly is it increasing? If it is increasing, is it speeding up or slowing down? etc.)

Formulas and graphs

In many applications, we have the formula for our functions, for example

$$f(x) = -3x^3 + 4x + 9.$$

How do we get from the formula to the graph? In light of the example above, what we really want to know is this:

Given the formula for a function, how do we find out is where it is increasing, where it is decreasing, and where it changes from one to the other?

The tool we use to answer this question is differentiation.

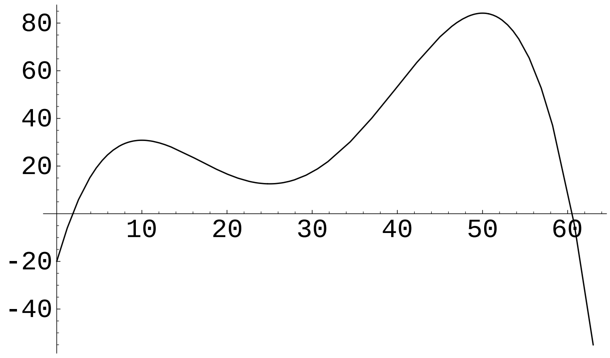

Figure 1: Graph of a function. The horizontal axis is the $x-$axis, and the vertical is the $y-$axis.

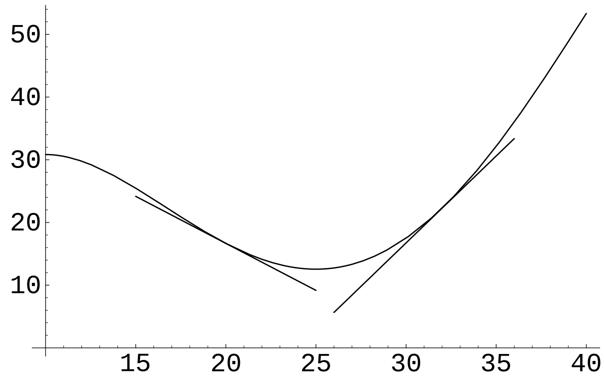

Figure 2: Tangents to the graph of Figure 1 at the points $(x, y) = (20, 16.67)$ and $(x, y) = (32, 22.28)$.

The slope of a graph

For linear functions, it is very easy to tell if the graph is increasing or decreasing. We just write the formula for the function in the form

$$y = ax + b$$

and check if a (the slope) is positive or negative. If a is positive, it means the graph - which is a straight line - slopes upwards. If a is negative, the line slopes downwards.

Consider the graph in Figure 1. This is the graph of the function

$$f(x) = -20 + \frac{25}{2}x - x^2 + \frac{17}{600}x^3 - \frac{1}{4000}x^4.$$

It increases in some places, and decreases in others. We would like to have some quantity, like the slope (which only works for straight lines), which tells us which one it is, increase or decrease.

We use this fact. The tangent to the graph at any given point captures the details of increase and decrease, including the rate of increase and rate of decrease. Remember, the tangent at a point (x, y) on a graph is the straight line which passes through the point (x, y) but which does not pass through any other (nearby) points of the graph. A couple of examples are shown in Figure 2. (This is a close-up of a portion of Figure 1.)

Definition

The slope of a curve at a point (x, y) is defined to be the slope of the tangent to the curve at that point.

Figure 2 shows that this quantity does indeed determine whether or not a function is increasing or decreasing at a given point. It also shows up the crucial fact that the slope of a curve changes from point to point on the curve, unlike the case of a straight line.

Calculating the slope of a curve

How do we calculate the slope of the tangent at a point on a given curve? See hand written notes.

Example

Calculate the slope of $f(x) = x^2 + 3x$ at the point on the graph with $x = 2$.
Solution: First, we calculate the $y-$value corresponding to $x = 2$. This is

$$y = f(2) = 2^2 + 3(2) = 10.$$

Next, we calculate the secant slope. We need first

$$\begin{aligned} f(x + \Delta x) &= (x + \Delta x)^2 + 3(x + \Delta x) \\ &= x^2 + 2\Delta x x + (\Delta x)^2 + 3x + 3\Delta x \\ &= x^2 + 3x + (2x + 3)\Delta x + (\Delta x)^2. \end{aligned}$$

Then

$$\begin{aligned} f(x + \Delta x) - f(x) &= [x^2 + 3x + (2x + 3)\Delta x + (\Delta x)^2] - [x^2 + 3x] \\ &= (2x + 3)\Delta x + (\Delta x)^2. \end{aligned}$$

Divide by Δx to get the secant slope:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x + 3 + \Delta x.$$

Now we let Δx shrink down to zero to get

$$\text{Slope of the tangent at } x = 2x + 3.$$

We let $x = 2$ to get the result:

$$\text{The slope of the graph at the point (2,10) is } m = 7.$$

Definition

The quantity

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is called the **derivative** of the function f at x. We denote this by $f'(x)$, or by $\dfrac{dy}{dx}$. The process of finding the derivative of a function is called differentiation.

Rules of differentiation

The example above was an example of what is called differentiation from first principles. It's pretty long-winded. In practice, we use this list of short-cut rules (with indicative examples) to calculate derivatives.

1. If $f(x) = x^n$ for any number n, then $f'(x) = nx^{n-1}$.
   (a) $f(x) = x^3 \Longrightarrow f'(x) = 3x^2$
   (b) $y = x^{12} \Longrightarrow \dfrac{dy}{dx} = 12x^{11}$
   (c) $y = x^{-4} \Longrightarrow \dfrac{dy}{dx} = -4x^{-5}$
   (d) $f(x) = \sqrt{x}$ First we have to remember that $\sqrt{x} = x^{\frac{1}{2}}$ and then apply the rule to get $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$.
   (e) $y = \dfrac{1}{x^2}$. We write this as $y = x^{-2}$ and apply the rule to get $\dfrac{dy}{dx} = -2x^{-3}$
2. If $g(x) = cf(x)$ for some constant c, then $g'(x) = cf'(x)$. That is, we leave the numbers in front of any $x-$terms alone when differentiating.
   (a) $f(x) = 4x^7 \Longrightarrow f'(x) = 4(7x^6) = 28x^6$
   (b) $y = -2x^{-5} \Longrightarrow f'(x) = 10x^{-6}$
3. If $f(x) = c$ for some constant c, then $f'(x) = 0$. (This actually follows from Rules 1 and 2. It is also in line with the fact that the graph is just the horizontal line $y = c$, which has slope equal to zero).
4. If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$. This means that to differentiate the sum of two functions, we differentiate the functions and the add the result. The same rule holds for functions which are added rather than subtracted.
   (a) $f(x) = 2x^3 + 5x^7 \Longrightarrow f'(x) = 6x^2 + 35x^6$ (We differentiate $2x^3$ and $5x^7$ separately using the previous rules, and then add the results.)
   (b) $h(x) = 3x^6 - 2x \Longrightarrow h'(x) = 18x^5 - 2$. (We differentiate $3x^6$ and $2x$ separately using the previous rules, and then subtract the results.)
   (c) The rule applies for any number of functions being added or subtracted, so if $y = -3x^2 + 5x^3 - 11x^6 + 15$, then $\dfrac{dy}{dx} = -6x + 15x^2 - 66x^5$.
5. The trigonometric functions:

$$\begin{aligned} f(x) = \sin x &\implies f'(x) = \cos x \\ g(x) = \cos x &\implies g'(x) = -\sin x \end{aligned}$$

6. The product rule.

$$\text{If } y = uv, \text{ then } \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

We also write this as

$$y = uv \Longrightarrow y' = uv' + vu'.$$

(a) $y = 2x^2(4x^9+15)$. Here we could take $u = 2x^2$ and $v = 4x^9+15$. Then $u' = 4x, v' = 36x^8$ and so

$$y' = 2x^2(36x^8) + (4x^9+15)(4x).$$

(b) $f(x) = 2x\sin x$. Let $u = 2x, v = \sin x$. Then $u' = 2, v' = \cos x$ and so

$$f'(x) = 2x\cos x + 2\sin x.$$

7. The quotient rule. (Quotient here just means fraction.)

$$y = \frac{u}{v} \Longrightarrow y' = \frac{vu' - uv'}{v^2}.$$

(a) $f(x) = \dfrac{2x+1}{x-2}$. Here we take $u = 2x+1$ and $v = x-2$. These give $u' = 2, v' = 1$. Filling out the formula gives

$$f'(x) = \frac{(x-2)(2)-(2x+1)(1)}{(x-2)^2} = \frac{-5}{(x-2)^2}.$$

(b) $y = \dfrac{\cos x}{x^2}$. Take $u = \cos x, v = x^2$. Then $u' = -\sin x, v' = 2x$. Filling out the formula gives

$$\frac{dy}{dx} = \frac{x^2(-\sin x)-(\cos x)(2x)}{x^4} = -\frac{x^2\sin x + 2x\cos x}{x^4}.$$

8. The chain rule. We can think of a function like

$$f : x \to \sin(2x)$$

as being 'a function of a function'. That is, the final output is calculated in two steps:

$$\begin{aligned} x &\to 2x; \\ 2x &\to \sin(2x). \end{aligned}$$

Giving these steps names, we could write

$$\begin{aligned} u : x &\to 2x \qquad \text{or} u(x) = 2x \\ g : u &\to \sin u \qquad \text{or} g(u) = \sin u. \end{aligned}$$

The final output is

$$y = f(x) = g(u) = \sin(u) = \sin(2x).$$

In a situation like this, the derivative is found by using

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

In the present example, we have

$$y = \sin u, \qquad u = 2x.$$

Then $\dfrac{dy}{du} = \cos u$ and $\dfrac{du}{dx} = 2$, so

$$\frac{dy}{dx} = \cos u \times 2 = 2\cos u = 2\cos(2x).$$

(a) $y = (4x+1)^3$. We can write this as $y = u^3$, where $u = 4x+1$. Then $\dfrac{dy}{du} = 3u^2$, $\dfrac{du}{dx} = 4$. Multiplying gives

$$\frac{dy}{dx} = 3u^2 \times 4 = 12u^2 = 12(4x+1)^2.$$

(b) $y = \cos(3x^2-1)$. We write this as $y = \cos u, u = 3x^2 - 1$. Then $\dfrac{dy}{du} = -\sin u$, $\dfrac{du}{dx} = 6x$. Multiplying gives

$$\frac{dy}{dx} = -6x\sin u = -6x\sin(3x^2-1).$$

Exercises

1. Find the derivative of each of the following functions.

   (a) $y = 4x^9 - 5x + 12$

   (b) $y = -6x^{-4} + 3x^3$

(c) $y = 3x + x^2 - 7x^3$

(d) $P = -3Q + 200$

(e) $y = \dfrac{1}{2x}$

(f) $y = \dfrac{3x - 4}{x + 11}$

(g) $y = \dfrac{1}{x} - \sqrt{x}$

(h) $f(x) = \dfrac{2}{\sqrt{x}}$

(i) $y = \dfrac{1}{\sqrt{3x^2 + 9}}$

(j) $y = 2\sqrt{x}\sin x$

(k) $s = \dfrac{2t^2 - 1}{3t + 4}$

2. Find the slope of the tangent to the graph of $y = -3x^2 + 4x + 17$ at $x = -2$.

3. At what values of x is the tangent to the graph of $y = 3x^3 + 4x^2 - 9x + 12$ horizontal? What is the significance of these points (they are called stationary points)? Calculate the slope of the graph 0.01 units to the left and and 0.01 units to the right of each stationary point. What does this information tell you?