DCU School of Mathematical Sciences

BASIC SKILLS WORKSHEET 2

Fractions, Decimals and Percentages

The aim of this worksheet is to study several different ways of writing numbers and calculating with them.

What is an integer?

An integer is a whole number. It can be positive or negative. For example, 1, -1, 0, 354 and -32456 are all integers.

What is a fraction?

A fraction is a number of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. p is called the numerator and q is called the denominator. For example,

- $\frac{1}{2}$ is a fraction with numerator 1 and denominator 2.
- $\frac{5}{4}$ is a fraction with numerator 5 and denominator 4.
- $\frac{65}{100}$ is a fraction with numerator 65 and denominator 100.

Equivalent Fractions

The idea of equivalent fractions is quite familiar. We all know that half a chocolate bar is equivalent to two quarters of the same bar. Two tenth shares of a lottery prize are equivalent to one fifth share of the same prize.

Fractions are equivalent if they have the same value.

$$\frac{1}{2} \quad \text{is equivalent to} \quad \frac{2}{4}$$

$$\frac{2}{10} \quad \text{is equivalent to} \quad \frac{1}{5}$$

It is possible to create an equivalent fraction by multiplying, or dividing the top and bottom of the given fraction by the same number.

$$\frac{3}{5} = \frac{6}{10} \quad \text{(Multiply top and bottom by 2)}$$

$$\frac{8}{3} = \frac{96}{36} \quad \text{(Multiply top and bottom by 12)}$$

$$\frac{8}{24} = \frac{1}{3} \quad \text{(Divide top and bottom by 3)}$$

In the last example, we have rewritten the fraction in its simplest form. That is, when the numerator (the top) and the denominator (the bottom) don't have any factors in common. To write a fraction in its simplest form, follow these steps:

1. Write both the top and the bottom in terms of their factors.
2. Divide both the top and bottom by any common factors.
3. Multiply out any remaining factors in both the top and bottom.

So, if we are asked to express $\frac{48}{60}$ in its simplest form, the first thing we do is to write the top and bottom in terms of their factors.

$$48 = 2 \times 2 \times 2 \times 2 \times 3$$

and

$$60 = 2 \times 2 \times 3 \times 5$$

Hence

$$\frac{48}{60} = \frac{2 \times 2 \times 2 \times 2 \times 3}{2 \times 2 \times 3 \times 5} = \frac{2 \times 2}{5} = \frac{4}{5},$$

and we see that $\frac{48}{60}$ in its simplest form is $\frac{4}{5}$.

Exercise 1

1. Fill in the missing number in each of the following:

   (a) $\frac{5}{12} = \frac{30}{}$

   (b) $\frac{24}{27} = \frac{8}{}$

   (c) $-\frac{12}{17} = -\frac{24}{}$

   (d) $\frac{3}{5} = \frac{}{20}$

   (e) $\frac{6}{11} = \frac{}{-33}$

   (f) $\frac{30}{-32} = \frac{15}{}$

2. Express the following fractions in their simplest form.

   (a) $\frac{25}{80}$

   (b) $\frac{12}{48}$

   (c) $\frac{13}{39}$

   (d) $\frac{6}{72}$

   (e) $\frac{9}{81}$

There are different types of fraction.

If the numerator p is smaller than the denominator q, then $\frac{p}{q}$ is called a proper fraction. If the numerator p is bigger than the denominator q, then $\frac{p}{q}$ is called an improper fraction. For example,

- $\frac{1}{2}$ is a proper fraction because $1 < 2$.
- $\frac{5}{4}$ is an improper fraction because $5 > 4$.
- $\frac{100}{200}$ is a proper fraction because $100 < 200$.

Whole numbers can be expressed as improper fractions by putting them over 1. So

$$3 = \frac{3}{1}.$$

Mixed numbers

A number like $2\frac{1}{4}$, which is written as a whole number and a fraction stuck together, is called a mixed number. Mixed numbers can be written as improper fractions. It is usually much easier to perform operations (addition, multiplication, etc.) on mixed numbers when they are written as improper fractions.
To write $2\frac{1}{4}$ as an improper fraction

$$2\frac{1}{4} = 2 + \frac{1}{4} = \frac{2}{1} + \frac{1}{4} = \frac{8}{4} + \frac{1}{4} = \frac{8+1}{4} = \frac{9}{4}.$$

To write $3\frac{2}{7}$ as an improper fraction

$$3\frac{2}{7} = 3 + \frac{2}{7} = \frac{3}{1} + \frac{2}{7} = \frac{21}{7} + \frac{2}{7} = \frac{21+2}{7} = \frac{23}{7}.$$

We can also go in the other direction. For example, to write $\frac{7}{4}$ as a mixed fraction

$$\frac{7}{4} = \frac{4+3}{4} = \frac{4}{4} + \frac{3}{4} = 1 + \frac{3}{4} = 1\frac{3}{4}$$

To write $\frac{29}{8}$ as a mixed fraction

$$\frac{29}{8} = \frac{24+5}{8} = \frac{24}{8} + \frac{5}{8} = 3 + \frac{5}{8} = 3\frac{5}{8}$$

Exercise 2

1. Express the following mixed fractions as improper fractions.
   (a) $3\frac{1}{2}$
   (b) $2\frac{2}{3}$
   (c) $10\frac{2}{7}$
   (d) $9\frac{3}{5}$
   (e) $3\frac{3}{4}$
   (f) $2\frac{5}{6}$
2. Express the following improper fractions as mixed fractions.
   (a) $\dfrac{7}{3}$
   (b) $\dfrac{9}{2}$
   (c) $\dfrac{7}{5}$
   (d) $-\dfrac{20}{7}$
   (e) $\dfrac{18}{7}$
   (f) $\dfrac{35}{9}$

Adding and Subtracting Fractions.

Fractions can only be added when their denominators (the bottom part) are the same.

If we want to add two fractions which have the same denominator, then we add the numerators (the top parts). For example,

$$\frac{5}{11} + \frac{2}{11} = \frac{5+2}{11} = \frac{7}{11}.$$

Similarly, subtracting one fraction from another fraction with the same denominator we just subtract one numerator from the other. For example,

$$\frac{5}{11} - \frac{2}{11} = \frac{5-2}{11} = \frac{3}{11}.$$

However, if our fractions don't have the same denominator, we can't do this. So we must express them in a form where they do have the same denominator. This common denominator is the smallest number that is divisible by both the two original denominators.
Suppose we have the following sum

$$\frac{1}{2} + \frac{1}{3}$$

The smallest number that is divisible by both 2 and 3 is 6. We need to express both fractions with a denominator of 6

From the section on equivalent fractions we know that

$$\frac{1}{2} = \frac{3}{6} \quad \text{(Multiply top and bottom by 3)}$$

and

$$\frac{1}{3} = \frac{2}{6} \quad \text{(Multiply top and bottom by 2)}$$

So

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

Let's look at another example.

$$\frac{3}{5}+\frac{2}{3}$$

The smallest number that is divisible by both 5 and 3 is 15. This is our common denominator.

$$\frac{3}{5}=\frac{9}{15} \quad \text{(Multiply top and bottom by 3)}$$

and

$$\frac{2}{3}=\frac{10}{15} \quad \text{(Multiply top and bottom by 5).}$$

So

$$\frac{3}{5}+\frac{2}{3}=\frac{9}{15}+\frac{10}{15}=\frac{19}{15}$$

When adding or subtracting mixed numbers, for example,

$$3\frac{2}{5}-4\frac{1}{2}$$

convert both mixed numbers into improper fractions and proceed as before.

$$3\frac{2}{5}=3+\frac{2}{5}=\frac{3}{1}+\frac{2}{5}=\frac{15}{5}+\frac{2}{5}=\frac{17}{5}$$

$$4\frac{1}{2}=4+\frac{1}{2}=\frac{4}{1}+\frac{1}{2}=\frac{8}{2}+\frac{1}{2}=\frac{9}{2}$$

So now

$$3\frac{2}{5}-4\frac{1}{2}=\frac{17}{5}-\frac{9}{2}$$

The smallest number that is divisible by both 5 and 2 is 10. This is our common denominator

$$\frac{17}{5}=\frac{34}{10} \quad \text{(Multiply top and bottom by 2)}$$

and

$$\frac{9}{2}=\frac{45}{10} \quad \text{(Multiply top and bottom by 5).}$$

So

$$3\frac{2}{5}-4\frac{1}{2}=\frac{17}{5}-\frac{9}{2}=\frac{34}{10}-\frac{45}{10}=-\frac{11}{10}$$

Exercise 3

1. $\frac{1}{4}+\frac{3}{5}$

2. $\frac{12}{15}-\frac{2}{7}$

3. $\frac{3}{8}-\frac{2}{7}$

4. $1\frac{2}{9}+\frac{1}{3}$

5. $\frac{9}{10}+4\frac{1}{6}+\frac{8}{5}$

6. $3\frac{2}{7}+\frac{1}{4}-\frac{3}{8}$

Multiplying Fractions.

To multiply two fractions together, multiply the numerators, and then multiply the denominators. For example,

$$\frac{2}{7}\times\frac{3}{8}=\frac{2\times 3}{7\times 8}=\frac{6}{56}.$$

This may now be expressed in its simplest form,

$$\frac{6}{56}=\frac{3}{28}\quad\text{(Divide top and bottom by 2).}$$

Hence

$$\frac{2}{7}\times\frac{3}{8}=\frac{3}{28}.$$

Another example

$$\frac{4}{9}\times\frac{15}{16}=\frac{4\times 15}{9\times 16}=\frac{60}{144}.$$

This may now be expressed in its simplest form,

$$\frac{60}{144}=\frac{5}{12}\quad\text{(Divide top and bottom by 12).}$$

Hence

$$\frac{4}{9}\times\frac{15}{16}=\frac{5}{12}.$$

Dividing Fractions

To divide one fraction by a second fraction, we invert the second fraction and then multiply. For example,

$$\frac{5}{12} \div \frac{5}{4} = \frac{5}{12} \times \frac{4}{5} = \frac{20}{60} = \frac{1}{3}.$$

$$\frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{21}{20}.$$

Exercise 4

1. Evaluate

   (a) $\dfrac{2}{3} \times \dfrac{6}{11}$

   (b) $\dfrac{18}{19} \times \dfrac{23}{2}$

   (c) $2\frac{1}{2} \times 3\frac{7}{15}$

2. Evaluate

   (a) $\dfrac{2}{3} \div \dfrac{6}{11}$

   (b) $\dfrac{18}{19} \div \dfrac{23}{2}$

   (c) $2\frac{1}{2} \div 3\frac{7}{15}$

3. Find

   (a) $\dfrac{3}{4}$ of 24

   (b) $\dfrac{2}{5}$ of $\dfrac{19}{23}$

   (c) $\dfrac{16}{19}$ of $\dfrac{156}{2}$

4. Is $\frac{7}{8}$ of $\frac{13}{15}$ the same as $\frac{13}{15}$ of $\frac{7}{8}$?

What is a percentage?

It is often convenient, when comparing two different proportions, to convert them into fractions with a standard denominator. This way, we can simply compare numerators. For example, if an exam runs over two years and consists of 19 questions in the first year and 17 questions in the second year, one would like to be able to compare the scores of those who take the exam in the first year to the scores of those who take the exam in the second year. It is conventional to choose 100 as the standard denominator. Thus, a percentage is a fraction whose denominator is 100. We use the percent symbol % to represent a percentage. For example,

$$\tfrac{65}{100} \text{ can be written as } 65\%.$$

Any fraction can be written in the form of a percentage. Let's take $\frac{3}{5}$ as an example. If we wish to write this as a percentage we must express it as a fraction with a denominator of 100. To do this we multiply the denominator by 100. In order not to change the value of the fraction we must also multiply the numerator by 100. So

$$\frac{3}{5} = \frac{3}{5} \times \frac{100}{100}.$$

We may write this last expression as

$$\begin{aligned} \frac{3}{5} \times \frac{100}{100} &= \frac{3}{5} \times \frac{100}{1} \times \frac{1}{100} \\ &= \frac{300}{5} \times \frac{1}{100} \\ &= 60 \times \frac{1}{100} \\ &= \frac{60}{100} \\ &= 60\% \end{aligned}$$

Once we have written a fraction in the form of a percentage, we can then express it as a decimal. For example,

- $27\% = \dfrac{27}{100} = 0.27$,
- $35.5\% = \dfrac{35.5}{100} = 0.355$,
- $6\% = \dfrac{6}{100} = 0.06$.

Exercise 5

1. Express the following fractions as percentages.
   (a) $\frac{13}{25}$
   (b) $\frac{36}{72}$
   (c) $\frac{12}{50}$
2. Thomas scores $\frac{13}{19}$ in a test. Edel scores $\frac{14}{17}$. Express the scores as percentages.
3. Write 27% as a decimal.
4. Write the following decimals as percentages.
   (a) 0.76
   (b) 0.345
   (c) 0.045
   (d) 0.112
   (e) 1.0

Calculating with Percentages.

If we want to calculate 30% of 120, we convert the percentage back into a fraction and multiply. So, 30% of 120 is

$$\frac{30}{100} \times \frac{120}{1} = 36.$$

We can also ask what 4.5 is when it is expressed as a percentage of 7.5. First, we attempt to write down a fraction. But $\frac{4.5}{7.5}$ is not a valid fraction because neither 4.5 nor 7.5 are integers (remember our definition of a fraction). So we multiply the top and the bottom by 2 to get $\frac{9}{15}$. We now proceed as before to convert this fraction into a percentage:

$$\frac{4.5}{7.5} = \frac{4.5}{7.5} \times \frac{2}{2} = \frac{9}{15} = \frac{9}{15} \times \frac{100}{100} = \left(\frac{900}{15}\right) \times \frac{1}{100} = \frac{60}{100} = 60\%$$

If we want to increase a number by a certain percentage, we calculate the value of that percentage, and add it to the number. For example, if we want to increase 230 by 5% we calculate 5% of 230:

$$\frac{5}{100} \times \frac{230}{1} = 11.5$$

and add the result to 230

$$230 + 11.5 = 241.5.$$

To decrease 230 by the same amount, we subtract:

$$230 - 11.5 = 208.5.$$

Exercise 6

1. What is 3.5 expressed as a percentage of 16?

2. If a loan of $1200 is decreased by 17% then what remains to be repaid?

3. A retailer sells a camera for $1200. If the cost price was $825, what is the profit as a percentage of the cost price?

4. A portfolio of shares is bought for $3600. In the first year its value increases by 24%. Over the second year its value increases by a further 20%. What is the value of the portfolio at the end of the second year?

Solutions to exercises

Exercise 1

1. (a) 72
   (b) 9
   (c) 34
   (d) 12
   (e) -18
   (f) -16
2. (a) $\frac{5}{16}$
   (b) $\frac{1}{4}$
   (c) $\frac{1}{3}$
   (d) $\frac{1}{12}$
   (e) $\frac{1}{9}$

Exercise 2

1. (a) $\frac{7}{2}$
   (b) $\frac{8}{3}$
   (c) $\frac{72}{7}$
   (d) $\frac{48}{5}$
   (e) $\frac{15}{4}$
   (f) $\frac{17}{6}$
2. (a) $2\frac{1}{3}$
   (b) $4\frac{1}{2}$
   (c) $1\frac{2}{5}$
   (d) $-2\frac{6}{7}$

(e) $2\frac{4}{7}$

(f) $3\frac{8}{9}$

Exercise 3

1. $\frac{17}{20}$
2. $\frac{18}{35}$
3. $\frac{5}{56}$
4. $\frac{14}{9}$
5. $\frac{20}{3}$
6. $\frac{177}{56}$

Exercise 4

1. (a) $\frac{12}{33}$

   (b) $\frac{207}{19}$

   (c) $\frac{26}{3}$
2. (a) $\frac{11}{9}$

   (b) $\frac{36}{437}$

   (c) $\frac{75}{104}$
3. (a) 18

   (b) $\frac{38}{115}$

   (c) $\frac{1248}{19}$
4. Yes. Both are equal to $\frac{7}{8} \times \frac{13}{15} = \frac{91}{120}$

Exercise 5

1. (a) 52%
   (b) 50%
   (c) 24%

2. Thomas: 68.4%. Edel: 82.4%.

3. 0.27

4. (a) 76%
   (b) 34.5%
   (c) 4.5%
   (d) 11.2%
   (e) 100%

Exercise 6

1. 21.9%

2. $996

3. 45.45%

4. $5356.80