DCU School of Mathematical Sciences

BASIC SKILLS WORKSHEET 3

Algebra I - Variables and Expressions

The aim of this worksheet is to revise some of the basic 'grammar' of maths that allows us to use mathematical language in science, business and engineering.

What is algebra?

This is the name given to the area of maths that deals with fiddling about with symbols and letters. The name alegbra comes from an Arabic book "al-jabr w'al-muqābala" which was written around 825AD. The title means something like "completion and balance". The author Muhammad Ibn Mūsa Al-Khwārizmī was probably born in what is know Uzbekistan. However he lived in Baghdad, where he worked in the House of Wisdom, an academy of science established by the Caliph (the spiritual leader of the Islamic world). Al-Khwārizmī also wrote books on geography and astronomy.

What is it for?

It's for making our lives easier. Up until the 14^{th} Century or so, people used to write things like this:

"It is desired that quantity for which thrice the quantity added to twice its own product has the magnitude twelve."

It's easier to say "Find x where $2x^2 + 3x = 12$."

What is a variable?

A variable is a symbol (usually a letter) that we use to represent a quantity whose value can change or whose value is not known. For example,

$$P = \text{price of a gallon of petrol}$$
$$x = \text{length of a piece of string}$$
$$y = \text{amount of salt in a saline solution}$$

It is also common to use letters to represent quantities whose values are fixed and are known. This is very common in science, where examples include

- c = speed of light in a vacuum
- G = Newton's gravitational constant
- The Greek letter pi, written π, is used to represent the constant 3.14159... which appears in the formula for the area of a circle.

Letters or symbols used in this way are called constants. We also use the word constant to describe fixed numbers, e.g. $3, \sqrt{2}, \frac{1}{2}$, etc.

Other Greek letters are frequently used and for reference the Greek alphabet is given below

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

What is an expression?

This means a collection of variables and numbers combined using addition, subtraction, multiplication etc. For example, if x and y represent unknown numbers, then the following are expressions involving x and y:

$$2x + y, \qquad xy, \qquad 3x^2 + 8y, \qquad \frac{2x - y}{x + 4y}.$$

Reminder

In order to save time and ink, we use these shorthand ways as substitutes for $\times$ (multiplication) and $\div$ (division):

two quantities written together means multiplication: $xy = x \times y$;

two quantities written above and below a horizontal or diagonal bar means division:

$$\frac{x}{y} = x/y = x \div y.$$

- Note that xy is the same as yx just as 3×5 is the same as 5×3.
- However, x/y is not the same as y/x (for example $10/5 = 2$ while $5/10 = 1/2$).
- Multiplication and division by 1 leaves a quantity unchanged. So $x \times 1 = x$ and $x/1 = x$.
- Multiplication by 0 always gives 0. Division by 0 is never allowed.

Here are some more definitions that will be useful to have.

Terms

A term is an individual part of an expression. For example the expression

$$2x^2 - 4xy + 18y^2 + 21$$

contains four terms; $2x^2$, $-4xy$, $18y^2$ and 21. These are called respectively "the term in x^2", "the term in xy" and "the term in y^2". The term which is just a number is called the "constant term".

Coefficients

A coefficient is a quantity (usually a number or a constant) that multiplies another quantity. For example, in the expression

$$-2ab + 4c^2 + 11bc$$

the coefficient of ab is -2; the coefficient of c^2 is 4 and the coefficient of bc is 11.

Equations

An equation is simply a statement that two expressions are equal to one another. For example,

$$x = 4$$
$$x^2 - 3x + 2 = 0$$
$$-2ab + 4c^2 + 11bc = 15b$$
$$E = mc^2$$

The difference between expressions and equations is that *expressions never contain an equals sign.*

Brackets

Suppose that we needed to multiply the expression $x - 2$ by 3. The result could be written as

$$3 \times x - 2.$$

However we don't usually use the $\times$ sign. Also, we need a way of indicating that *all* of the expression $x - 2$ is being multiplied by 3. We use brackets to do this:

$$3 \times x - 2 = 3(x - 2).$$

This means that every term in the brackets gets multiplied by 3, and so the result is

$$3(x - 2) = 3x - 6.$$

Exercise 1

In the following expressions, write down the required quantity:

1. $2x^2 - 4x + 3$ write down the coefficient of x:
2. $3bc + 5b^2 - ac$ write down the term in bc:
3. $13x^3 - 43$ write down the term in x^3:
4. $-7\alpha\beta - 2\beta^2 - 16$ write down the term in $\alpha\beta$:
5. $-P^2 + 4P - 32$ write down the constant term:

Substituting in expressions

- This means replacing variables with actual numbers. For example consider the following expressions in u and v:

$$u + v$$
$$u - v$$
$$2u^2 + v^2$$
$$\frac{2u - v}{v + 3u}$$
$$\sqrt{uv + 3} - v^2.$$

If $u = 2, v = 3$ then these expressions take the values

$$2 + 3 = 5$$
$$2 - 3 = -1$$
$$2(2)^2 + (3)^2 = 2(4) + 9 = 8 + 9 = 17$$
$$\frac{2(2) - 3}{3 + 3(2)} = \frac{1}{9}$$
$$\sqrt{(2)(3) + 3} - 3^2 = \sqrt{9} - 9 = 3 - 9 = -6.$$

- Often we may be given a formula, for example $y = x^3 + 4$, and be asked to evaluate y when $x = 2$. This is just another way of asking us to find the value of y when $x = 2$. So for this example

$$\begin{aligned} y &= 2^3 + 4 \\ &= 8 + 4 \\ &= 12. \end{aligned}$$

Exercise 2

Evaluate the following expressions when $x = 3$, $y = -2$, $z = 1$.

1. $x + y$
2. $2x + 3yz$
3. $z - y$
4. $x^2 + y^2 - z^2$
5. $\dfrac{2x}{y + 4z}$
6. $\dfrac{3xy - 2z^2}{4xyz}$
7. $\sqrt{x^2 - 4yz}$.

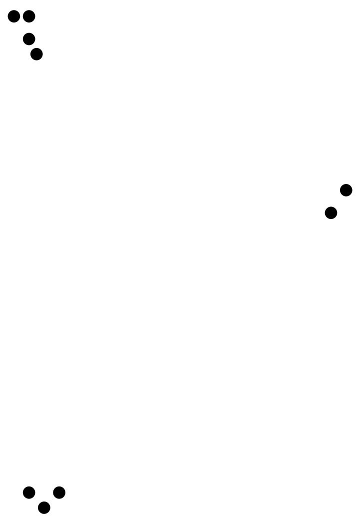

Figure 1: Some dots

Simplifying expressions

If you were counting the dots in Figure 1 , chances are you wouldn't say that there are 4 dots and 2 dots and 3 dots. You would probably say there are 9 dots. Similarly, in an expression like

$$2x + 4y^2 + 6x - 3x + 3y^2$$

it make more sense to put all the same objects together. So the expression is better written as

$$5x + 7y^2.$$

When simplifying expressions in this way it is crucial to put only the same objects together.

Multiplying brackets

We know that ab means $a \times b$. Similarly, $2(a+b)$ means

$$2 \times (a+b) = 2a + 2b.$$

and $a(2b-c)$ means

$$a \times (2b-c) = 2ab - ac.$$

When two bracketed terms are multiplied, it means that all terms of the first bracket are multiplied by all terms of the second. So

$$\begin{aligned} (x+2)(x-3) &= x(x-3) + 2(x-3) \\ &= x^2 - 3x + 2x - 6 \\ &= x^2 - x - 6 \end{aligned}$$

$$\begin{aligned} (a-3)(b+17) &= a(b+17) + (-3)(b+17) \\ &= a(b+17) - 3(b+17) \\ &= ab + 17a - 3b - 51 \end{aligned}$$

and

$$\begin{aligned} (a-2b)(c+1) &= a(c+1) - 2b(c+1) \\ &= ac + a - 2bc - 2b \end{aligned}$$

After multiplying out brackets in this way, we always collect like terms together.

Exercise 3

1. Simplify the following expressions by collecting together all like terms.

   (a) $-4x + 2y + 6x - 18y$

   (b) $2(-3x + 5y) - 4(x + 3y)$

   (c) $2a^2b + 3a^2b^2 - ab^2 + 8a^2b$

   (d) $-3(a - b) + 3(-a + b)$

2. In each case, multiply out the brackets and simplify the resulting expressions by collecting together like terms.

   (a) $(x + 4)(x + 1)$

   (b) $(3y - 4)(y + 1)$

   (c) $(2x - y)(3x - 5y)$

   (d) $(2a + b - 3)(a + b)$

Factoring expressions

To obtain the product of two numbers they are multiplied together. For example the product of 3 and 4 is 3×4 which equals 12. The numbers which are multiplied together are called factors. We say that 3 and 4 are both factors of 12.

- The product of x and y is xy. So x and y are both factors of xy.
- The product of $5x$ and $3y$ is $15xy$. So $5x$ and $3y$ are both factors of $15xy$.
- $2x$ and $5y$ are factors of $10xy$ since when we multiply $2x$ by $5y$ we obtain $10xy$.
- $(x+1)$ and $(x+2)$ are factors of x^2+3x+2 because when we multiply $(x+1)$ by $(x+2)$ we obtain x^2+3x+2.
- 3 and $x-5$ are factors of $3x-15$ because $3(x-5) = 3x-15$.

Sometimes, if we study two expressions to find their factors, we might note that some of the factors are the same. These factors are called common factors.

- Consider the numbers 18 and 12. Both 6 and 3 are factors of 18 because $6 \times 3 = 18$. Both 6 and 2 are factors of 12 because $6 \times 2 = 12$. So, 18 and 12 share a common factor, namely 6. In fact 18 and 12 share other common factors. Can you find them ?
- The number 10 and the expression $15x$ share a common factor of 5. Note that $10 = 5 \times 2$, and $15x = 5 \times 3x$. Hence 5 is a common factor.
- $3a^2$ and $5a$ share a common factor of a since $3a^2 = 3a \times a$ and $5a = 5 \times a$. Hence a is a common factor.
- $8x^2$ and $12x$ share a common factor of $4x$ since $8x^2 = 4x \times 2x$ and $12x = 3 \times 4x$. Hence $4x$ is a common factor.

To factorise an expression containing two or more terms it is necessary to look for factors which are common to the different terms. Once found, these common factors are written outside a bracketed term. It is ALWAYS possible to check your answers when you factorise by simply multiplying the brackets again.

For example if we are asked to factorise

$$15x + 10.$$

We first look for any factors which are common to both $15x$ and 10. The common factor here is 5. So the original expression can be written as

$$15x + 10 = 5(3x) + 5(2)$$

which shows clearly the common factor.

This common factor is written outside a bracketed term, the remaining quantities being placed inside the bracket:

$$15x + 10 = 5(3x + 2)$$

and the expression has been factorised. We say that the factors of $15x + 10$ are 5 and $3x + 2$. The answer can be checked by showing

$$5(3x + 2) = 5(3x) + 5(2) = 15x + 10$$

.

Let's consider another example, this time with three terms

$$16a + 40b + 24.$$

The common factor here is 8. The original expression can be written as

$$8(2a) + 8(5b) + 8(3).$$

Again, we write the common factor outside a bracketed term, with the remaining quantities placed inside the bracket:

$$16a + 40b + 24 = 8(2a + 5b + 3).$$

Exercise 4

Factorise the following expressions

1. $10x - 5y$

2. $21 + 7x$

3. $xy - 5x$

4. $4y - 12xy$

5. $16xy - 24yz$

6. $28x + 21y + 70z$

7. $ab + bc - bd$

8. $6x^2 - 12x - 30xy$

Quadratic expressions

A quadratic expression is one of the form

$$ax^2 + bx + c$$

where a, b and c are numbers. The number a is the coefficient of x^2, b is the coefficient of x and c is known as the constant term. The numbers b or c in a quadratic expression can be zero, however, the coefficient a of the x^2 term can never be zero. The following are all quadratic expressions:

$$x^2 + 4x - 12, \qquad 2x^2 - 3, \qquad 3x^2 - 4x, \qquad 9x^2.$$

The variable in a quadratic expression need not always be x. The following are also quadratic expressions:

$$y^2 - 7y - 1\ , \qquad 3t^2 - 5\ , \qquad 3Q^2 + 4Q - 9$$

Factorising quadratic expressions

Consider the expression $(x + 1)(x + 2)$. Multiplying the brackets and simplifying gives the following quadratic expression $x^2 + 3x + 2$.
However, if we were given the quadratic expression first, how would we factorise it? We look at some examples to see how this is done. Note that not all quadratic expressions can be factorised.

- Suppose we wish to factorise $x^2 + 4x - 5$. To start, we write

  $$x^2 + 4x - 5 = (\quad)(\quad)$$

  In order to get the x^2 term we place x and x in the brackets to get

  $$x^2 + 4x - 5 = (x+\ \ )(x+\ \ )$$

  In order to obtain the constant term -5, we consider the factors of -5. These are -1, 5 and 1, -5. Now two possibilities exist:

  $$(x - 1)(x + 5) \qquad \text{and} \qquad (x + 1)(x - 5).$$

  To find the correct one, multiply each out to see which gives the correct middle term, $4x$. So the correct factorisation is

  $$x^2 + 4x - 5 = (x - 1)(x + 5)$$

  This is a trial and error process. However, with practice, you will be able to carry out this process quite easily.

- Let's consider another quadratic expression $x^2 + 8x + 7$. To start, we write

  $$x^2 + 8x + 7 = (\quad)(\quad)$$

  In order to get the x^2 term we place x and x in the brackets to get

  $$x^2 + 8x + 7 = (x+\ \ )(x+\ \ )$$

  In order to obtain the constant term $+7$, we consider the factors of $+7$. These are 1, 7 and -1, -7. Now two possibilities exist:

$$(x+1)(x-7) \qquad \text{and} \qquad (x-1)(x-7).$$

To find the correct one, multiply each out to see which gives the correct middle term, $8x$. So the correct factorisation is

$$x^2 + 8x + 7 = (x+1)(x+7)$$

- Finally we consider a quadratic expression where the coefficient of x^2 is not equal to 1, $3x^2 + 7x + 2$. Again, to start, we write

  $$3x^2 + 7x + 2 = (\quad)(\quad)$$

  In order to get the $3x^2$ term we now place $3x$ and x in the brackets to get

  $$3x^2 + 7x + 2 = (3x+\ \ )(x+\ \ )$$

  In order to obtain the constant term $+2$, we consider the factors of $+2$. These are 1, 2 and -1, -2. Now four possibilities exist:

  $$(3x+2)(x+1), \qquad (3x+1)(x+2), \qquad (3x-2)(x-1) \qquad \text{or} \qquad (3x-1)(x-2).$$

  To find the correct one, multiply each out to see which gives the correct middle term, $7x$. So the correct factorisation is

  $$3x^2 + 7x + 2 = (3x+1)(x+2)$$

- On occasions you will meet expressions of the form $x^2 - y^2$. Such an expression is known as the difference of two squares, as we are finding the difference between two squared terms. It is easy to verify by multiplying out the brackets that this factorises as

  $$x^2 - y^2 = (x+y)(x-y)$$

  So if you can recognise that an expression is in this form, it is quite easy to factorise it:

  $$x^2 - 36 = x^2 - 6^2 = (x+6)(x-6)$$

  $$25x^2 - 9z^2 = (5x)^2 - (3z)^2 = (5x+3z)(5x-3z)$$

  $$\alpha^2 - 1 = \alpha^2 - 1^2 = (\alpha+1)(\alpha-1)$$

Exercise 5

Factorise the following quadratic expressions

1. $x^2 + 6x - 7$

2. $x^2 - 6x + 9$

3. $x^2 + 9x + 14$

4. $2x^2 + 3x + 1$

5. $6x^2 + 7x - 5$

6. $5x^2 - 4x - 1$

7. $x^2 - 144$

8. $4z^2 - 16$

Solutions to exercises

Exercise 1

1. -4
2. $3bc$
3. $13x^2$
4. -7
5. -32.

Exercise 2

1. 1
2. 0
3. 3
4. 12
5. 3
6. $\frac{5}{6}$
7. $\sqrt{17}$.

Exercise 3

1. (a) $2x - 16y$
   (b) $-10x - 2y$
   (c) $10a^2b + 3a^2b^2 - ab^2$
   (d) $-6a + 6b$
2. (a) $x^2 + 5x + 4$
   (b) $3y^2 - y - 4$
   (c) $6x^2 - 13xy + 5y^2$
   (d) $2a^2 + 3ab + b^2 - 3a - 3b$.

Exercise 4

1. $5(2x - y)$
2. $7(3 + x)$
3. $x(y - 5)$
4. $4y(1 - 3x)$
5. $8y(2x - 3z)$
6. $7(4x + 3y + 10z)$
7. $ab + bc + bd = b(a + c + d)$
8. $6x(x - 2 - 5y)$

Exercise 5

1. $(x + 7)(x - 1)$
2. $(x - 3)(x - 3)$
3. $(x + 2)(x + 7)$
4. $(2x + 1)(x + 1)$
5. $(3x + 5)(2x - 1)$
6. $(5x + 1)(x - 1)$
7. $(x + 12)(x - 12)$
8. $(2z + 4)(2z - 4)$