DCU School of Mathematical Sciences

BASIC SKILLS WORKSHEET 5

Algebra III - Equations

The aim of this worksheet is to revise solving linear equations, quadratic equations and some equations involving simple algebraic fractions

What is equation?

An equation is a statement that two expressions are equal to each other. An equation always contains some unknown quantity which we wish to find. For example

$$x + 6 = 1.$$
$$x^2 - 4x + 4 = 0.$$

What does it mean to solve an equation?

To solve an equation involves finding all values of the unknown quantity so that when we substitute each of these values into the equation, the left side equals the right side. These values are called the solutions or roots of the equation. For example, $x = 2$ solves the equation $4x - 8 = 0$ since

$$4(2) - 8 = 0.$$

The equation $x^2 + x - 6 = 0$ is solved by both $x = 2$ and $x = -3$ since

$$(2)^2 + 2 - 6 = 0 \qquad \text{and} \qquad (-3)^2 + (-3) - 6 = 0.$$

Linear Equations

An equation of the form $ax + b = 0$ where a and b are real numbers and x is an unknown quantity is called a linear equation. For example

$$3x + 12 = 0.$$

We see that in these equations the unknown quantity only occurs to the first power (i.e. there are no x^2 or $\sqrt{x}$ or x^3 terms). Linear equations may also appear in forms that seem different from this definition but are actually equivalent. The following are all linear equations as we could write them in the form $ax + b = 0$ if necessary

$$3x + 7 = 1 \qquad (\text{equivalent to } 3x + 6 = 0).$$
$$6 - 2x = 0 \qquad (\text{equivalent to } -2x + 6 = 0).$$
$$3x + 4 = 2x - 5 \qquad (\text{equivalent to } x + 9 = 0).$$

Solving Linear Equations

We solve a linear equation by trying to get the unknown quantity by itself on the left-hand side of the equation. Using the terminology from Algebra II this is equivalent to making the unknown quantity the subject of the equation. We can always check our answer by substitution into the original equation. If our answer does not satisfy the given equation then a mistake has been made.

Solving linear equations - examples

1. Solve the equation

$$3x + 15 = 0.$$

Solution: We wish to make the unknown quantity x the subject of the equation

$$\begin{aligned} 3x + 15 &= 0 && \text{start here} \\ 3x = 0 - 15 &= -15 && \text{subtract 15 from both sides} \\ x &= -5 && \text{dividing both sides by 3.} \end{aligned}$$

x is now the subject of the equation and therefore our solution is $x = -5$. We can check our answer by substitution into the original equation

$$3(\text{-}5) + 15 = 0.$$

2. Solve the equation

$$4x + 7 = 2x + 11.$$

Solution:

$$\begin{aligned} 4x + 7 &= 2x + 11 && \text{start here} \\ 4x - 2x + 7 &= 11 && \text{subtract } 2x \text{ from both sides so get all } x\text{'s on the left} \\ 2x + 7 &= 11 && \text{adding the terms involving } x \text{ together} \\ 2x = 11 - 7 &= 4 && \text{subtracting 7 from both sides} \\ x &= 2 && \text{divide both sides by 2.} \end{aligned}$$

Again we can check our answer by substitution into the original equation

$$4(2) + 7 = 2(2) + 11.$$

3. Solve the equation
$$\frac{7x+12}{4} = 10.$$

Solution:

$$\begin{aligned} \frac{7x+12}{4} &= 10 && \text{start here} \\ 7x+12 &= 40 && \text{multiply both sides by 4} \\ 7x = 40-12 &= 28 && \text{subtract 12 from each side} \\ x &= 4 && \text{divide both sides by 7.} \end{aligned}$$

Again we can check our answer by substitution into the original equation
$$\frac{7(4)+12}{4} = 10.$$

4. Solve the equation
$$3(2t+6) = 4(2-t)$$

Solution:

$$\begin{aligned} 3(2t+6) &= 4(2-t) && \text{start here} \\ 6t+18 &= 8-4t && \text{multiply out the brackets on both sides} \\ 6t &= 8-4t-18 && \text{subtract 18 from each side} \\ 6t+4t &= 8-18 && \text{add } 4t \text{ to each side} \\ 10t &= -10 && \\ t &= -1 && \text{divide both sides by 10.} \end{aligned}$$

Again we can check our answer by substitution into the original equation
$$3(2(-1)+6) = 3(-2+6) = 3(4) = 12 = 4(3) = 4(2+1) = 4(2-(-1)).$$

Exercise 1

1. Which of the following are linear equations.

   (a) $2x + 12 = 14$.

   (b) $5x^2 - 12 = 0$.

   (c) $\dfrac{2y+3}{4} = 5$.

   (d) $3(x+4) = 7x - 2$.

   (e) $2t + 5 = t(t-1)$.

2. Solve the following linear equations

   (a) $4x = 12$.

   (b) $\dfrac{y}{7} = 2$.

   (c) $7 - 3x = 1$.

(d) $5x - 12 = 3x - 8$.

(e) $\dfrac{3t - 7}{2} = 1$.

(f) $3(x - 1) = 2(1 - x)$.

(g) $7(1 - 2x) = 3(3 - 5x)$.

Quadratic Equations

A quadratic equation is an equation of the form

$$ax^2 + bx + c = 0$$

where a,b and c are real numbers and x represents the unknown quantity we wish to find. Notice that the left-hand side of the equation is a quadratic expression where the number a is the coefficient of x^2, b is the coefficient of x and c is known as the constant term. Again, the numbers b or c in a quadratic equation can be zero, however, the coefficient a of the x^2 term can never be zero.

Solving Quadratic equations

We will look at solving quadratic equations by two different methods:

- by factorisation
- by a formula

We will frequently encounter the symbol $\pm$ in this section and it simply means plus *or* minus ($x = \pm 3$ means $x = 3$ or $x = -3$).

Solving quadratic equations by factorisation- examples

1. Solve the equation

$$x^2 - 16 = 0.$$

   Solution:

$$\begin{aligned} x^2 - 16 &= 0 && \text{start here} \\ x^2 &= 16 && \text{add 16 to both sides} \\ x &= \pm 4 && \text{both } 4^2 \text{ and } (-4)^2 \text{ are equal to 16.} \end{aligned}$$

2. Solve the equation

$$x^2 - 2x = 0$$

   Solution:

$$\begin{aligned} x^2 - 2x &= 0 && \text{start here} \\ x(x-2) &= 0 && \text{take out the common factor of } x \text{ on the left-hand side} \end{aligned}$$

   It is important here to remember that if the product of two quantities is zero, then one or both of the quantities must be zero. It follows that either

$$x = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{that is} \quad x = 2$$

3. Solve the equation

$$x^2 - 5x + 6 = 0.$$

Solution: We first note that if we factorise $x^2 - 5x + 6$ we get $(x-3)(x-2)$. If you wish to convince yourself of this just multiply out the brackets. We can now continue as with the last example

$$\begin{aligned} x^2 + 6x + 9 &= 0 && \text{start here} \\ (x-3)(x-2) &= 0 && \text{factorising the quadratic expression on the left} \\ x - 3 = 0 \quad &\text{or} \quad x - 2 = 0 \\ x = 3 \quad &\text{or} \quad x = 2 && \text{solving } x - 3 = 0 \text{ and } x - 2 = 0. \end{aligned}$$

4. Solve the equation

$$x^2 - 5x + 6 = 0.$$

Solution: We first note that if we factorise $2x^2 + 3x - 2$ we get $(2x-1)(x+2)$. If you wish to convince yourself of this just multiply out the brackets. We can now continue as with the previous examples

$$\begin{aligned} 2x^2 + 3x - 2 &= 0 && \text{start here} \\ (2x-1)(x+2) &= 0 && \text{factorising the quadratic expression on the left} \\ 2x - 1 = 0 \quad &\text{or} \quad x + 2 = 0 \\ x = \frac{1}{2} \quad &\text{or} \quad x = -2 && \text{solving } 2x - 1 = 0 \text{ and } x + 2 = 0. \end{aligned}$$

Exercise 2

Solve the following quadratic equations:

(a) $x^2 = 64$.

(b) $x^2 - 36 = 0$.

(c) $x^2 + 7x = 0$.

(d) $2x^2 - 16x = 0$.

(e) $x^2 + x - 2 = 0$

(f) $x^2 - 7x + 12 = 0$

(g) $x^2 - x - 20 = 0$

(h) $2x^2 - 5x + 2 = 0$

(i) $6x^2 - x - 1 = 0$

Quadratic Formula

Sometimes it is difficult or impossible to factorise a quadratic expression. For example, try to factorise

$$2x^2 + 3x - 6.$$

However, in these cases, it may be possible to solve the equation using the following formula:
If $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula gives two solutions, one taking the positive square root and the other taking the negative square root.

Using the quadratic formula - examples

1. Solve the equation

$$x^2 + 7x + 12 = 0.$$

Solution: In this case $a = 1$, $b = 7$ and $c = 12$. Substituting these values into our formula gives

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4(1)(12)}}{2(1)} && \text{substitution into the formula} \\ x &= \frac{-7 \pm \sqrt{1}}{2} && \text{simplifying the term under the square root} \\ x &= \frac{-7+1}{2} = -3 \text{ or } x = \frac{-7-1}{2} = -4 && \text{evaluating the square root.} \end{aligned}$$

2. Solve the equation

$$y^2 + 5y - 3 = 0.$$

Solution: Note that a quadratic does not always have to be written in terms of x, here the equation is written in terms of y. In this case $a = 1$, $b = 5$ and $c = -3$. Substituting these values into our formula gives

$$\begin{aligned} y &= \frac{-5 \pm \sqrt{5^2 - 4(1)(-3)}}{2(1)} && \text{substitution into the formula} \\ y &= \frac{-5 \pm \sqrt{37}}{2} && \text{simplifying the term under the square root} \\ y &= 0.541 \text{ or } y = -5.541 && \text{evaluating correct to three decimal places.} \end{aligned}$$

3. Solve the equation
$$x^2 - 8x + 16 = 0.$$

Solution: In this case $a = 1$, $b = -8$ and $c = 16$. Substituting these values into our formula gives

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(16)}}{2(1)} \qquad \text{substitution into the formula}$$

$$x = \frac{8 \pm \sqrt{0}}{2} \qquad \text{simplifying the term under the square root}$$

$$x = 4(\text{twice}) \qquad \text{we get a single repeated root.}$$

Looking at the above example we see that because $b^2 - 4ac = 0$ the quadratic equation had just one repeated root . The effect has $b^2 - 4ac$ in determining the type of roots / solutions we get is summarised below

1. $b^2 - 4ac > 0$ two distinct real roots.
2. $b^2 - 4ac = 0$ a single root known as a repeated root.
3. $b^2 - 4ac < 0$ no real roots.

Exercise 3

1. Say whether each of these equations has two distinct real roots, one repeated root or no real roots.

   (a) $x^2 + 4x - 12 = 0$.

   (b) $5x^2 - 12x + 24 = 0$.

   (c) $2x^2 + 13x - 4 = 0$.

   (d) $x^2 - 6x + 9 = 0$.

   (e) $2x^2 - 3x - 4 = 0$.

2. Using the quadratic formula solve the following quadratic equations

   (a) $2x^2 - 7x - 15 = 0$.

   (b) $6x^2 + 11x - 2 = 0$.

   (c) $4x^2 - 12x + 9 = 0$.

(d) $2x^2 - 7x + 4 = 0.$

(e) $9x^2 - 30x + 25 = 0.$

Solving equations involving simple rational functions

Now that we can solve linear and quadratic equations we have all the tools we need to solve equations involving some simple rational functions. (A rational function is a function which has the form of a fraction. There may be terms involving x both above and below the line.) In most of these problems all we do is multiply across by an appropriate term in order to remove the denominator. We then proceed as with solving linear equations .

Examples

1. Solve the equation
$$\frac{x}{3} = \frac{x+1}{2} - 7$$
Solution:
$$\begin{array}{rcll} \frac{x}{3} & = & \frac{x+1}{2} - 7 & \text{start here} \\ x & = & 3\frac{(x+1)}{2} - 7(3) & \text{multiply across by 3} \\ 2x & = & 3(x+1) - 7(3)(2) & \text{multiply across by 2} \\ 2x & = & 3x + 3 - 42 & \text{removing the brackets} \\ x & = & 39 & \text{solving the resulting linear equation.} \end{array}$$

2. Solve the equation
$$\frac{3}{x+2} = \frac{5}{2x-4}$$
Solution:
$$\begin{array}{rcll} \frac{3}{x+2} & = & \frac{5}{2x-4} & \text{start here} \\ 3 & = & \frac{5}{2x-4}(x+2) & \text{multiply across by } (x+2) \\ 3(2x-4) & = & 5(x+2) & \text{multiply across by } 2x-4) \\ 6x-12 & = & 5x+10 & \text{removing the brackets} \\ x & = & 22 & \text{solving the resulting linear equation.} \end{array}$$

3. Solve the equation
$$\frac{3x-2}{x+7} = 2$$
Solution:
$$\begin{array}{rcll} \frac{3x-2}{x+7} & = & 2 & \text{start here} \\ 3x-2 & = & 2(x+7) & \text{multiply across by } (x+7) \text{ to remove the denominator} \\ 3x-2 & = & 2x+14 & \text{removing the brackets} \\ x & = & 16 & \text{solving the resulting linear equation.} \end{array}$$

4. Solve the equation

$$\frac{2x^2 - 3x + 5}{4} = \frac{x^2 - 2x + 3}{3}$$

Solution:

$$\begin{array}{rcll}
\dfrac{2x^2 - x + 5}{4} & = & \dfrac{x^2 - 2x + 3}{3} & \text{start here} \\
(2x^2 - x + 5) & = & 4\dfrac{(x^2 - 2x + 3)}{3} & \text{multiply across by 4} \\
3(2x^2 - x + 5) & = & 4(x^2 - 2x + 3) & \text{multiply across by 3} \\
6x^2 - 3x + 15 & = & 4x^2 - 8x + 12 & \text{removing the brackets} \\
2x^2 - 3x + 15 & = & -8x + 12 & \text{subtract } 4x^2 \text{ from both sides} \\
6x^2 + 5x + 15 & = & 12 & \text{add } 5x \text{ to both sides} \\
2x^2 + 5x + 3 & = & 0 & \text{subtract 12 from both sides} \\
x = -1 & \text{or} & x = -\dfrac{3}{2} & \text{solving using the quadratic formula.}
\end{array}$$

5. Solve the equation

$$\frac{x^2 + 2x + 1}{x + 1} = 4$$

Solution:

$$\begin{array}{rcll}
\dfrac{x^2 + 2x + 1}{x + 1} & = & 4 & \text{start here} \\
x^2 + 2x + 1 & = & 4(x + 1) & \text{multiply across by } x + 1 \\
x^2 + 2x + 1 & = & 4x + 4 & \text{multiply out the bracket} \\
x^2 - 2x + 1 & = & 4 & \text{add } 4x \text{ to both sides} \\
x^2 - 2x - 3 & = & 0 & \text{add 4 to both sides} \\
x = -1 & \text{or} & x = 3 & \text{solving using the quadratic formula or factorisation}
\end{array}$$

Exercise 4

Solve the following equations

1. $\dfrac{x+2}{3} - x = 4.$

2. $\dfrac{3x+20}{7x+1} = 3.$

3. $\dfrac{2x-3}{3} = \dfrac{x+4}{2}$.

4. $\dfrac{3}{x-7} = \dfrac{2}{x+3}$.

5. $\dfrac{7x+4}{x+3} = 8.$

6. $\dfrac{x^2+x-6}{7} = \dfrac{x^2+x-8}{6}.$

7. $\dfrac{x^2+5x+1}{x+3}=3.$

Solutions to exercises

Exercise 1

1. (a) linear
   (b) not linear
   (c) linear
   (d) linear
   (e) not linear
2. (a) $x = 3$
   (b) $y = 14$
   (c) $x = 2$
   (d) $x = 2$
   (e) $t = 3$
   (f) $x = 1$
   (g) $x = 2$

Exercise 2

1. (a) $x = \pm 8$
   (b) $x = \pm 6$
   (c) $x = 0$ or $x = -7$
   (d) $x = 0$ or $x = 8$
   (e) $x = -2$ or $x = 1$
   (f) $x = 3$ or $x = 4$
   (g) $x = -4$ or $x = 5$
   (h) $x = \frac{1}{2}$ or $x = 2$

Exercise 3

2. (a) two distinct real roots
   (b) no real roots
   (c) two distinct real roots
   (d) one repeated root
   (e) two distinct real roots

3. (a) $x = 5$ or $x = -\frac{3}{2}$
   (b) $x = -2$ or $x = \frac{1}{6}$
   (c) $x = \frac{3}{2}$ twice
   (d) $x = 2.781$ or $x = 0.719$
   (e) $x = \frac{5}{3}$ twice.

Exercise 4

1. $x = -5$.
2. $x = \frac{17}{18}$
3. $x = 18$
4. $x = -23$
5. $x = -20$
6. $x = 4$ or $x = -5$
7. $x = -4$ or $x = 2$