DCU School of Mathematical Sciences

BASIC SKILLS WORKSHEET 6

Indices and powers

The aim of this worksheet is to revise the maths of powers ($x^2, \sqrt{x}$ etc).

What are indices?

Recall that we use this shorthand notation:

$$\begin{aligned} 2 \times 2 &= 2^2, \\ 2 \times 2 \times 2 &= 2^3, \\ 2 \times 2 \times 2 \times 2 &= 2^4, \end{aligned}$$

and so on. The numbers 2,3,4 here are called *indices* (this is the plural of 'index'). So the word index is just another word for 'power'. In all cases here, the number 2 is called the base. Thus a^m has base a and index m.

What are they for?

They are for making life easier. Not only do they save time in writing down certain pieces of maths, but they can also simplify certain problems. The aim here is to learn how to deal with indices in order to exploit this.

The basics

In line with what we have above,

$$2^m = \underbrace{2 \times 2 \times \cdots \times 2}_{2 \text{ appears } m \text{ times.}}$$

This works for any positive whole number m. So

$$\begin{aligned} 3^3 &= 3 \times 3 \times 3 = 27, \\ 2^8 &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256. \end{aligned}$$

The first law of indices

This says that

$$a^m \times a^n = a^{m+n}.$$

Let's check this:

$$\begin{aligned} 2^3 \times 2^4 &= (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) \\ &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^7 \\ &= 2^{3+4}. \end{aligned}$$

This allows us to simplify certain expressions, and so make any equations involving these expressions easier to solve.

Examples

1. Simplify $4^6 \times 4^7$.
   Solution: $4^6 \times 4^7 = 4^{6+7} = 4^{13}$.

2. Simplify $a^2b^3a^4b^5$.
   Solution: The key here is that we can only combine powers attached to the same base. So

$$\begin{aligned} a^2b^3a^4b^5 &= (a^2 \times a^4) \times (b^3 \times b^5) \\ &= a^{2+4} \times b^{3+5} \\ &= a^6b^8. \end{aligned}$$

3. Simplify $3x^2 \times 4x^4$.
   Here, we must keep the numerical part and the x's separate.

$$\begin{aligned} 3x^2 \times 4x^4 &= (3 \times 4) \times (x^2 \times x^4) \\ &= 12x^6. \end{aligned}$$

The second law of indices

This says that

$$\frac{a^m}{a^n} = a^{m-n}.$$

Let's check this:

$$\begin{aligned} \frac{3^8}{3^5} &= \frac{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3 \times 3} \\ &= 3 \times 3 \times 3 \\ &= 3^3 = 3^{8-5}. \end{aligned}$$

Examples

1. Simplify $\dfrac{5^7}{5^3}$.
   Solution: $\dfrac{5^7}{5^3} = 5^{7-3} = 5^4$.

2. Simplify $\dfrac{a^4b^5}{a^2b^2}$.
   Solution: Again, the key is that we can only combine powers attached to the same base. So

$$\begin{aligned} \frac{a^4b^5}{a^2b^2} &= \frac{a^4}{a^2} \times \frac{b^5}{b^2} \\ &= a^{4-2} \times b^{5-2} \\ &= a^2b^3. \end{aligned}$$

3. Simplify $\dfrac{-4y^7}{2y^3}$.

   We must keep the numerical part and the y's separate.

$$\begin{aligned}\frac{-4y^7}{2y^3} &= \frac{-4}{2} \times \frac{y^7}{y^3} \\ &= -2y^4.\end{aligned}$$

Some special cases

Any number divided by itself gives one. So for example,

$$\frac{2^6}{2^6} = 1.$$

But according to the second law,

$$\frac{2^6}{2^6} = 2^{6-6} = 2^0.$$

This would work for any number a, not just 2. So we have this fact:

For any number a, $a^0 = 1$.

Similarly, by cancelling out powers of 2, we have

$$\begin{aligned}\frac{2^5}{2^4} &= \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2} = 2 \\ &= 2^{5-4} = 2^1, \qquad \text{using the second law.}\end{aligned}$$

Again, this would work for any number a, not just 2. So we have

For any number a, $a^1 = a$.

The third law of indices

This is about 'powers of powers' and says

$$(a^m)^n = a^{m \times n}.$$

Let's check this:

$$\begin{aligned}(5^2)^3 &= \underbrace{5^2 \times 5^2 \times 5^2}_{(5^2 \text{ appears three times})} \\ &= 5^{2+2+2} \qquad \text{using 1st law} \\ &= 5^{2\times 3}.\end{aligned}$$

$$\begin{aligned}(4^3)^4 &= \underbrace{4^3 \times 4^3 \times 4^3 \times 4^3}_{(4^3 \text{ appears four times})} \\ &= 4^{3+3+3+3} \qquad \text{using 1st law} \\ &= 4^{3\times 4}.\end{aligned}$$

$$
\begin{aligned}
(a^2b^3)^4 &= \underbrace{a^2b^3 \times a^2b^3 \times a^2b^3 \times a^2b^3}_{(a^2b^3 \text{ appears four times})} \\
&= a^2 \times b^3 \times a^2 \times b^3 \times a^2 \times b^3 \times a^2 \times b^3 \\
&= (a^2 \times a^2 \times a^2 \times a^2) \times (b^3 \times b^3 \times b^3 \times b^3) \\
&= a^{2+2+2+2}b^{3+3+3+3} \qquad \text{using 1st law} \\
&= a^{2\times 4}b^{3\times 4}.
\end{aligned}
$$

$$
\begin{aligned}
(2x^2)^5 &= \underbrace{2x^2 \times 2x^2 \times 2x^2 \times 2x^2 \times 2x^2}_{(2x^2 \text{ appears five times})} \\
&= 2 \times x^2 \times 2 \times x^2 \times 2 \times x^2 \times 2 \times x^2 \times 2 \times x^2 \\
&= (2 \times 2 \times 2 \times 2 \times 2) \times (x^2 \times x^2 \times x^2 \times x^2 \times x^2) \\
&= 2^{1+1+1+1+1}x^{2+2+2+2+2} \qquad \text{using 1st law} \\
&= 2^{1\times 5}x^{2\times 5}.
\end{aligned}
$$

Exercise 1

1. Use the three laws of indices to simplify these expressions.

   (a) $12^7 \times 12^3$

   (b) $5^3 \times 5^{13}$

   (c) $\dfrac{8^5}{8^3}$

   (d) $\dfrac{t^5}{t^5}$

   (e) $(a^3b^5)(a^7b^2)$

   (f) $(x^2y^3)(x^6y^2)$

   (g) $\dfrac{x^5y^6}{x^3y^2}$

   (h) $(x^2y^3)^4$

   (i) $(3xy^2)^3$

Negative powers

We have this rule for negative powers:

$$a^{-m} = \frac{1}{a^m}.$$

So for example $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$.

Fractional powers

What does $3^{\frac{1}{2}}$ mean? To find out, multiply this number by itself:

$$\begin{aligned} 3^{\frac{1}{2}} \times 3^{\frac{1}{2}} &= 3^{\frac{1}{2}+\frac{1}{2}} \\ &= 3^1 = 3, \end{aligned}$$

using the first law of indices. So $3^{\frac{1}{2}}$ is the number which when multiplied by itself gives 3, that is $3^{\frac{1}{2}} = \sqrt{3}$.

Similarly, what does $4^{\frac{1}{3}}$ mean? This time we multiply this number by itself three times:

$$\begin{aligned} 4^{\frac{1}{3}} \times 4^{\frac{1}{3}} \times 4^{\frac{1}{3}} &= 4^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= 4^1 = 4, \end{aligned}$$

using the first law of indices. So $4^{\frac{1}{3}}$ is the number which when multiplied by itself three times gives 4. This can also be written as $4^{\frac{1}{3}} = \sqrt[3]{4}$.

In general, for any number a,

$$\begin{aligned} a^{\frac{1}{n}} &= \sqrt[n]{a} \\ &= n^{th}\text{root of } a. \end{aligned}$$

That is $a^{\frac{1}{n}}$ is the number which when multiplied by itself n times is a.

Examples

1. $3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$.
2. $2^{-5} = \dfrac{1}{2^5} = \dfrac{1}{32}$.
3. $64^{\frac{1}{3}} = 4$, since $4^3 = 64$.
4. $81^{\frac{1}{4}} = 3$, since $3^4 = 81$.
5. $16^{-\frac{1}{2}} = \dfrac{1}{16^{\frac{1}{2}}} = \dfrac{1}{\sqrt{16}} = \dfrac{1}{4}$.
6. $125^{-\frac{1}{3}} = \dfrac{1}{125^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{125}} = \dfrac{1}{5}$.

The important results of this sheet are summarised below.

$$a^m \times a^n = a^{m+n}$$
$$\frac{a^m}{a^n} = a^{m-n}$$
$$(a^m)^n = a^{m \times n}$$
$$a^0 = 1$$
$$a^{-m} = \frac{1}{a^m}$$
$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Exercise 2

1. Evaluate these numbers.

   (a) $25^{\frac{1}{2}}$

   (b) $27^{\frac{1}{3}}$

   (c) $64^{-\frac{1}{3}}$

   (d) 10^{-3}

2. Explain the meaning of the number $x^{\frac{1}{4}}$.

3. Simplify the following expressions.

   (a) $\dfrac{x^4x^{-3}}{x^5x^{-2}}$

   (b) $y^3\sqrt{y}$

   (c) $\sqrt[3]{a^4b^5}$

Applications

Suppose we want to solve the equation

$$(1+x)^7 = 1.35$$

This type of equation arises in financial maths and elsewhere. We need to isolate the x. No amount of adding, subtracting or dividing across will help here. We need to use indices. The key is to remember that $a^1 = a$ for any number a. So how could we get $(1+x)^1$? Here's how:

$$((1+x)^7)^{\frac{1}{7}} = (1+x)^{7\times\frac{1}{7}} = (1+x)^1 = 1+x.$$

So to solve the equation, we take the 1/7th power of both sides. This gives

$$1+x = (1.35)^{\frac{1}{7}} = 1.0438 \qquad \text{(correct to 4 decimal places)}.$$

Then we subtract 1 to get the answer,

$$x = 0.0438$$

Exercise 3

1. Use the method above to solve these equations.

   (a) $(1+x)^5 = 32$

   (b) $(1+\frac{x}{100})^{12} = 1.5$

   (c) $(1-x)^3 = 0.2335$

   (d) $(1-\frac{y}{24})^{18} = 100$

Solutions to exercises

Exercise 1

1. (a) 12^{10}
   (b) 5^{16}
   (c) 8^2
   (d) 1
   (e) $a^{10}b^7$
   (f) x^8y^5
   (g) x^2y^4
   (h) x^8y^{12}
   (i) $27x^3y^6$

Exercise 2

1. (a) 5
   (b) 3
   (c) $\frac{1}{4}$
   (d) 0.001
2. $x^{\frac{1}{4}}$ is the fourth root of the number x. That is, when $x^{\frac{1}{4}}$ is multiplied by itself four times, the result if x.
3. (a) x^{-2}
   (b) $y^{\frac{7}{2}}$
   (c) $a^{\frac{4}{3}}b^{\frac{5}{3}}$

Exercise 3

1. (a) 1
   (b) 3.4366
   (c) 0.3842
   (d) -6.9972