DCU School of Mathematical Sciences

BASIC SKILLS WORKSHEET 7

Logarithms

The aim of this worksheet is to introduce the basics of logarithms and their applications.

What are logarithms?

The equation

$$2^3 = 8$$

expresses a certain relationship between the numbers 2, 3 and 8. There is no reason why we shouldn't express this relationship in a different way. In the 17th century, the Scots mathematician John Napier discovered/invented a different and very useful way of writing this simple relationship. He defined logarithms (logs) by rewriting the equation above in the form

$$\log_2 8 = 3.$$

We read this as "log to the base 2 of 8 equals 3". That's all there is to logs, it's just a different way of expressing the relationship between certain numbers.

Let me repeat that.

That's all there is to logs, it's just a different way of expressing the relationship between certain numbers.
Every equation which can be written using an index or power (this is called an exponential form) can also be written in a corresponding logarithmic form.

Examples

- $\log_2 8 = 3$ because $2^3 = 8$.
- $\log_4 16 = 2$ because $4^2 = 16$
- $\log_2 64 = 6$ because $2^6 = 64$
- $\log_2 \frac{1}{8} = -3$ because $2^{-3} = \frac{1}{8}$
- $\log_5 15625 =?$ because $5^? = 15625$
- $\log_3 \frac{1}{9} =?$ because $3^? = \frac{1}{9}$
- $\log_3 81 =?$ because $3^? = 81$
- $\log_4 256 =?$ because $4^? = \frac{?}{256}$

Exercise 1

Write down the value of

1. $\log_3 27$

2. $\log_{10} 1000$

3. $\log_2 \frac{1}{4}$

4. $\log_5 625$

Note

$\log_{10}$ is used very frequently. For this reason, the 10 is usually omitted, so

$$\log = \log_{10} .$$

This is the "log" key that appears on your calculator. Practice using it; you should find that $\log 10 = 1$, $\log 100 = 2$, $\log 34.2 = 1.534026...$

Laws of logarithms

The three laws of indices can be translated into three corresponding laws of logarithms. As with the laws of indices, these can be used to simplify expressions and to shorten calculations.

$$1.\ \log(ab) = \log a + \log b.$$
$$2.\ \log \tfrac{a}{b} = \log a - \log b.$$
$$3.\ \log a^m = m \log a.$$

These laws apply for logs to any base.

Examples

- $\log 4 + \log 3 = \log 12$. You can check this result on your calculator.
- $\log_2 6 + \log_2 x = \log_2 6x$.
- $\log_3 25 - \log_3 5 = \log_3 5$.
- $\log 16 = \log 4^4 = 4 \log 4$. You can check this result on your calculator.

Exercise 2

Simplify the following

1. $\log_4 3 + \log_4 6$

2. $\log 4x + \log x$

3. $\log 14 - \log 7$

4. $\log_2 15x - \log_2 5$

5. $\log 1000$

6. $\log_4 x^4$

We now look at one of the reasons people bother studying logs. Consider the equation

$$2^x = 15.$$

(This equation wouldn't be any problem if it were $2^x = 16$. Then it would simply be a matter of choosing the correct whole number value for x, which would be $x = 4$ in this case.) Back to $2^x = 15$. We have $2^3 = 8$ and $2^4 = 16$, so $x = 3$ is too small and $x = 4$ is too big. How do we find the value of x that is just right?

The answer is to use logs, and especially the third law. Let's take log of both sides of the equation

$$\begin{aligned} 2^x &= 15 \\ \Rightarrow \log 2^x &= \log 15 \\ \Rightarrow x \log 2 &= \log 15 \qquad \text{using the third law} \\ \Rightarrow x &= \frac{\log 15}{\log 2} \\ &= 3.91 \qquad \text{correct to 2 decimal places.} \end{aligned}$$

Let's look at another example:

$$4^x = 20.$$

Again we take the log of both sides of the equation

$$\begin{aligned} 4^x &= 20 \\ \Rightarrow \log 4^x &= \log 20 \\ \Rightarrow x \log 4 &= \log 20 \qquad \text{using the third law} \\ \Rightarrow x &= \frac{\log 20}{\log 4} \\ &= 2.16 \qquad \text{correct to 2 decimal places.} \end{aligned}$$

Exercise 3

Solve the following equations for x.

1. $3^x = 25$

2. $(1.025)^x = 2.654$

3. $100(1.05)^x = 1650$

4. $10^{2x} = 39$

Solutions to exercises

1.
   1. 3
   2. 3
   3. -2
   4. 4
2.
   1. $\log_4 18$
   2. $\log 4x^2$
   3. $\log 2$
   4. $\log_2 5x$
   5. $3\log 10$
   6. $4\log_4 x$
3.
   1. 2.9300
   2. 39.5287
   3. 57.4575
   4. 0.7955