DCU School of Mathematical Sciences

BASIC SKILLS WORKSHEET 8

Functions and graphs

The aim of this worksheet is to revise some maths relating to functions, their graphs and some applications.

What is a function?

A function is a rule that takes an input and generates an output. It is like a machine: we throw the input into one end of the machine, it gets to work, and then spits out the output at the other end:

INPUT x — ACME DEVICES — OUTPUT y

Figure 1: Representation of a function

For example, the rule might be:

Add 6 to the input.

Then if the input is 4, the rule tells us that the output would be 10. If the input is 5, then the output is 11.

What is it for?

Functions are one of the most important tools in maths because they are used for describing how different quantities depend upon one another. For example the rule

Multiply the input by itself and then multiply by π

is a function that tells us how to find the area (output) of a circle in terms of its radius (input).

What is a graph?

This is basically a picture of a function that can tell us, at a glance, lots of useful information about the function.

Functions and formulae

In most applications, the inputs for functions are numbers (represented by letters; i.e. the inputs are variables), and the outputs are also numbers. We often have a formula which tells use the value of the output once we know the value of the input. For example, the function

Add 6 to the input

takes as input any number x and generates the output $x + 6$. If we give the function a name - let's call it f - then we can represent this as follows:

$$f : x \rightarrow x + 6.$$

This should be read

f takes the input x and generates the output $x + 6$.

We also use the notation

$$f(x) = x + 6$$

to mean exactly the same thing. Note that

$$f(t) = t + 6$$

is also exactly the same function: it takes the input and adds 6 to generate the output.

It is very common to use the name x for inputs and y for outputs, so we could use the formula

$$y = x + 6$$

to describe the function above.

Example

Write down a formula to describe the function

multiply the input by 3 and then add 7.

Evaluate the output for the inputs (a) 2, (b) 3, (c) -4.

Let's call the input t, which can represent any number, and call the function g. Then

$$g : t \rightarrow 3t + 7$$

(read: "g takes the input t and generates the output $3t + 7$") or

$$g(t) = 3t + 7.$$

The three different outputs are

$$\begin{aligned} g(2) &= 3(2) + 7 = 6 + 7 = 13, \\ g(3) &= 3(3) + 7 = 9 + 7 = 16, \\ g(-4) &= 3(-4) + 7 = -12 + 7 = -5. \end{aligned}$$

Note: $g(2)$ is read as 'g of 2' and means g evaluated at $t = 2$, not g multiplied by 2.

A warning

Not all functions have formulae which can be written down as a collection of $x's$ and numbers. For example, the rule

write down the height of the input

has "people" as inputs, and their heights as outputs. So "functions" and "formulae" are not exactly the same.

Exercise 1

1. Write down a formula to describe these functions, and calculate the outputs for each of the different functions when the inputs are 1, 3, and -2.

   (a) Double the input and subtract 8.

   (b) Multiply the input by itself and then divide by 4.

   (c) Multiply the input by 5, add 12 and then divide the result by 7.

   (d) Add 1 to the input and then take the square root of the result.

2. A function h is defined by $h(x) = 4x + 9$. Calculate

   (a) $h(0)$

   (b) $h(2)$

   (c) $h(t)$

   (d) $h(2x)$.

The domain and range of a function

In Exercise 1.1(d), we found the value $\sqrt{-1}$ for one of the outputs. This is not a real number. The problem is that we used an "illegal" input value. We should only use input values that lead to outputs which are real numbers. Given a function f, the set of all allowed input values is called the *domain* of the function. The set of all possible output values is called the *range* of the function. In other words

- The domain of a function is the set of all the numbers you can plug into the function.
- The range of a function is the set of all the numbers you can get out of the function.

Let's do an example

$$f(x) = 2x + 1$$

What's the domain? Well, you can plug any real number you want into this function: 4, or -7, 1.01738, etc., and the world doesn't blow up. So $f(x)$ has as its domain the set of all real numbers. This is also written as $-\infty < x < \infty$ or $x \in \mathbb{R}$.

What's the range? From the definition of $f(x)$ we see that any number can be obtained as an output. So $f(x)$ has as its range the set of all real numbers. This is written as $-\infty < y < \infty$, where we are using y to represent the outputs of the function.

Now let's consider

$$g(x) = x^2$$

What's the domain? Again you can plug any real number you want into this function. So the domain is the set of all real numbers,$-\infty < x < \infty$.

What's the range? Well, let's think about it. If we plug any number into this function, are we ever going to be able to get a negative number out of it? The answer is no. So it looks like the range of this function is the set of all non-negative numbers (the positive numbers plus zero) or $0 \leq y < \infty$.

Another example

$$h(x) = \sqrt{x} \quad \text{(where we only consider the positive square root of a number)}$$

What's the domain? As we can't get the square root of a negative number, we can only plug non-negative numbers in. So the domain is $0 \leq x < \infty$.

What's the range? As we are only considering the positive square root, the outputs will not contain any negative numbers. So the range is $0 \leq y < \infty$.

One more example

$$k(x) = \frac{1}{x-5}$$

What's the range? What numbers can we plug into this function? Well, it looks like any number will do. But wait! What happens if we plug in 5? We divide by zero, and the world blows up. So let's not plug in 5. Since any other number is fine, the domain of this function is: all real numbers except 5 or $x \in \mathbb{R} \setminus \{5\}$.

What's the range? I'll leave that one to you

In general, when you're trying to find the domain of a function, there are two things you should look out for. They are

1. Look for potential division by zero.

2. Look for places where you might take the square root of a negative number.

There might be other reasons for restricting the domain of a function other than finding 'illegal' outputs. For example, if our input represented the length of a piece of string, then the domain should only have positive numbers in it. If our input represented the size of the workforce, or the number of molecules in a jar, then the domain should only have positive whole numbers.

Graphs I

A graph is a picture of a function. What we need is a picture that will tell us the output values corresponding to different input values. To do this, we need some machinery.

The coordinate plane

This is a chart on which we draw graphs. It is constructed as follows.

- Draw a horizontal number line to mark the input values. If our input is called x, we call this the "x-axis".
- Draw a vertical number line, crossing the horizontal line, to mark the output values. If our output is called y, we call this the "y-axis".
- Draw a scale on each axis to mark the position of different numbers. For each axis, put 0 at the point where the axes intersect. We call this point the *origin*.
  - On the horizontal axis, mark negative numbers to the left and positive numbers to the right of the origin.
  - On the vertical axis, mark negative numbers below and positive numbers above the origin.

We call this the $x - y$ coordinate place or just the $x - y$ plane. (Here $x - y$ does not mean "x minus y"; it is read as "x y plane".)

Points in the plane

The $x - y$ plane is basically a map for locating any point on the plane. We do this as follows. To label the location of the point A (to give its "address")

- Draw a vertical line from A and mark off the value of x where this hits the horizontal axis.
- Draw a horizontal line from A and mark off the value of y where this hits the vertical axis.

The first number is called "the x-coordinate of the point A" and the second number is called "the y-coordinate of the point A". We write $A : (2, 6)$ or $A = (2, 6)$ to indicate that A is the point with these coordinates.

On the other hand, given the coordinates, we can draw the corresponding point in the diagram as follows. For example, to locate the point B with coordinates $(4, 3)$,

- Start at the origin.
- Move horizontally along the x-axis to the correct x-value ($x = 4$ in this case).
- Move vertically from here to the correct y- value ($y = 3$ in this case).

This gives us the point B in the diagram.

Other examples: point C has coordinates $(-3, 5)$. Point D has coordinates $(6.4, -3)$.

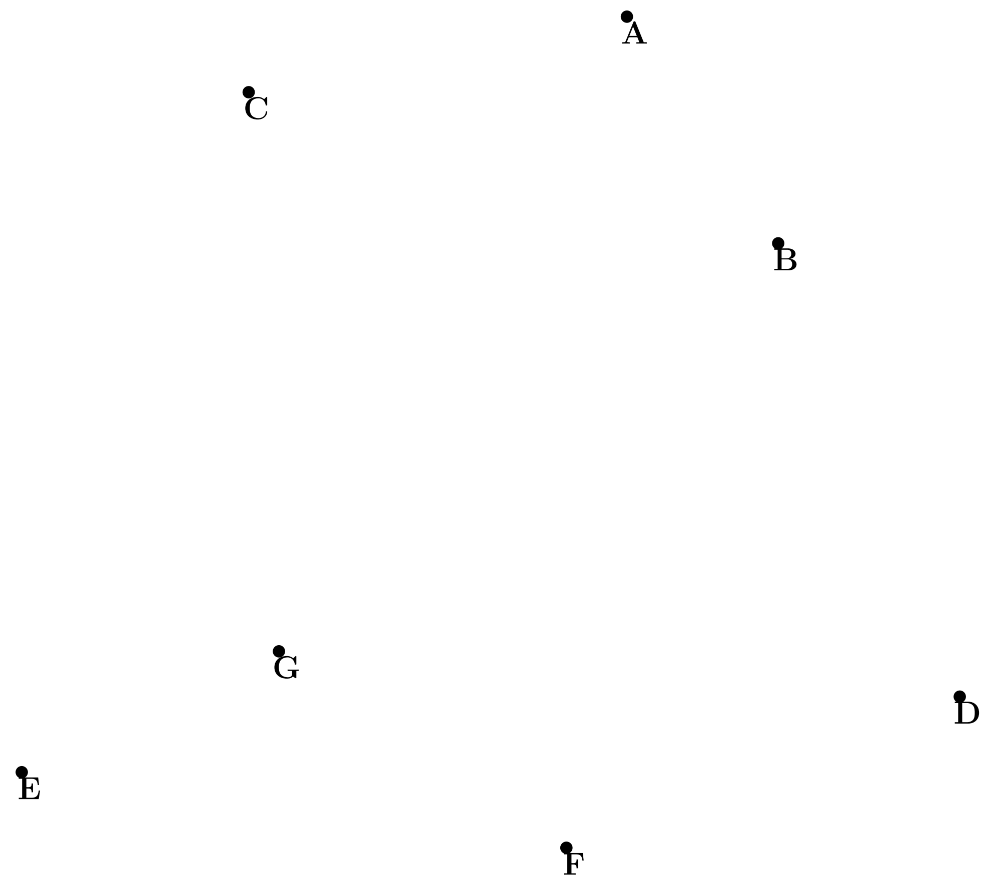

Figure 2: The coordinate plane

Exercise 2

1. Write down the coordinates of the points E, F, G in Figure 2.
2. Mark the points $(1, -1), (3, 1), (-1, -3), (-3, -5)$ in the diagram.

Graphs II

In the last exercise, you should notice that the four points $(1,-1),(3,1),(-1,-3),(-3,-5)$ all lie on the same straight line. This suggests that the $x-$ and $y-$coordinates for these four points all have something in common. In fact it is easy to check that each pair of (x,y) values satisfies the equation

$$y = x - 2.$$

In other words, these four pairs of numbers are input-output pairs for the function

$$f : x \to x - 2.$$

The line that we found is the *graph* of this function:

- **Every** point (x,y) on this line satisfies the rule $y = x - 2$;
- **Only** points on this line satisfy the rule.

This is the definition of the graph of a function f. So in general, the graph of a function f is a curve in the $x-y$ plane such that

- **Every** point (x,y) on this curve satisfies the rule $y = f(x)$;
- **Only** points on this curve satisfy this rule.

Example

The purpose of this example is to demonstrate some different uses of graphs. Figure 3 shows the graph of a function

$$P : Q \to f(Q)$$

which describes the profit P made by a firm by selling Q units of a certain commodity. We can use this diagram to answer the following questions.

1. Estimate the profit P when sales are at $Q = 10$.
2. For what values of Q will the firm break even?
3. For what range of values of Q will the firm make a profit?
4. For what range of values of Q will the firm make a loss?
5. What level of sales Q will give the largest profit?
6. Suppose that sales are currently $Q = 40$. Should the firm attempt to increase sales or decrease sales in order to increase profit?
7. Suppose that sales are currently $Q = 16$. Should the firm attempt to increase sales or decrease sales in order to increase profit?

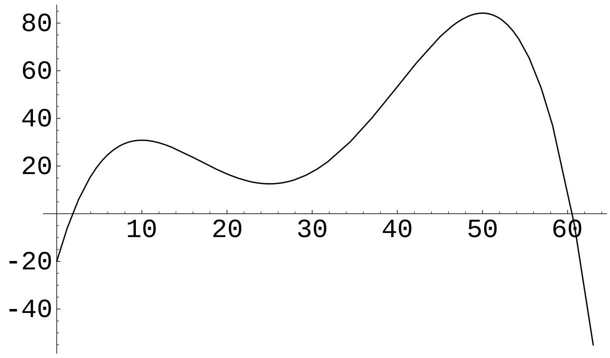

Figure 3: Graph of a profit function. The horizontal axis is the $Q-$axis, and the vertical is the $P-$axis.

Answers.

1. On the diagram, move up from $Q = 10$ to hit the graph. Moving horizontally across, we hit the vertical axis at (roughly) $P = 30$.
2. The firm breaks even when $P = 0$. This is where the graph hits the horizontal axis, which is at (roughly) $Q = 2$ and $Q = 60$.
3. Profit means $P > 0$. This is where the graph lies above the horizontal axis, which is for Q between 2 and 60.
4. Loss means $P < 0$. This is where the graph lies below the horizontal axis, which is for Q less than 2 or Q bigger than 60.
5. The highest value of P occurs at (roughly) $Q = 50$.
6. At the portion of the graph nearest to $Q = 40$, we can see that the graph is increasing from left to right. Thus to get to higher vales of P, we should move Q to the right, i.e. we should increase Q.
7. Here the graph *decreases* from left to right. So to get to higher values of P, we should move to the left, i.e. we should decrease Q.

This example shows us that we can use the graph of a function to do the following useful things.

- Find output values for given input values.
- Solve equations. To solve $f(x) = 0$ using a graph, we look for the points where the graph crosses the horizontal axis.
- Find the largest (or smallest) values that a function can have.
- Find the inputs for which a function has positive output values and for which it has negative output values.
- Find where a function is increasing and where it is decreasing.

Exercise 3

1. Draw a new coordinate diagram like that of Figure 2. Plot the points $(2, 5)$, $(-1, 1)$ and draw the line that passes through these two points. Find two other points that lie on this line.

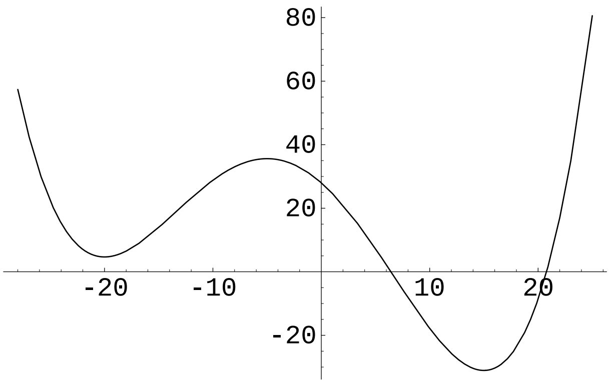

Figure 4: Graph of the function f. The horizontal axis is the $x-$axis, and the vertical is the $y-$axis.

2. Consider the graph of the function $f : x \to f(x)$ in Figure 4 and answer the following questions.

   (a) For what values of x is $f(x) = 0$?

   (b) For what values of x is $f(x)$ positive? negative?

   (c) What is the smallest value of $f(x)$ for the values of x which appear in the diagram? At what value of x does this occur?

   (d) What is the largest value of $f(x)$ for the values of x which appear in the diagram? At what value of x does this occur?

   (e) For what values of x is the function increasing? decreasing?

Solutions to exercises

Exercise 1

1. In each case, I am using x as the name of the input.

   (a) $f(x) = 2x - 8$. $f(1) = -6, f(3) = -2, f(-2) = -12$.

   (b) $g(x) = \dfrac{x^2}{4}$. $g(1) = \frac{1}{4}, g(3) = \frac{9}{4}, g(-2) = 1$.

   (c) $h(x) = \dfrac{5x + 12}{7}$. $h(1) = \frac{17}{7}, h(3) = \frac{27}{7}, h(-2) = \frac{2}{7}$.

   (d) $j(x) = \sqrt{x+1}$. $j(1) = \sqrt{2}, j(3) = 2, j(-2) = \sqrt{-1}$.

2. (a) $h(0) = 9$

   (b) $h(2) = 17$

   (c) $h(t) = 4t + 9$

   (d) $h(2x) = 4(2x) + 9 = 8x + 9$.

Exercise 3

2. All answers can only be given roughly, since we are reading them off a diagram.

   (a) $x = 6, x = 21$.

   (b) Positive for x less than 6 and for x bigger than 21. Negative for x between 6 and 21.

   (c) Smallest value for $f(x)$ is roughly -30, and occurs at $x = 15$.

   (d) Largest value for $f(x)$ is roughly 80, and occurs at $x = 25$. Notice however than there would be larger values if we moved farther to the right. The value 80 is only the largest value for the values of x which appear in the diagram.

   (e) Increasing: From $x = -20$ to $x = -5$; for x bigger than 15.
   Decreasing: For x less than -20; for x between -5 and 15.