DCU School of Mathematical Sciences
BASIC SKILLS WORKSHEET 9
The Straight Line

The aim of this worksheet is to revise some maths and geometry about straight lines. Worksheet 6 is a prerequisite for this worksheet.

Linear equations.

Recall that an equation for y in terms of x is called linear if it can be written in the form

$$y = ax + b$$

for some numbers a, b. Similarly, the function[1]

$$f : x \to ax + b \qquad \text{or} \qquad f(x) = ax + b$$

is called a linear function. They are called 'linear' because their graphs are straight lines. So if we talk about "the line $y = ax + b$" what we mean is the graph of the function $f(x) = ax + b$.

Straight line graphs

These are easy enough to draw. All we need to do is find two different points that lie on the graph and draw the straight line that passes through these two points.

Example

Sketch the graph of $f(x) = -3x + 4$.

To find two different points on the graph, we take two different input values (two different values for x) and use the formula to calculate the corresponding output values (values for y).

$$\text{If } x = 0, \text{ then } y = -3(0) + 4 = 4,$$

so our first point has coordinates $(0, 4)$.

$$\text{If } x = 1, \text{ then } y = -3(1) + 4 = 1,$$

so our second point is $(1, 1)$. We plot these points and draw the line through them to get the graph (labelled L_1 in Figure 1).

Exercise 1

1. Sketch the lines
   (a) $y = 4x + 1$
   (b) $-2y = x - 1$
   (c) $y = 4$

[1] Recall that the notation $f : x \to ax + b$ means that f is the function (or rule) that takes as input the number x and uses it to generate the output value $ax + b$. See Worksheet 6.

Figure 1: Some lines

Special cases

The last example gave the equation of the line as $y = 4$. We should think of this as

$$y = 0 \times x + 4$$

to emphasise that we are dealing with functions and graphs, and not just saying that the variable named y has the value 4. To draw its graph, we use the standard method of finding two points on the line:

$$\text{If } x = 0, \text{ then } y = 0(0) + 4 = 4,$$

so the point $(0, 4)$ lies on the line.

$$\text{If } x = 1, \text{ then } y = 0(1) + 4 = 4,$$

so the point $(1, 4)$ lies on the line. This works for any input value of x. The result is that the line $y = 4$ is horizontal, passing through all points which have $y-$coordinate equal to 4.

Let k be any number. The line $y = k$ is horizontal, passing through all points which have $y-$coordinate equal to k.

A similar argument tells us this:

Let k be any number. The line $x = k$ is vertical, passing through all points which have $x-$coordinate equal to k.

The slope of a line

How steep is the line $y = 3x + 1$? In everyday life, we say that a road (or hill or path...) is steep if it changes a lot in the vertical direction over a short horizontal distance. So we could (and will) use something similar to measure the steepness of a line. Let's ask this question:

What is the change in the vertical direction ($y-$coordinate) brought about by an increase of one unit in the horizontal direction ($x-$coordinate)?

Suppose we start at $x = 2$ and increase by one unit to $x = 3$. Then y changes from

$$y = 3(2) + 1 = 7 \text{ to } y = 3(3) + 1 = 10,$$

so y increases by 3 units.
Now suppose we start at $x = 7$ and increase by one unit to $x = 8$. Then y changes from

$$y = 3(7) + 1 = 22 \text{ to } y = 3(8) + 1 = 25,$$

so again y increases by 3 units.
It is a property of this line that the $y-$coordinate always increases by 3 units when the $x-$coordinate increases by 1 unit.

Definition

The **slope** of a straight line is the change in the value of the $y-$coordinate brought about by a one unit increase in the $x-$coordinate.

This immediately tells us that a line with positive slope increases from left to right; a line with negative slope decreases from left to right and a line with slope equal to zero is horizontal.

Some facts

For a challenge, try to prove these facts. Don't worry if you can't.

1. If we write the equation of a line in the form $y = ax + b$, then the slope of the line is a.
2. If (x_1, y_1) and (x_2, y_2) are two different points on a line, then the line has slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

3. A line which has slope m and contains the point (x_1, y_1) has equation

$$y - y_1 = m(x - x_1).$$

These facts give us ways of

1. finding the slope of a line given an equation for the line;
2. finding the slope of a line given two points on the line;
3. finding an equation of a line given its slope and one point on the line;
4. (combining 2 and 3:) finding an equation of a line given two points on the line.

Parallel and perpendicular lines

Two lines (L_1 with slope m_1 and L_2 with slope m_2) are

parallel if and only if $m_1 = m_2$
perpendicular if and only if $m_1 m_2 = -1$.

Exercise 2

1. Write down the slope of the lines

   (a) $y = -2x + 4$

   (b) $4y + x = -3$

   (c) $-3y + 2x = 2(y + x) - 5$

2. Find the slope and then an equation for the lines

   (a) through $(0, 3)$ and $(-2, 5)$;

   (b) parallel to $-2y = x + 4$ and passing through $(2, 2)$;

(c) perpendicular to $y = 4x - 3$ and passing through the origin;

(d) through $(-5, 3)$ and $(1, 1)$.

3. The line $y + k = 4x - 3$ passes through the point $(1, -3)$. What is the value of k?

Intersecting lines - simultaneous equations

Here's a little economics to motivate this problem.

How does the price P of a typical mobile phone depend upon the quantity of phones Q that are available?

On the customer's side (the demand side), it typically happens that the price goes down as more and more items become available. So if we think of Q as the input and P as the output, we could use a straight line with negative slope to model the dependence of P upon demand Q. Let's take the example

$$P = -2Q + 200.$$

However on the manufacturer's side (the supply side), they like to increase the price as they manufacture more and more items, in order to maximize their profit. So here a line with positive slope is appropriate, let's say

$$P = 3Q - 400.$$

In a situation like this where there are different influences pulling P in different directions, what typically happens is that an equilibrium state is reached. That is, the price P settles down to a value where the 'pull' from both sides is exactly the same. This also dictates a particular value for Q.

To find these values we have to find the values of Q and P which satisfy *both* of the equations above. That is, we have to solve the pair of simultaneous equations

$$\begin{aligned} P &= -2Q + 200 \\ P &= 3Q - 400. \end{aligned}$$

There are two ways to do this. We can use algebra, i.e. fiddle about with these two equations until we ge the answer. We can also use geometry. If we draw the straight lines, then the equilibrium point (Q_1, P_1) we are looking for

- Lies on the line $P = -2Q + 200$ and
- lies on the line $P = 3Q - 400$.

In other words, the point we are looking for is the point of intersection of the two lines.

Exercise

Sketch the two lines and use the diagram to find the point of intersection.

The problem with this method is that it can be inaccurate, since we are trying to read numbers off a diagram.

Solving simultaneous linear equations

Using algebra is usually better. We use the method of elimination, described in the steps below.

Example

Find the point of intersection of the lines $2y = 3x + 4$ and $y - 4x = 5$.

1. Write down the two equations, one above the other, with all the x, y terms on the left and just numbers on the right:

$$\begin{aligned} 2y - 3x &= 4 \qquad (1) \\ y - 4x &= 5. \qquad (2) \end{aligned}$$

2. Multiply all of the first equation by the coefficient of y in the second equation.
   Multiply all of the second equation by $-1\times$(the coefficient of y in the first equation).

$$\begin{aligned} \text{Equation}(1) \times 1: \quad 2y - 3x &= 4 \\ \text{Equation}(2) \times -2: \quad -2y + 8x &= -10. \end{aligned}$$

3. Add these two equations:

$$5x = -6$$

4. Solve this equation for x. Here we get $x = -6/5$.

5. Return to one of the original equations, substitute in our value for x and then solve for y:

$$\text{Equation}(1) : 2y - 3(\frac{-6}{5}) = 2y + \frac{18}{5} = 4,$$

   which gives $y = \frac{1}{5}$.

6. Check that these values for x and y also satisfy the second of the original two equations:

$$\text{Equation}(2) : \frac{1}{5} - 4(\frac{-6}{5}) = \frac{25}{5} = 5.\checkmark$$

Special cases

- If the result of Step 3 is the equation "0=0", then the two lines are identical and every point on the lines is a point of intersection.

- The result of Step 3 may be an equation of the form

$$0 = \text{ a non-zero number.}$$

  Which is nonsense. This indicates that the two lines are parallel, and so have no intersection points. (The fact that we are looking for something that does not exist leads to the nonsensical equation.)

Exercise 3

1. Find the points of intersection of the given pairs of lines, or show that there are none.

   (a) $y = 3x + 1$, $y + x = 4$.

   (b) $2y - 3x + 4 = 0$, $x + 4y = 6$.

(c) $y + 1 = -4x - 2$, $2y + 8x - 7 = 0$

(d) $y = 5$, $x = y$.

(e) $\dfrac{x}{y} = 7,\ -x + 3y = 10.$

2. Find an equation for the line perpendicular to the line $y = -2x + 3$ and which passes through the point $(2, 5)$. Then find the point of intersection of the two lines.

3. Find an equation for the line with slope -1 which passes through the point of intersection of $-3y + x = 4$ and $5y + 3 = -2x$.

Solutions to exercises

Exercise 1

1. Shown in Figure 2 (see next page) as L_1.
2. Shown in Figure 2 as L_2.
3. Shown in Figure 2 as L_3.

Exercise 2

1. (a) $m = -2$

   (b) $m = -\frac{1}{4}$

   (c) $m = 0$

2. (a) Slope is $m = -1$, equation for line is $y = -x + 3$.

   (b) Slope is $m = -\frac{1}{2}$, equation for line is $y = -\frac{1}{2}x + 3$.

   (c) Slope is $m = -\frac{1}{4}$, equation for line is $y = -\frac{1}{4}x$.

   (d) Slope is $m = -\frac{1}{3}$, equation for line is $y = -\frac{1}{2}x + \frac{4}{3}$.

3. $k = 4$.

Exercise 3

1. (a) $(x, y) = (3/4, 13/4)$

   (b) $(x, y) = (2, 1)$

   (c) No intersection.

   (d) $(x, y) = (5, 5)$

   (e) $(x, y) = (-35/2, -5/2)$.

2. The line is $y = \frac{1}{2}x + 4$, the point of intersection is $(x, y) = (-2/5, 19/5)$.
3. The point of intersection is $(x, y) = (1, -1)$, the line is $y = -x$.

$\mathbf{L_3}$

$\mathbf{L_1}$

$\mathbf{L_2}$

Figure 2: Solution for exercise 1