A parameter robust numerical method for a system of singularly perturbed ordinary differential equations.

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Abstract

In this paper a two-point boundary value problem for a system of singularly perturbed ordinary differential equations is considered. It is shown that a parameter robust method can be constructed which gives first order convergence in the maximum norm.

1 Introduction.

Let Ω = (0, 1) and consider the two-point boundary value problem for the singularly perturbed system of ordinary differential equations

\[
\begin{pmatrix}
-\varepsilon \frac{d^2}{dx^2} & 0 \\
0 & -\varepsilon \frac{d^2}{dx^2}
\end{pmatrix}
\begin{pmatrix}
\tilde{u}_\varepsilon \\
A
\end{pmatrix}
+ f, \quad x \in \Omega \quad (1a)
\]

\[
\tilde{u}_\varepsilon(0) = \tilde{u}_0, \quad \tilde{u}_\varepsilon(1) = \tilde{u}_1 \quad (1b)
\]

where \( \tilde{u}_\varepsilon = \begin{pmatrix} u_{\varepsilon,1} \\ u_{\varepsilon,2} \end{pmatrix} \), \( A = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \), \( f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \)

, the functions \( a_{11}, a_{12}, a_{21}, a_{22}, f_1, f_2 \in C^2(\Omega) \) satisfy the following inequalities:

\[
(i) \quad a_{11}(x) > |a_{12}(x)| \quad \text{and} \quad a_{22}(x) > |a_{21}(x)|, \quad (1c)
\]

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(ii) \( a_{12}(x) < 0, \ a_{21}(x) < 0 \ \forall x \in \Omega \) \hfill (1d)

and the singular perturbation parameter satisfies \( 0 < \varepsilon \ll 1 \). The equations are coupled through the off-diagonal elements in the matrix \( A \). Writing the differential operator as

\[
L_{\varepsilon} \vec{\psi} \equiv \begin{pmatrix} -\varepsilon \frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon \frac{d^2}{dx^2} \end{pmatrix} \vec{\psi} + A \vec{\psi}
\]

with \( \vec{\psi} = (\psi_1, \psi_2)^T \). The operator \( L_{\varepsilon} \) satisfies the following maximum principle.

**Theorem 1** Assume that \( \vec{\psi}(0) \geq 0 \), \( \vec{\psi}(1) \geq 0 \) and \( L_{\varepsilon} \vec{\psi}(x) \geq 0 \ \forall \ x \in \Omega \), then \( \vec{\psi}(x) \geq 0 \ \forall \ x \in \Omega \).

**2 The Continuous Problem.**

**Lemma 1** Let \( u_{\varepsilon} \) be the solution of (1), then for \( k = 0, 1, 2, 3 \) and \( i = 1, 2 \)

\[
|u_{\varepsilon,i}|_k \leq C(1 + \varepsilon^{-k/2})
\]

where \( C \) is a constant independent of \( \varepsilon \).

The sharper bounds on the derivatives of the solution that are required in the proof of the error estimates given in this paper, are obtained by decomposing the solution \( u_{\varepsilon} \) into smooth and singular components.

**Lemma 2** The solution \( u_{\varepsilon} \) of (1) can be written in the form

\[
u_{\varepsilon} = u_{\varepsilon} + w_{\varepsilon} + \bar{w}_{\varepsilon}
\]

where for all \( 0 \leq k \leq 3 \) and \( i = 1, 2 \),

\[
||v_{\varepsilon,i}^{(k)}|| \leq C(1 + \varepsilon^{2-k/2})
\]

and for all \( x \in \Omega \) and \( i = 1, 2 \)

\[
|w_{\varepsilon,i}^{(k)}(x)| \leq C \varepsilon^{-k/2} e^{-x\sqrt{\alpha/\varepsilon}}, \ |w_{\varepsilon,i}^{(k)}(x)| \leq C \varepsilon^{-k/2} e^{-(1-x)\sqrt{\alpha/\varepsilon}}
\]

where \( \alpha = \min_0 \{a_{11} + a_{12}, a_{21} + a_{22}\} \) and \( C \) is a constant independent of \( \varepsilon \).

**Proof:** The solution \( u_{\varepsilon} \) can be decomposed as follows,

\[
u_{\varepsilon} = u_{\varepsilon} + \bar{w}_{\varepsilon}
\]

where \( L_{\varepsilon} \bar{w}_{\varepsilon} = \tilde{f}, \ u_{\varepsilon}(0) = A^{-1} \tilde{f}(0), \ u_{\varepsilon}(1) = A^{-1} \tilde{f}(1) \)
\[ L_{\varepsilon}\vec{w}_{\varepsilon} = 0, \quad \vec{w}_{\varepsilon}(0) = \vec{u}_{\varepsilon}(0) - \vec{v}_{\varepsilon}(0), \quad \vec{w}_{\varepsilon}(1) = \vec{u}_{\varepsilon}(1) - \vec{v}_{\varepsilon}(1) \]

Write \( \vec{v}_{\varepsilon} = \vec{v}_0 - \varepsilon\vec{v}_1 \) where \( \vec{v}_0 \) is the solution to the reduced problem:

\[ \vec{v}_0 = A^{-1}\vec{f}, \quad \vec{v}_0(0) = A^{-1}\vec{f}(0), \quad \vec{v}_0(1) = A^{-1}\vec{f}(1) \]

Now

\[ L_{\varepsilon}\vec{v}_1 = -\frac{d^2}{dx^2}\vec{v}_0, \quad \vec{v}_1(0) = \vec{0}, \quad \vec{v}_1(1) = \vec{0} \]

We have \( ||v_{\varepsilon,i}^{(k)}|| \leq C \), as \( A^{-1} \) exists. Thus we can apply the classical bounds used previously to bound \( \vec{v}_1 \), and hence the smooth component satisfies

\[ ||v_{\varepsilon,i}^{(k)}|| \leq C(1 + \varepsilon^{2+k}) \]

We divide the singular component of the solution into its left and right subcomponents

\[ \vec{w}_{\varepsilon} = \vec{w}_l + \vec{w}_r. \]

The boundary layer functions \( \vec{w}_l \) and \( \vec{w}_r \) are defined as the solutions of the two problems:

\[ L_{\varepsilon}\vec{w}_l = 0, \quad \vec{w}_l(0) = \vec{w}_{\varepsilon}(0), \quad \vec{w}_l(1) = 0 \]

\[ L_{\varepsilon}\vec{w}_r = 0, \quad \vec{w}_r(0) = 0, \quad \vec{w}_l(1) = \vec{w}_{\varepsilon}(1) \]

The bound on \( \vec{w}_l \) is obtained using the maximum principle as follows. Introduce the barrier functions:

\[ \varphi^{\pm}(x) = \left( \begin{array}{c} e^{-x\sqrt{\alpha/\varepsilon}} \\ e^{-x\sqrt{\alpha/\varepsilon}} \end{array} \right) \pm \vec{w}_l(x) \]

Then \( \varphi^{\pm}(0) \geq 0, \quad \varphi^{\pm}(1) \geq 0 \) and,

\[ L_{\varepsilon}\varphi^{\pm}(x) = \left( \begin{array}{cc} -\varepsilon \frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon \frac{d^2}{dx^2} \end{array} \right) \left( \begin{array}{c} e^{-x\sqrt{\alpha/\varepsilon}} \\ e^{-x\sqrt{\alpha/\varepsilon}} \end{array} \right) + \left( \begin{array}{cc} a_{11} & a_{21} \\ a_{12} & a_{22} \end{array} \right) \left( \begin{array}{c} e^{-x\sqrt{\alpha/\varepsilon}} \\ e^{-x\sqrt{\alpha/\varepsilon}} \end{array} \right) \pm L\vec{w}_l(x) \]

\[ = \left( \begin{array}{c} (a_{11} + a_{12} - \alpha)e^{-x\sqrt{\alpha/\varepsilon}} \\ (a_{21} + a_{22} - \alpha)e^{-x\sqrt{\alpha/\varepsilon}} \end{array} \right) \]

If we choose \( \alpha = \min\{a_{11} + a_{12}, a_{21} + a_{22}\} \) then \( L_{\varepsilon}\varphi^{\pm}(x) \geq 0 \) and by the maximum principle, for \( i=1,2 \),

\[ \varphi^{\pm}(x) \geq 0 \Rightarrow |w_{\varepsilon,i}(x)| \leq Ce^{-x\sqrt{\alpha/\varepsilon}} \]

We bound \( \vec{w}_r(x) \) analogously. The bounds on the derivatives of the singular parts are obtained using similar methods to those in [2].
3 The discrete problem.

The problem given in (1) is now discretised using a fitted mesh method composed of a standard finite difference operator on a fitted piecewise uniform mesh. The finite difference operator is the centered difference operator

\[ L^N \bar{\psi}_i = \begin{pmatrix} -\varepsilon \delta^2 & 0 \\ 0 & -\varepsilon \delta^2 \end{pmatrix} \begin{pmatrix} \psi_{1i} \\ \psi_{2i} \end{pmatrix} + A_i \begin{pmatrix} \psi_{1i} \\ \psi_{2i} \end{pmatrix} \]

where

\[ \delta^2 \psi_i = \frac{D^+ - D^-}{2} \psi_i, \quad D^+ \psi_i = \frac{\psi_{i+1} - \psi_i}{x_{i+1} - x_i}, \quad D^- \psi_i = \frac{\psi_i - \psi_{i-1}}{x_i - x_{i-1}} \]

and

\[ A_i = \begin{pmatrix} a_{11}(x_i) & a_{12}(x_i) \\ a_{21}(x_i) & a_{22}(x_i) \end{pmatrix} \]

The fitted piecewise uniform mesh \( \Omega^N_{\varepsilon} \), \( N \geq 4 \) is constructed by dividing \( \Omega \) into three subintervals \( \Omega_l = (0, \sigma) \), \( \Omega_c = (\sigma, 1-\sigma) \), \( \Omega_r = (1-\sigma, 1) \) so that \( \Omega = \Omega_l \cup \Omega_c \cup \Omega_r \). The transition parameter \( \sigma \) is chosen to be

\[ \sigma = \min \left\{ \frac{1}{4}, C\sqrt{\varepsilon \ln N} \right\}, \quad \text{where } C \geq \frac{1}{\sqrt{\alpha}} \]

and \( \alpha = \min\{a_{11} + a_{12}, a_{21} + a_{22}\} \). In the numerical results in the next section \( C = \frac{2}{\sqrt{\alpha}} \) is chosen. Then \( \Omega^N_{\varepsilon} \) is obtained by putting a uniform mesh with \( N/4 \) mesh elements on both \( \Omega_l \) and \( \Omega_r \), and a uniform mesh with \( N/2 \) elements on \( \Omega_c \). Put \( \Omega^N_{\varepsilon} = \{x_i\}^N_0 \). The resulting fitted mesh finite difference method for problem (1) is then

\[ L^N \bar{U}^N(x_i) = \bar{f}(x_i), \quad x_i \in \Omega^N_{\varepsilon} \bar{U}^N(0) = \bar{u}_\varepsilon(0), \quad \bar{U}^N_{\varepsilon}(1) = \bar{u}_\varepsilon(1) \]

(3)

With the restrictions on the matrix \( A \) given in (1c) and (1d), we have the following discrete maximum principle.

**Lemma 3** Assume that the vector \( \bar{\psi}_i \) with mesh function components \( \psi_{1i}, \psi_{2i} \) satisfies \( \psi_{10} \geq 0, \psi_{20} \geq 0, \psi_{1N} \geq 0, \psi_{2N} \geq 0 \) and given \( L^N \bar{\psi}_i \geq 0 \) for \( i=1...N-1 \), we have \( \bar{\psi}_i \geq 0 \) for \( i = 0...N \)

An immediate consequence of this discrete maximum principle is the following \( \varepsilon \)-uniform stability result.

**Lemma 4** If \( \bar{z}_i = \begin{pmatrix} z_{1i} \\ z_{2i} \end{pmatrix} \) is any mesh function such that \( \bar{z}_0 = \bar{z}_N = 0 \) then

\[ |\bar{z}_i| \leq \alpha^{-1} \max_i |L^N \bar{z}_i| \]

The main result of this paper is contained in the following theorem.
Theorem 2  The fitted mesh finite difference method (3) with the standard centered finite difference operator and the piecewise uniform fitted mesh \( \Omega^N_\epsilon \) is \( \epsilon \)-uniform for the problem (1) provided that \( \sigma \) is chosen to satisfy (2). Moreover the solutions \( \vec{u}_\epsilon \) of (1) and the solutions \( \vec{U}^N_\epsilon \) of (3) satisfy the following \( \epsilon \)-uniform error estimate

\[
\sup_{0<\epsilon<1} ||U^N_{\epsilon,i} - u_{\epsilon,i}||_{\Omega^N_\epsilon} \leq CN^{-1}\ln N
\]

for \( i=1,2 \) and \( C \) is a constant independent of \( \epsilon \) and \( N \).

Further theoretical results on singularly perturbed systems of parabolic equations may be found in [3].

4  Numerical Results.

Numerical results are presented in this section which validate the theoretical results established in the previous section. All computations were performed in C double precision using a Pentium PC. The following is the problem to be solved numerically:

\[
\begin{pmatrix}
-\epsilon \frac{d^2}{dx^2} & 0 \\
0 & -\epsilon \frac{d^2}{dx^2}
\end{pmatrix}
\begin{pmatrix}
\vec{u}_\epsilon \\
A\vec{u}_\epsilon
\end{pmatrix}
= \vec{f},
\vec{u}_\epsilon(0) = \vec{0},\, \vec{u}_\epsilon(1) = \vec{0}
\]

For various choices of \( A \) and \( \vec{f} \). In solving this problem, the following \( 2(N - 1) \times 2(N - 1) \) system of equations had to be solved:

\[
\begin{pmatrix}
T_1 & D_1 \\
D_2 & T_2
\end{pmatrix}
\begin{pmatrix}
Z \\
W
\end{pmatrix}
= \begin{pmatrix}
C_1 \\
C_2
\end{pmatrix}
\]

where \( T_1 \) and \( T_2 \) are tri-diagonal matrices, \( D_1 \) and \( D_2 \) are diagonal matrices. This matrix system was solved iteratively using the following simultaneous iterative scheme.

\[
T_1Z_{(k)} = C_1 - D_1W_{(k)}
\]

\[
T_2W_{(k+1)} = C_2 - D_2Z_{(k)}
\]

An initial guess for \( W \) of \((1, 1, \ldots 1)^T\) was inputed. The iterative scheme terminated when successive solutions were sufficiently close. In this case a tolerance of \( 10^{-12} \) was used. The maximum pointwise error for each value of \( N \) in each component of the solution is taken to be

\[
E^N_{\epsilon,i} = \max_{\Omega^N_\epsilon} |\vec{U}^{4096}_{\epsilon,i} - \vec{U}^N_{\epsilon,i}|
\]

where \( \vec{U}^{4096}_{\epsilon,i} \) is the linear interpolent of the solution with 4096 mesh points. In the first example, a constant co-efficient matrix is examined.

Example 1 \( A = \begin{pmatrix}
3 & -1 \\
-1 & 3
\end{pmatrix}, \quad \vec{f} = \begin{pmatrix}
2 \\
3
\end{pmatrix} \)
Table 1: The computed maximum pointwise errors $E_{N,1}^N$ in applying method (3) to Example 1 for various values of $\varepsilon$ and $N$.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
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<td>8.674e-05</td>
<td>2.167e-05</td>
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The computed maximum pointwise errors $E_{N,1}^N$ in applying method (3) to example 1, for each $N$ are displayed in Table 1. It is clear that the method is parameter-robust for this example with the errors for each $N$ stabilising as $\varepsilon \to 0$.

In the second example a variable coefficient example is examined.

Example 2

$$A = \begin{pmatrix} (x + 1)^2 & -(x + 0.5) \\ -1 & 2 \end{pmatrix}, \quad f = \begin{pmatrix} x^5 - 0.08 \\ \sin(\pi x) \end{pmatrix}$$

Table 2 shows the errors computed for the two components of the solution to example 2. Table 3 illustrates the errors when a uniform mesh is used rather than a fitted mesh. The maximum value of the error in each column of Table 3 is highlighted. These values lie along a diagonal of the $\varepsilon - N$ plane. The method used on this mesh is not $\varepsilon$-uniform.

References


Table 2: The computed maximum pointwise errors $E_{\varepsilon,1}^N$ (top table) and $E_{\varepsilon,2}^N$ in applying method (3) to Example 2 for various values of $\varepsilon$ and $N$.

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Table 3: The computed maximum pointwise errors $E_{\varepsilon,1}^N$ in applying a classical numerical method (consisting of the standard centered finite difference operator on a uniform mesh) to Example 2 for various values of $\varepsilon$ and $N$.

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