A class of singularly perturbed semilinear differential equations with interior layers. *

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Abstract

In this paper singularly perturbed semilinear differential equations with a discontinuous source term are examined. A numerical method is constructed for these problems which involves an appropriate piecewise-uniform mesh. The method is shown to be uniformly convergent with respect to the singular perturbation parameter. Numerical results are presented which validate the theoretical results.

1 Introduction

In this paper a class of singularly perturbed semilinear ordinary differential equations in one dimension is considered on the unit interval $\Omega = (0, 1)$. A single discontinuity in the source term is assumed to occur at a point $d \in \Omega$. It is convenient to introduce the notation $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$ and to denote the jump at d in any function with $[\omega](d) = \omega(d+) - \omega(d-)$. The problem is

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Find $u_\varepsilon \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ such that

$$-\varepsilon u_\varepsilon'' + b(u)u_\varepsilon = f \quad \text{for all} \quad x \in \Omega^- \cup \Omega^+ \tag{1.1a}$$

$$u_\varepsilon(0) = A, \quad u_\varepsilon(1) = B, \tag{1.1b}$$

$$f(d-) \neq f(d+), \quad b(0) > 0, \tag{1.1c}$$

$$b \in C^4(-\infty, \infty), \quad f \in C^4(\bar{\Omega} \setminus \{d\}). \tag{1.1d}$$

Below we impose further restrictions (2.2), (2.11) on the magnitudes of $\|f\|_{\bar{\Omega}}$, the boundary values $|u_\varepsilon(0)|$, $|u_\varepsilon(1)|$ and the class of nonlinear functions $b(\cdot)$ that will be examined. These restrictions are introduced at appropriate locations in the paper. Because f is discontinuous at d the solution u_ε of (1.1) does not necessarily have a continuous second order derivative at the point d. Thus $u_\varepsilon \notin C^2(\Omega)$, but the first derivative of the solution exists and is continuous. If $f \in C^1(\Omega)$, then under certain restrictions on the nonlinearity, only boundary layers would appear in the solution of (1.1). The asymptotic structure of the solutions of singularly perturbed semilinear differential equations with both boundary and interior layers is given in [1].

D'Annunzio [2] examined semi-linear problems, whose solutions displayed both boundary and interior layer phenomena; but D'Annunzio placed restrictions on the mesh size so that the number of mesh intervals employed depended adversely on the small parameter. In this paper, our goal is to design numerical methods, which are parameter-uniform. That is, if u_ε is a solution of (1.1) and U_ε is a numerical approximation, then

$$\|u_\varepsilon - U_\varepsilon\|_\infty \leq Cg(N), \quad g(N) \to 0 \text{ as } N \to \infty$$

where the number of mesh intervals N is independent of ε; and C is a generic constant independent of ε and N. Shishkin [12] established parameter-uniform convergence for a class of quasi-linear parabolic equations with smooth data using finite difference schemes based on piecewise-uniform meshes. The numerical method presented in this paper is also based on piecewise–uniform meshes. Singularly perturbed linear problems with discontinuous data were treated in [13]. A linear version of (1.1) was studied in [7], where a parameter–uniform numerical method based on a suitably designed piecewise–uniform mesh adapted to the interior layer was shown to converge with $g(N) = N^{-1} \ln N$. The methodology in [7] is extended in this paper to the nonlinear problem (1.1). In [5], it was shown that numerical methods based on uniform meshes cannot be parameter–uniform for semilinear singularly perturbed problems. Sun and Stynes [14] constructed finite difference schemes based on piecewise-uniform meshes for semilinear problems, whose solutions exhibit only boundary layer structure. In this paper, we are primarily interested in the interior layer behaviour introduced by the discontinuity of f.

2 The Continuous problem.

We introduce the concepts of upper and lower solutions, which are useful in establishing existence and in determining the character of the solution.

Definition 1: A function $\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ is a lower solution of (1.1) if

$$-\varepsilon\alpha'' + b(\alpha)\alpha \leq f, x \neq d \tag{2.1a}$$

$$\alpha'(d+) \geq \alpha'(d-) \tag{2.1b}$$

$$\alpha(0) \leq u_\varepsilon(0); \quad \alpha(1) \leq u_\varepsilon(1). \tag{2.1c}$$

An upper solution β is defined in an analogous fashion, with all inequalities reversed.

Theorem 1 *[10] If $\alpha, \beta \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ are, respectively, lower and upper solutions for the problem (1.1) and $\alpha(x) \leq \beta(x), \quad \forall x \in \bar{\Omega}$, then there exists a solution to (1.1) and*

$$\alpha(x) \leq u_\varepsilon(x) \leq \beta(x), \quad \forall x \in \bar{\Omega}.$$

Hence, to establish existence, we are only required to construct, a lower and upper solution. First we place a restriction on the magnitude of the boundary conditions and $\|f\|$.

Assumption 1 Assume that there exists a $\theta > 0$ such that ,

$$b(y) \geq \theta > 0, \ \forall y \in D_{\theta,K} = [-\frac{K}{\theta}, \frac{K}{\theta}] \tag{2.2a}$$

where

$$K = \max\{\|f\|, \theta|u_\varepsilon(0)|, \ \theta|u_\varepsilon(1)|\}. \tag{2.2b}$$

Note that since $b(0) > 0$ and b is smooth then there exists a neighbourhood $[-\delta, \delta]$ such that $b(y) \geq \theta > 0, \ \forall y \in [-\delta, \delta]$. Requiring that $b(z) \geq \theta > 0, \ \forall z \in (-\infty, \infty)$ is considerably more stringent on the extent of the class of nonlinear problems under consideration.

Theorem 2 *Problem (1.1), (2.2) has a solution $u_\varepsilon \in C^1([0,1], D_{\theta,K})$ and*

$$\|u_\varepsilon\| \leq \frac{K}{\theta}.$$

Proof: Let $\alpha(x) = -\frac{1}{\theta}K = -\beta(x)$. Then $\alpha(0) \leq u_\varepsilon(0) \leq \beta(0)$ and $\alpha(1) \leq u_\varepsilon(1) \leq \beta(1)$, with $\varepsilon\alpha'' = \varepsilon\beta'' = 0$. Note also that, by virtue of (2.2)

$$-\varepsilon\alpha''(x) + b(\alpha)\alpha = b(-\frac{1}{\theta}K)(-\frac{1}{\theta}K) \leq f$$

and

$$-\varepsilon\beta''(x) + b(\beta)\beta \geq f.$$

Hence α and β are lower and upper solutions with $\alpha(x) \leq \beta(x), \ \forall x \in [0,1]$. By the previous theorem, there exists a solution to problem (1.1), (2.2) and

$$\alpha(x) \leq u_\varepsilon(x) \leq \beta(x), \ \forall x \in [0,1].$$

Theorem 3 *Let α, β be lower and upper solutions. Assume that*

$$y < z \quad \text{implies that} \quad b(y)y < b(z)z, \quad \forall y, z \in [-\max\{\|\alpha\|, \|\beta\|\}, \max\{\|\alpha\|, \|\beta\|\}]. \tag{2.3}$$

With this assumption,

$$\alpha(x) \leq \beta(x), \quad \forall x \in [0, 1].$$

Proof: Let p be any point at which $\omega = \alpha - \beta$ attains its maximum value in $\bar{\Omega}$. Assume that $\omega(p) > 0$. If $p \neq d$ and $p \in \Omega^- \cup \Omega^+$ then $\omega''(p) \leq 0$ and at this point $x = p$

$$\varepsilon\alpha''(p) \geq b(\alpha)\alpha - f > b(\beta)\beta - f \geq \varepsilon\beta''(p)$$

which implies that $\omega''(p) > 0$, which is a contradiction. If $p = d$, then the argument depends on whether or not ω is differentiable at d. If $\omega'(d)$ does not exist, then $[\omega'](d) \neq 0$ and because $\omega'(d-) \geq 0, \quad \omega'(d+) \leq 0$ it is clear that $[\omega'](d) < 0$. However, because α and β are lower and upper solutions, we also have that $[\alpha'](p) \geq 0$ and $[\beta'](p) \leq 0$, which contradicts $[\omega'](d) < 0$. If $\omega'(d)$ does exist, then $\omega'(d) = 0$ and one can follow the argument as in the linear problem [7] to arrive again at a contradiction. Hence the assumption that $\omega(p) > 0$ always leads to a contradiction.

This result and assumption (2.3) guarantee uniqueness of the solution of (1.1), (2.2). Let u_1, u_2 be two solutions of problem (1.1), (2.2). Then, by Theorem 2, we have that

$$\|u_i\| \leq \frac{K}{\theta}, \quad i = 1, 2.$$

Assuming (2.3), u_1, u_2 can be viewed as lower and upper solutions and so $u_1 \leq u_2$. Reversing the roles of u_1, u_2 provides uniqueness.

Follow the arguments in [9] to get

$$|u_\varepsilon|_{k,\Omega^-\cup\Omega^+} \leq C\frac{K}{\theta}(1 + \varepsilon^{-k/2}), \quad 0 \leq k \leq 4 \tag{2.4}$$

where the semi-norms $|\cdot|_{k,D}$ are defined by

$$|y|_{k,\Omega^-\cup\Omega^+} = \|\frac{d^k y}{dx^k}\|_{\Omega^-\cup\Omega^+}.$$

The reduced problem (i.e., set $\varepsilon = 0$ in (1.1a)) is

$$b(y_0)y_0 = f(x), \quad x \neq d. \tag{2.5}$$

Note that if $f \equiv 0$ on the subinterval $x < d$ (or $x > d$) then by (1.1c), $y_0 \equiv 0$ is a solution to the reduced problem on the subinterval $x < d$ (or $x > d$). By (2.2) we know that, if $\|f\| \neq 0$ then

$$b(-\frac{\|f\|}{\theta})(-\frac{\|f\|}{\theta}) - f \leq 0 \leq b(\frac{\|f\|}{\theta})\frac{\|f\|}{\theta}) - f. \tag{2.6}$$

Hence in the interval $D_{\theta,\|f\|} = [-\frac{\|f\|}{\theta}, \frac{\|f\|}{\theta}]$, for each $x \neq d$, there exists an associated y such that

$$b(y)y - f(x) = 0, \quad x \neq d.$$

Thus it is sufficient for existence of a reduced solution that

$$b(y) \geq \theta > 0, \ \forall y \in [-\frac{\|f\|}{\theta}, \frac{\|f\|}{\theta}]. \tag{2.7}$$

This is implied by (2.2), and hence a reduced solution y_0 exists and by (2.3) is unique within the interval $D_{\theta,\|f\|} \subset D_{\theta,K}$ and

$$b(y_0) \geq \theta > 0. \tag{2.8}$$

We now impose a further condition on the strength of the nonlinearity. Assume that, given θ in (2.2), there exists a $\gamma > 0$ such that

$$\frac{\partial b(y)y}{\partial y} \geq \gamma > 0, \quad \forall y \in D_{\theta,\|f\|} = [-\frac{\|f\|}{\theta}, \frac{\|f\|}{\theta}] \tag{2.9}$$

Assuming (2.9) guarantees (via the Implicit Function Theorem) that if $f \in C^k(\Omega^- \cup \Omega^+)$ a reduced solution $y_0 \in C^k(\Omega^- \cup \Omega^+)$ exists and is unique, with $\|y_0\| \leq \frac{\|f\|}{\theta}$. For u_ε to be unique, we can assume that

$$\frac{\partial b(y)y}{\partial y} \geq \gamma > 0, \quad \forall y \in D_{\theta,K} = [-\frac{K}{\theta}, \frac{K}{\theta}]. \tag{2.10}$$

Note that (2.10) implies (2.3), which yields uniqueness of the solution u_ε.

To establish the parameter-robust properties of the numerical methods involved in this paper, the following decomposition of u_ε into regular v_ε and singular w_ε components will be used. The regular component v_ε is defined as the solution of

$$\begin{aligned} -\varepsilon v_\varepsilon'' + b(v_\varepsilon)v_\varepsilon &= f \quad x \neq d \\ b(v_0)v_0 &= f \quad x \neq d \\ v_\varepsilon(0) = v_0(0), \quad v_\varepsilon(d-) &= v_0(d-) \\ v_\varepsilon(d+) = v_0(d+), \quad v_\varepsilon(1) &= v_0(1). \end{aligned}$$

Note that $\|v_0\| \leq \frac{\|f\|}{\theta}$, which implies that $|v_\varepsilon(0)|, |v_\varepsilon(d-)| \leq \|f\|/\theta$. Hence, using the arguments in Theorem 2 on Ω^-, Ω^+ separately, we deduce that v_ε exists and

$$\|v_\varepsilon\| \leq \|f\|/\theta.$$

The singular component w_ε is given implicitly by $u_\varepsilon = w_\varepsilon + v_\varepsilon$ and

$$\begin{aligned} -\varepsilon u_\varepsilon'' + b(u_\varepsilon)u_\varepsilon &= f \quad x \neq d \\ u_\varepsilon &\in C^1(0,1) \\ u_\varepsilon(0) = A, \quad u_\varepsilon(1) &= B. \end{aligned}$$

Since the solution of (1.1), (2.2), (2.10) is unique

$$\|w_\varepsilon\| = \|u_\varepsilon - v_\varepsilon\| \leq \frac{\|f\| + K}{\theta}.$$

In order to derive sharp pointwise bounds on the singular component w_ε, we are required to strengthen the restriction given in (2.10) to the following assumption.

Assumption 2. Assume that, given θ in (2.2), there exists a $\gamma > 0$ such that

$$\frac{\partial(b(y)y)}{\partial y} \geq \gamma > 0, \quad \forall y \in D_{\theta,2\|f\|+K} = [-\frac{2\|f\| + K}{\theta}, \frac{2\|f\| + K}{\theta}] \tag{2.11a}$$

where

$$K = \max\{\|f\|, \theta|u_\varepsilon(0)|,\ \theta|u_\varepsilon(1)|\}. \tag{2.11b}$$

Theorem 4 *Let u_ε be the solution of the problem (1.1), (2.2), (2.11). Then $u_\varepsilon = v_\varepsilon + w_\varepsilon$ and for each integer j, satisfying $0 \leq j \leq 4$, the components v_ε and w_ε satisfy the following bounds, for ε sufficiently small,*

$$|v_\varepsilon(x)|_j \leq \begin{cases} C(1 + \varepsilon^{1-\frac{j}{2}}), & x \in \Omega^- \\ C(1 + \varepsilon^{1-\frac{j}{2}}), & x \in \Omega^+ \end{cases}$$

$$|w_\varepsilon(x)|_j \leq \begin{cases} C(\varepsilon^{-\frac{j}{2}}(e^{-x\sqrt{\gamma/\varepsilon}} + e^{-(d-x)\sqrt{\gamma/\varepsilon}})), & x \in \Omega^- \\ C(\varepsilon^{-\frac{j}{2}}(e^{-(x-d)\sqrt{\gamma/\varepsilon}} + e^{-(1-x)\sqrt{\gamma/\varepsilon}})), & x \in \Omega^+ \end{cases}$$

where C is a constant independent of ε and $|\cdot|_j$ denotes the maximum pointwise norm of the j^{th} derivative.

Proof: Note that $|v_0|_{j,\Omega^-\cup\Omega^+} \leq C$. Introduce the notation $g(y) = (b(y)y)_y$ and by assumption (2.11), $g(y) \geq \gamma > 0, \forall y \in D_{\theta,2\|f\|}$. We have that

$$-\varepsilon(v - v_0)'' + (b(v)v - b(v_0)v_0) = -\varepsilon(v - v_0)'' + g(v_0 + t(v - v_0))(v - v_0) = \varepsilon v_0'',$$
$$(v - v_0)(0) = (v - v_0)(d-) = (v - v_0)(d+) = (v - v_0)(1) = 0.$$

Note that both $v, v_0 \in D_{\theta,\|f\|}$ and hence

$$g(v_0 + t(v - v_0)) \geq \gamma > 0.$$

Consider the nonlinear problem

$$-\varepsilon y'' + g(v_0 + ty)y = \varepsilon v_0'',\ x \in \Omega^-$$
$$y(0) = A_1, \quad y(d) = B_1.$$

Use $\alpha = -\varepsilon\|v_0''\|/\gamma = -\beta$ as lower and upper solutions. For ε sufficiently small,

$$v_0 \pm \varepsilon t\|v_0''\|/\gamma \in D_{\theta,2\|f\|}, \quad 0 < t < 1$$

and, hence, $g(v_0 + t\alpha) \geq \gamma$ and $g(v_0 + t\beta) \geq \gamma$. We thus have that

$$\|v - v_0\|_{\Omega^- \cup \Omega^+} \leq C\varepsilon.$$

Then, follow the argument in [9] to get the bounds

$$|v - v_0|_{j, \Omega^- \cup \Omega^+} \leq C\varepsilon(1 + \varepsilon^{-j/2}), \quad 1 \leq j \leq 4.$$

The singular component is the solution of

$$\begin{aligned} -\varepsilon w_\varepsilon'' + b(u)(v + w) = b(v)v, \quad x \neq d \\ w_\varepsilon = u_\varepsilon - v_\varepsilon, \quad x = 0, 1 \\ [w_\varepsilon](d) = -[v_\varepsilon](d), \quad [w_\varepsilon'](d) = -[v_\varepsilon'](d). \end{aligned}$$

Note that

$$-\varepsilon w_\varepsilon'' + (b(u)u - b(v)v) = -\varepsilon w_\varepsilon'' + g(u_\varepsilon + t(v - u_\varepsilon))w_\varepsilon = 0.$$

We have that $u_\varepsilon \in D_{\theta,K}, v_\varepsilon \in D_{\theta,\|f\|}$ and hence $g(u + t(v - u_\varepsilon)) = g(v + (1-t)w_\varepsilon) \geq \gamma > 0$ from (2.11). On the interval Ω^-, consider the nonlinear problem

$$\begin{aligned} -\varepsilon y'' + g(v + (1-t)y)y = 0, \; x \in \Omega^- \\ y(0) = A_1, \quad y(d) = B_1. \end{aligned}$$

Consider the barrier function

$$\phi(x) = \max\{|w_\varepsilon(0)|, |w_\varepsilon(d-)|\} \frac{e^{-x\sqrt{\gamma/\varepsilon}} + e^{-(d-x)\sqrt{\gamma/\varepsilon}}}{1 + e^{-d\sqrt{\gamma/\varepsilon}}}.$$

Note that $\varepsilon\phi'' = \gamma\phi \geq 0$ and $\|\phi\| \leq \frac{\|f\|+K}{\theta}$. Hence $v_\varepsilon \pm (1-t)\phi \in D_{\theta, 2\|f\|+K}$ and, by (2.11), this gives

$$g(v_\varepsilon \pm (1-t)\phi) \geq \gamma.$$

Let $\alpha = -\phi = \beta$. Then

$$-\varepsilon\alpha'' + g(v_\varepsilon + (1-t)\alpha)\alpha \leq 0 \leq -\varepsilon\beta'' + g(v_\varepsilon + (1-t)\beta)\beta.$$

Then $-\phi \leq w_\varepsilon \leq \phi, x \in \Omega^-$. Follow the arguments in [9] to get the bounds on the derivatives of w_ε.

3 Discrete Problem

On $\Omega^- \cup \Omega^+$ a piecewise-uniform mesh of N mesh intervals is constructed as follows. The interval $\overline{\Omega}^-$ is subdivided into the three subintervals

$$[0, \sigma_1], \quad [\sigma_1, d - \sigma_1] \quad \text{and} \quad [d - \sigma_1, d].$$

for some σ_1 that satisfies $0 < \sigma_1 \leq \frac{d}{4}$. On $[0, \sigma_1]$ and $[d - \sigma_1, d]$ a uniform mesh with $\frac{N}{8}$ mesh-intervals is placed, while on $[\sigma_1, d - \sigma_1]$ has a uniform mesh with $\frac{N}{4}$ mesh-intervals. The subintervals $[d, d+\sigma_2], [d+\sigma_2, 1-\sigma_2], [1-\sigma_2, 1]$ are treated analogously for some σ_2 satisfying $0 < \sigma_2 \leq \frac{1-d}{4}$. The interior points of the mesh are denoted by

$$\Omega_\varepsilon^N = \{x_i : 1 \leq i \leq \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \leq i \leq N - 1\}$$

Clearly $x_{\frac{N}{2}} = d$ and $\overline{\Omega}_\varepsilon^N = \{x_i\}_0^N$. Note that this piecewise–uniform mesh is a uniform mesh when $\sigma_1 = \frac{d}{4}$ and $\sigma_2 = \frac{1-d}{4}$. It is fitted to the singular perturbation problem (1.1) by choosing σ_1 and σ_2 to be the following functions of N and ε

$$\sigma_1 = \min\left\{\frac{d}{4},\ M\sqrt{\varepsilon}\ln N\right\},\ \sigma_2 = \min\left\{\frac{1-d}{4},\ M\sqrt{\varepsilon}\ln N\right\}, \quad M \geq \frac{1}{\sqrt{\gamma}} \tag{3.1}$$

where γ is specified in (2.11). On the piecewise-uniform mesh $\overline{\Omega}_\varepsilon^N$ a standard centred finite difference operator is used. Then the fitted mesh method for problem (1.1) is:

Find a mesh function U_ε such that

$$-\varepsilon\delta^2 U_\varepsilon(x_i) + b(U_\varepsilon(x_i))U_\varepsilon(x_i) = f(x_i) \quad \text{for all} \quad x_i \in \Omega_\varepsilon^N \tag{3.2a}$$

$$U_\varepsilon(0) = u_\varepsilon(0), \quad U_\varepsilon(1) = u_\varepsilon(1) \tag{3.2b}$$

$$D^- U_\varepsilon(x_{\frac{N}{2}}) = D^+ U_\varepsilon(x_{\frac{N}{2}}) \tag{3.2c}$$

where

$$\delta^2 Z_i = \left(\frac{Z_{i+1} - Z_i}{x_{i+1} - x_i} - \frac{Z_i - Z_{i-1}}{x_i - x_{i-1}}\right)\frac{1}{x_{i+1} - x_{i-1}}$$

Let $G : R^{N+1} \to R^{N+1}$ be a mapping associated with this finite difference scheme. For mesh function Y we have an associated vector $Y \in R^{N+1}$, where $Y_i = Y(x_i)$. Let

$$(GY)_i = \begin{cases} Y(0), \quad i = 0 \\ -\varepsilon\delta^2 Y_i + b(Y_i)Y_i, \quad i \neq N/2, 1 \leq i \leq N \\ -\varepsilon\delta^2 Y_i, \quad i = N/2, \\ Y(1), \quad i = N+1 \end{cases}$$

We also define a vector F by

$$F_i = \begin{cases} A, 0, B, \quad i = 0, N/2, N+1 \\ f(x_i), \quad \text{otherwise} \end{cases}$$

The finite difference scheme (3.2) can then be written in the form

$$GU_\varepsilon = F.$$

Definition 2. Given any vector $H \in R^{N+1}$, a lower mesh solution V for the problem $GW = H$ is a mesh function which satisfies

$$GV \leq H.$$

There is an analogous definition for an upper mesh solution to $GW = H$.

Theorem 5 *If Φ, Ψ are, respectively, lower and upper mesh solutions for the problem $GW = H$ with $\Phi(x_i) \leq \Psi(x_i), \quad \forall x_i \in \bar{\Omega}^N$, then there exists a solution to $GW = H$ such that*

$$\Phi(x_i) \leq W(x_i) \leq \Psi(x_i), \quad \forall x_i \in \bar{\Omega^N}.$$

Proof We follow the argument from Lorentz [3]. Let Φ_1, Φ_2 be two lower mesh functions. Define the mesh function Φ_3 by $\Phi_3(x_i) = \max\{\Phi_1(x_i), \Phi_2(x_i)\}$. At some point x_j, we assume w.l.o.g. that $\Phi_3(x_j) = \Phi_1(x_j)$. Note that $-\Phi_3(x_i) \leq -\Phi_1(x_i), \forall x_i$ and

$$\begin{aligned} -\varepsilon\delta^2\Phi_3(x_j) + b(\Phi_3)\Phi_3(x_j) &\leq -\varepsilon\delta^2\Phi_1(x_j) + b(\Phi_1)\Phi_1(x_j) \leq H(x_j), \quad x_j \neq d \\ -\varepsilon\delta^2\Phi_3(x_j) &\leq -\varepsilon\delta^2\Phi_1(x_j) \leq H(d), \quad x_j = d \\ \Phi_3(0) \leq H(0), & \qquad \Phi_3(1) \leq H(1). \end{aligned}$$

Then Φ_3 is also a lower mesh solution. Let $L = \{\phi : G\phi \leq H, \Phi \leq \phi \leq \Psi\}$. Define $U(x_i) = \sup_{\phi \in L}\{\phi(x_i)\}$. First note that $U \in L$ exists and $GU \leq H$. Assume that we do not have equality, then there exists some j such that $GU(x_j) < F(x_j)$. If $U \neq \Psi$, construct a new vector $Y = U + \eta\delta_{i,j}, \quad \eta > 0$. Then η can be chosen sufficiently small so that $Y \in L,\ U < Y, GY < H$. This is a contradiction. Note that if $U = \Psi$, then U is both an upper and a lower solution and so we are done.

Theorem 6 *Let Φ, Ψ be lower and upper mesh solutions of $GW = H$ and $M_2 = \max\{\|\Phi\|, \|\Psi\|\}$. Assume that (2.3) holds over the interval $[-M_2, M_2]$ then*

$$\Phi(x_i) \leq \Psi(x_i), \quad \forall x_i \in \bar{\Omega^N}.$$

Proof: Let x_j be that mesh point at which $\Phi - \Psi$ attains its maximum value in $\bar{\Omega^N}$. Assume that $\Phi(x_j) > \Psi(x_j)$. If $x_j \neq d$ then, since Φ, Ψ are lower and upper mesh solutions,

$$\varepsilon\delta^2\Phi(x_j) \geq b(\Phi)\Phi - H > b(\Psi)\Psi - H \geq \varepsilon\delta^2\Psi(x_j)$$

which contradicts $\delta^2(\Phi-\Psi)(x_j) \leq 0$, which would occur if $\Phi-\Psi$ had it's maximum at x_j. If $x_j = d$, then since Φ, Ψ are lower and upper mesh solutions, $\delta^2(\Phi-\Psi)(d) \geq 0$. To avoid a contradiction

$$(\Phi-\Psi)(d) = (\Phi-\Psi)(x_{N/2-1}) = (\Phi-\Psi)(x_{N/2+1})$$

and apply the first part of the argument to $(\Phi-\Psi)(x_{N/2-1})$.

Corollary 7 *Assuming (2.2) and (2.11), there exists a unique solution U_ε to the problem (3.1),(3.2) and*

$$\|U_\varepsilon\| \leq \frac{K}{\theta}.$$

Proof: Follow an analogous argument to that used in the proof of Theorem 2.

4 Error Analysis

We begin by looking at the truncation error. By classical estimates, for all $x_i \in \Omega^N \cap \Omega^-$

$$|-\varepsilon(\frac{d^2}{dx^2}-\delta^2)v_\varepsilon(x_i)| \leq \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})|v_\varepsilon|_3 \ \leq C\sqrt{\varepsilon}N^{-1}$$

and from [8] we have

$$|-\varepsilon(\frac{d^2}{dx^2}-\delta^2)w_\varepsilon(x_i)| \leq \begin{cases} \varepsilon(x_{i+1}-x_{i-1})|w_\varepsilon|_3 & (a) \\ 2\varepsilon \max\limits_{x\in[x_{i-1},x_{i+1}]} |w''_\varepsilon(x)| & (b) \end{cases}$$

Using (b) outside the layers and at $x = \sigma_1$, $x = d - \sigma_1$ gives

$$|-\varepsilon(\frac{d^2}{dx^2}-\delta^2)w_\varepsilon(x_i)| \leq \varepsilon C\varepsilon^{-1} \max_{x\in[x_{i-1},x_{i+1}]} e_1(x) \leq CN^{-1}.$$

Using (a) inside the layers gives, as above for v_ε,

$$|-\varepsilon(\frac{d^2}{dx^2}-\delta^2)w_\varepsilon(x_i)| \leq C\varepsilon\frac{\sigma_1}{N}\varepsilon^{-\frac{3}{2}}e_1(x_i) \ \leq CN^{-1}\ln N.$$

Similarly for all $x_i \in \Omega^N \cap \Omega^+$. Hence

$$|-\varepsilon(\frac{d^2}{dx^2}-\delta^2)u_\varepsilon(x_i)| \leq CN^{-1}\ln N, \quad x_i \neq d \tag{4.1a}$$

At the point $x_i = d$,

$$(D^+ - D^-)(U_\varepsilon - u_\varepsilon)(d) = -(D^+ - D^-)u_\varepsilon(d).$$

Let $h^{\pm}$ be the mesh interval sizes on either side of the point $x = d$ and $h = \max\{h^-, h^+\}$.

$$\begin{aligned}|(D^+ - D^-)u_\varepsilon(d)| &\le |(D^+ - \frac{d}{dx})u_\varepsilon(d)| + |(D^- - \frac{d}{dx})u_\varepsilon(d)| \\ &\le \frac{1}{2}h^+|u_\varepsilon|_2 + \frac{1}{2}h^-|u_\varepsilon|_2.\end{aligned}$$

Thus,

$$|(D^+ - D^-)(U_\varepsilon - u_\varepsilon)(d)| \le \frac{Ch}{\varepsilon} \tag{4.1b}$$

We are now ready to bound the nodal error $|(u_\varepsilon - U_\varepsilon)(x_i)|$.

Theorem 8 *Let u_ε be the solution of problem (1.1), (2.2), (2.11) and U_ε the solution of (3.1),(3.2). Then, for ε sufficiently small,*

$$\max_{x_i \in \bar{\Omega}_\varepsilon^N} |U_\varepsilon(x_i) - u_\varepsilon(x_i)| \le CN^{-1}\ln N$$

where C is a constant independent of ε and N.

Proof: At the internal mesh points,

$$\begin{aligned}-\varepsilon\delta^2 U_\varepsilon(x_i) + b(U_\varepsilon(x_i))U_\varepsilon(x_i) &= (-\varepsilon u_\varepsilon'' + b(u_\varepsilon)u_\varepsilon)(x_i) \\ -\varepsilon\delta^2(U_\varepsilon - u_\varepsilon)(x_i) + b(U_\varepsilon(x_i))U_\varepsilon(x_i) - b(u_\varepsilon)u_\varepsilon &= (-\varepsilon u_\varepsilon'' + \varepsilon\delta^2 u_\varepsilon)(x_i)\end{aligned}$$

Note that by (2.11),

$$b(U_\varepsilon(x_i))U_\varepsilon(x_i) - (b(u_\varepsilon)u_\varepsilon)(x_i) = g(Z)(U_\varepsilon(x_i) - u_\varepsilon(x_i))$$

where, since $u_\varepsilon, U_\varepsilon \in D_{\theta,K}$,

$$Z = u_\varepsilon + t(U_\varepsilon - u_\varepsilon) \in D_{\theta,K}, \text{ and } g(Z) \ge \gamma > 0, \ \forall\, 0 < t < 1.$$

Define the linear operator L_U^N by : For any mesh function V

$$\begin{aligned}L_U^N V(x_i) &= -\varepsilon\delta^2 V(x_i) + g(u_\varepsilon(x_i) + t(U - u))V(x_i) \ x_i \ne d \\ L_U^N V(d) &= D^- V(d) - D^+ V(d).\end{aligned}$$

From [7], we have the following discrete comparison principle: *If V is a mesh function such that $V(0) \ge 0, V(1) \ge 0;\ L_U^N V \ge 0,\ x_i \in \Omega^N$ and $D^+V(d) - D^-V(d) \le 0$ then $V(x_i) \ge 0$, for all $x_i \in \bar{\Omega}^N$.* By the truncation error estimates (4.1),

$$|L_U^N(U - u)| \le CN^{-1}\ln N, \quad x_i \ne d,$$

and at the mesh point $x_i = d$

$$|L_U^N(u - U)(d)| \le C\frac{h}{\varepsilon}$$

Consider the mesh function

$$\Xi(x_i) = C_1 N^{-1} \ln N + C_2 \frac{h}{\sqrt{\varepsilon}} \Phi_d(x_i) \pm U - u$$

where C_1 and C_2 are suitably large constants and Φ_d is defined as the solution of

$$-\varepsilon\delta^2 \Phi_d(x_i) + \gamma \Phi_d(x_i) = 0 \quad \text{for all} \quad x_i \in \Omega_\varepsilon^N$$
$$\Phi_d(0) = 0, \quad \Phi_d(d) = 1, \quad \Phi_d(1) = 0.$$

On the interval $[0, d]$, consider the barrier function

$$\omega(x_j) = \frac{\Pi_{i=1}^{j}(1 + \sqrt{\gamma} h_i / \sqrt{2\varepsilon})}{\Pi_{i=1}^{N/2}(1 + \sqrt{\gamma} h_i / \sqrt{2\varepsilon})}; \quad \omega(0) = 0, \ \omega(d) = 1$$

where $h_i = x_i - x_{i-1}$. Note that

$$D^+\omega(x_i) = \frac{\sqrt{\gamma}}{\sqrt{2\varepsilon}} \omega(x_i); \ D^-\omega(x_i) = \frac{\sqrt{\gamma}}{\sqrt{2\varepsilon}(1 + \sqrt{\gamma} h_i / \sqrt{2\varepsilon})} \omega(x_i)$$

which implies that

$$-\varepsilon\delta^2 \omega(x_i) + \gamma \omega(x_i) < 0, \quad 0 < x_i < d.$$

Hence, $\Phi_d(x_i) \leq \omega(x_i)$ and then

$$D^- \Phi_d(d) = \frac{1 - \Phi_d(d-h)}{h} \geq \frac{1 - \omega(d-h)}{h} = \frac{\sqrt{\gamma}/\sqrt{2\varepsilon})}{(1 + \sqrt{\gamma} h / \sqrt{2\varepsilon})} \geq \frac{C}{\sqrt{\varepsilon}}.$$

From this and using an analogous argument on the interval $[d, 1]$, we have that $\sqrt{\varepsilon}(D^+\Phi_d(d) - D^-\Phi_d(d)) \leq -C_2$. We conclude that

$$\|u_\varepsilon - U_\varepsilon\| \leq C N^{-1} \ln N.$$

5 Numerical results

In this section we present numerical results, which validate the theoretical results established in the previous section. In order to solve the nonlinear difference scheme we use a variant of the continuation method from [6, §10.3].

$$(-\varepsilon\delta_x^2 + b(U_\varepsilon(x_i, t_{j-1})) + D_t^-) U_\varepsilon(x_i, t_j) = f(x_i), \ x_i \neq d \ , j = 1, \dots J, \quad (5.1a)$$
$$D_x^- U_\varepsilon(d, t_j) = D_x^+ U_\varepsilon(d, t_j), j = 1, \dots J, \quad (5.1b)$$
$$U_\varepsilon(0, t_j) = u_\varepsilon(0), \ U_\varepsilon(1, t_j) = u_\varepsilon(1) \text{ for all } j, \quad (5.1c)$$
$$U_\varepsilon(x, 0) = u_{\text{init}}(x). \quad (5.1d)$$

In all cases, in this paper, the initial guess for the nonlinear solver is taken to be $u(0) + (u(1) - u(0))x$. We can interpret (5.1) as a discretization of the following time–dependent version of the problem

$$\text{Find} \quad u \in C^1([0,1] \times [0,T]) \quad \text{such that} \tag{5.2a}$$

$$-\varepsilon u_{xx} + b(u(x,t))u + u_t = f(x), \quad (x,t) \in (0,1) \setminus \{d\} \times (0,T] \tag{5.2b}$$

$$u(0,t) = u_\varepsilon(0) \quad u(1,t) = u_\varepsilon(1) \quad t \geq 0 \tag{5.2c}$$

$$u(x,0) = u_{\text{init}}(x) \quad 0 < x < 1. \tag{5.2d}$$

The choices of the uniform time-like step $k = t_j - t_{j-1}$ and the number of iterations J are determined as follows. Defining

$$e(j) \equiv \max_{1 \leq i \leq N} |U_\varepsilon(x_i, t_j) - U_\varepsilon(x_i, t_{j-1})|/k, \quad \text{for} \quad j = 1, 2, \cdots, J \tag{5.3a}$$

the time-like step k is chosen sufficiently small so that

$$e(j) \leq e(j-1), \quad \text{for all } j \text{ satisfying } 1 < j \leq J. \tag{5.3b}$$

Then the number of iterations J is chosen such that

$$e(J) \leq \text{ TOL }, \tag{5.3c}$$

where TOL is a suitably prescribed small tolerance. In the case of this paper, the tolerance TOL is chosen to be 10^{-7}. The numerical solution is computed using the following algorithm. Start from t_0 with the initial timestep $k = 1.0$. If, at some value of j, (5.3b) is not satisfied, then discard the timestep from t_{j-1} to t_j and restart from t_{j-1} with half the time step, that is $k^{new} = k/2$, and continue halving the timestep until one finds a k for which (5.3b) is satisfied. Assuming that (5.3b) is satisfied at each timestep, continue until either (5.3c) is satisfied or $t_j = 1000.$ or the $t_j - t_{j-1} < .000001$. If (5.3c) is not satisfied, we assume that the timestepping process stalled due to a too large choice of the initial timestep. In this case we repeat the entire process again from t_0, halving the initial timestep k to $k = 0.5$. If the process stalls again, we restart from t_0 again halving the initial timestep. If (5.3c) is satisfied the resulting values of $U_\varepsilon(x_i, t_J)$ are taken as the approximations to the solution of the continuous problem. Numerical results are presented for the problem

$$\varepsilon u_\varepsilon''(x) - (1 - u_\varepsilon^2) u_\varepsilon(x) = f(x), \tag{5.4a}$$

$$f(x) = \begin{cases} \delta_1 + x(0.5 - x), & x < 0.5; \\ -\delta_2 + (x - 0.5)(x - 1), & x > 0.5, \end{cases} \tag{5.4b}$$

$$u_\varepsilon(0) = A, \quad u_\varepsilon(1) = B. \tag{5.4c}$$

Let us examine in more detail the effects of the various constraints, such as those necessary for existence, in this particular case. Note first that for the problem (5.4)

$$b(y) = 1 - y^2 \geq \theta > 0, \quad \text{for} \quad |y| \leq \sqrt{1 - \theta}.$$

In this case, the restriction (2.7) on $\|f\|$ sufficient for existence of the reduced solution is

$$\|f\| \leq \theta\sqrt{1-\theta}. \tag{5.5a}$$

The range of f allowed by this constraint is maximized when $\theta = 2/3$, in which case it becomes:

$$\|f\| \leq \frac{2}{3\sqrt{3}} \approx 0.3849.$$

We remark that in order to guarantee that the solution u_ε exist, for all ε, we require in addition that (2.2) be satisfied, that is:

$$|A|, |B| \leq \sqrt{1-\theta}, \tag{5.5b}$$

which for this choice of $\theta = 2/3$ gives

$$|A|, |B| \leq \frac{1}{\sqrt{3}} \approx 0.57735026919.$$

However the restriction (2.11) when $K = \|f\|$, required to prove convergence of the numerical method, imposes the additional condition

$$\|f\| \leq \frac{\theta\sqrt{1-\gamma}}{3\sqrt{3}}, \ \ 0 < \gamma < 1. \tag{5.5c}$$

Note that the parameter M in the definition of the transition points of the mesh is bounded below by

$$M \geq \frac{1}{\sqrt{\gamma}}.$$

There is a trade-off between two competing constraints on γ. To allow maximum flexibility on choice of the mesh, we would like γ to be as large as possible. However to maximize the range of f values, it is better to choose γ smaller. To maximize the acceptable range of f, while keeping γ as large as possible, we choose to make (5.5a) and (5.5c) equal. Requiring this, we obtain:

$$\theta\sqrt{1-\theta} = \frac{\theta\sqrt{1-\gamma}}{3\sqrt{3}},$$

which implies, using the fact that $\gamma > 0$, that

$$\theta > \frac{26}{27} \approx 0.962963.$$

Assume $\theta = \frac{26}{27}$ then the restrictions (2.2), (2.11) on the data for the particular problem (5.4),that is (5.5a), (5.5b) and (5.5c) become

$$\|f\| < \tfrac{26}{81\sqrt{3}} \approx 0.1853 \tag{5.6a}$$

$$\max\{|u(0), |u(1)|\} \leq \tfrac{1}{3\sqrt{3}} \approx 0.19245. \tag{5.6b}$$

With these restrictions, the parameter in the transition point for the mesh is given by

$$M \geq \frac{1}{\sqrt{\gamma}}, \quad \text{where } \gamma = 1 - \frac{27^3}{26^2}\|f\|^2. \tag{5.7}$$

Figure 1 shows the solution of problem (5.4) for homogeneous boundary conditions $A = B = 0$ and $\delta_1 = -0.1, \delta_2 = 0.15$ on the left, and $A = -0.57735, B = 0.57735$ and $\delta_1 = -0.3849001794597, \delta_2 = 0.15$ on the right. The leftmost plot satisfies conditions (5.6). The data for the rightmost plot are close to the extremes of the condition (2.2) for the existence of the solution u_ε.

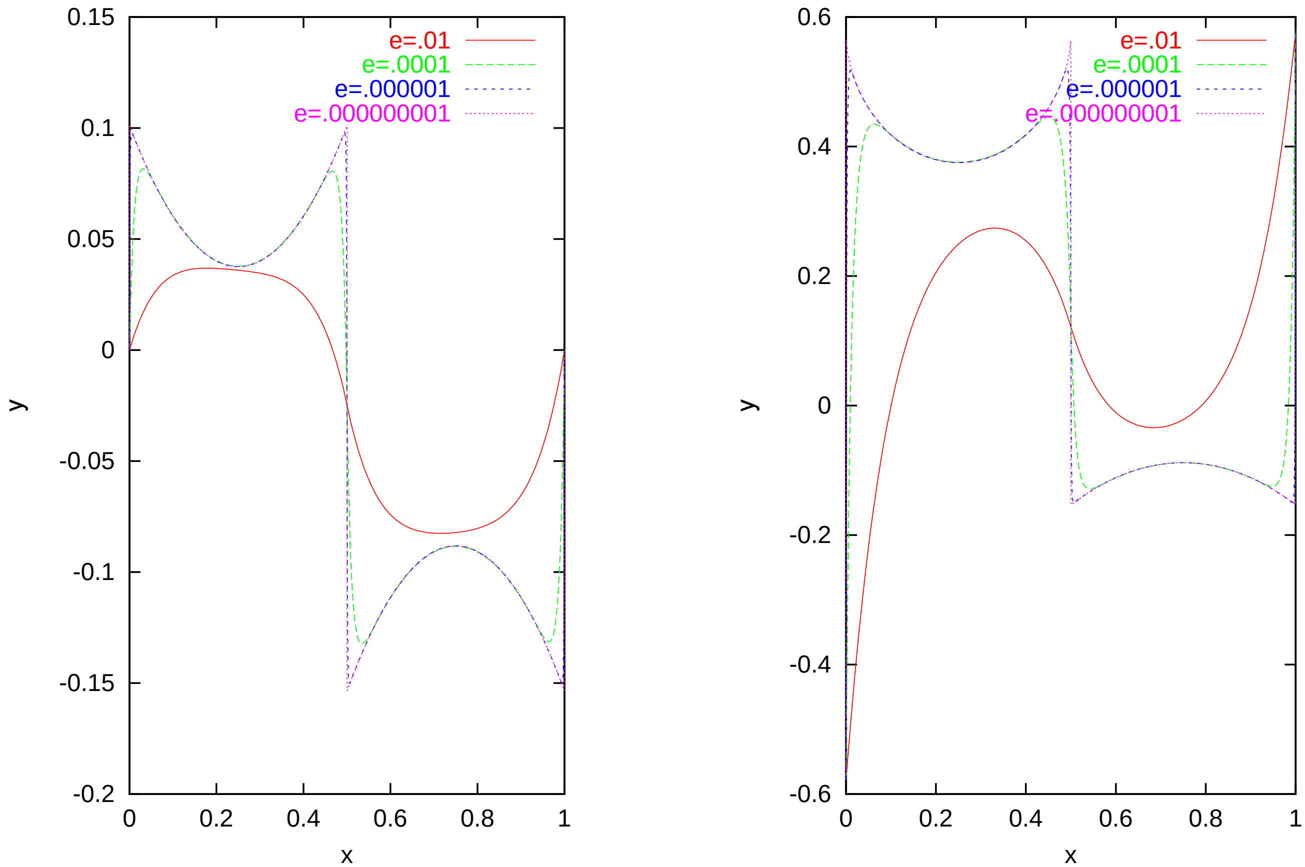

Figure 1: Solutions of (5.4) with (a) $A = B = 0$ and $\delta_1 = -0.1$, $\delta_2 = 0.15$ and (b) $-A = B = 0.57735$ and $\delta_1 = -0.3849001794597$, $\delta_2 = 0.15$ for $\varepsilon = 10^{-2}, 10^{-4}, 10^{-6}, 10^{-9}$.

Tables 1,2,3 respectively give the errors E_ε^N and the uniform errors E^N with respect to the finest mesh, the computed rates of convergence p_ε^N with respect to the finest mesh (where $N = 4096$), and the uniform rates of convergence p^N (see [6] for details on how these are calculated) and the number of iterations for problem (5.4) with $A = -0.14, B = 0.12$ and $\delta_1 = -0.1, \delta_2 = 0.15$, using the scheme (3.2c) where

$$\sigma_1 = \min\left\{\frac{1}{8},\ M\sqrt{\varepsilon}\ln N\right\} \quad \text{and} \quad \sigma_2 = \min\left\{\frac{1}{8},\ M\sqrt{\varepsilon}\ln N\right\}$$

and $M = 1.75$. Note that in this case the condition on γ (5.7) is:

$$\gamma = 1 - \frac{27^3}{26^2}\|f\|^2 \approx 0.344871, \quad \text{and} \quad M \geq \frac{1}{\sqrt{\gamma}} \approx 1.7028$$

	Number of Mesh Points N					
ϵ	32	64	128	256	512	1024
2^{-1}	0.000161	0.000079	0.000039	0.000019	0.000009	0.000004
2^{-2}	0.000236	0.000115	0.000056	0.000027	0.000013	0.000005
2^{-3}	0.000281	0.000136	0.000066	0.000032	0.000015	0.000006
2^{-4}	0.000258	0.000123	0.000059	0.000028	0.000013	0.000006
2^{-5}	0.000237	0.000081	0.000039	0.000018	0.000008	0.000004
2^{-6}	0.000338	0.000086	0.000022	0.000007	0.000003	0.000001
2^{-7}	0.000566	0.000142	0.000036	0.000009	0.000002	0.000001
2^{-8}	0.001046	0.000265	0.000066	0.000017	0.000004	0.000001
2^{-9}	0.001895	0.000517	0.000130	0.000032	0.000008	0.000002
2^{-10}	0.003859	0.001019	0.000258	0.000065	0.000016	0.000004
2^{-11}	0.006815	0.001881	0.000513	0.000129	0.000032	0.000008
2^{-12}	0.007478	0.003205	0.001016	0.000257	0.000064	0.000015
2^{-13}	0.007477	0.003204	0.001141	0.000370	0.000116	0.000030
2^{-14}	0.007477	0.003208	0.001132	0.000373	0.000117	0.000034
2^{-15}	0.007481	0.003209	0.001133	0.000373	0.000117	0.000034
2^{-16}	0.007483	0.003211	0.001133	0.000373	0.000117	0.000034
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-25}	0.007491	0.003215	0.001135	0.000374	0.000117	0.000034
E^N	0.007491	0.003215	0.001141	0.000374	0.000117	0.000034

Table 1: Computed maximum pointwise errors with respect to the finest mesh for $A = -0.14, B = 0.12$ and $\delta_1 = -0.1, \delta_2 = 0.15$ and $M = 1.75$.

To determine the sensitivity of the rates of convergence to the choice of the value M, in Table 4 we present a summary of the maximum errors over all values of ε above with respect to the finest mesh, and the associated rates of convergence, for various values of M for the same problem. We remark that in all cases the number of iterations taken was the same as in Table 3.

It is of some interest to the computational scientist to see if the method still converges in the region where there is no formal theoretical proof. This is, in fact, the case. Although the theory guarantees existence and uniqueness of the solution and convergence of the numerical method if (5.6) holds; in practice, the numerical method converges for a wider range of choices of f, $u(0)$, and $u(1)$. Tables 5 and 6 give examples of problems which just satisfy condition (2.2) but not (5.6a) and/or (5.6b) and hence (2.11) which was required for the proof of the convergence of the numerical solutions. Table 5 is for the problem (5.4) with homogeneous boundary

ε	Number of Mesh Points N 32	64	128	256	512
2^{-1}	1.02	1.03	1.05	1.10	1.22
2^{-2}	1.03	1.03	1.05	1.10	1.22
2^{-3}	1.05	1.04	1.06	1.10	1.22
2^{-4}	1.07	1.05	1.06	1.11	1.23
2^{-5}	1.55	1.07	1.07	1.11	1.23
2^{-6}	1.98	1.97	1.56	1.12	1.23
2^{-7}	1.99	2.00	2.00	2.02	2.07
2^{-8}	1.98	2.00	2.00	2.02	2.05
2^{-9}	1.87	1.99	2.00	2.02	2.07
2^{-10}	1.92	1.98	2.00	2.02	2.07
2^{-11}	1.86	1.88	1.99	2.01	2.07
2^{-12}	1.22	1.66	1.99	2.01	2.07
2^{-13}	1.22	1.49	1.62	1.67	1.94
2^{-14}	1.22	1.50	1.60	1.67	1.80
2^{-15}	1.22	1.50	1.60	1.67	1.80
2^{-16}	1.22	1.50	1.60	1.67	1.80
2^{-17}	1.22	1.50	1.60	1.67	1.80
2^{-18}	1.22	1.50	1.60	1.67	1.80
2^{-19}	1.22	1.50	1.60	1.67	1.80
2^{-20}	1.22	1.50	1.60	1.67	1.80
2^{-21}	1.22	1.50	1.60	1.67	1.80
2^{-22}	1.22	1.50	1.60	1.67	1.80
2^{-23}	1.22	1.50	1.60	1.67	1.80
2^{-24}	1.22	1.50	1.60	1.67	1.80
2^{-25}	1.22	1.50	1.60	1.67	1.80
p^N	1.22	1.50	1.61	1.67	1.80

Table 2: Computed rates of convergence for $A = -0.14, B = 0.12$ and $\delta_1 = -0.1, \delta_2 = 0.15$ and $M = 1.75$.

conditions $A = B = 0$ and $\delta_1 = -0.3849, \delta_2 = 0.15$, and thus the condition (5.6a) on f is violated. Table 6 is for the problem (5.4) with $A = 0.0, B = -0.3849$, and $\delta_1 = -0.1, \delta_2 = 0.15$, which satisfy condition (2.2) with $\theta = 2/3$.

We remark that in these cases where the right hand side and the boundary conditions are such that condition (2.2) is close to being violated, the number of iterations also increases dramatically. Table 7 gives the iteration counts for $M = 2.5$ for the case $A = B = 0$ and $\delta_1 = -0.3849, \delta_2 = 0.15$. Table 8 gives the iteration counts for the case $A = -0.55735, B = 0.55735$, $\delta_1 = -0.3849001794597, \delta_2 = 0.15$, and $M = 2.5$. The iteration counts for other values of M also increased to similar numbers. In these cases however the number of iteration did vary somewhat with the choice of M. On the other hand, if the sufficient condition for existence of the reduced solution that is:

$$|f| \leq \frac{2}{3\sqrt{3}} = 0.38490017945975050967,$$

is exceeded then for sufficiently small ε the algorithm does not converge. For example, for $A = 0.0, B = 0.0$, $\delta_1 = -0.39, \delta_2 = 0.15$, and $M = 2.5$. the algorithm fails to converge for $\varepsilon < 2^{-19}$.

	Number of Mesh Points N					
ε	32	64	128	256	512	1024
2^{-2}	3	3	3	3	3	3
2^{-4}	4	4	4	4	4	4
2^{-6}	4	4	4	4	4	4
2^{-8}	5	5	5	5	5	5
2^{-10}	5	5	5	5	5	5
2^{-12}	6	6	6	6	6	6
2^{-14}	6	6	6	6	6	6
2^{-16}	6	6	6	6	6	6
2^{-18}	6	6	6	6	6	6
2^{-20}	6	6	6	6	6	6
2^{-22}	7	7	7	7	6	6
2^{-24}	7	7	7	7	7	7

Table 3: Number of iterations for $A = -0.14, B = 0.12$ and $\delta_1 = -0.1, \delta_2 = 0.15$ and $M = 1.75$.

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N	32	64	128	256	512
			$M=0.5$		
E_N	0.049406	0.034813	0.024206	0.016368	0.010443
p_N	0.51	0.52	0.56	0.65	0.83
			$M=1.0$		
E_N	0.009245	0.004683	0.002345	0.001147	0.000534
p_N	0.98	1.00	1.03	1.10	1.26
			$M=1.5$		
E_N	0.006024	0.002331	0.000836	0.000275	0.000086
p_N	1.37	1.48	1.60	1.68	1.79
			$M=1.75$		
E_N	0.007491	0.003215	0.001141	0.000374	0.000117
p_N	1.22	1.50	1.61	1.67	1.80
			$M=2.0$		
E_N	0.008770	0.004153	0.001458	0.000490	0.000153
p_N	1.08	1.51	1.57	1.68	1.79
			$M=2.5$		
E_N	0.010592	0.006047	0.002178	0.000743	0.000239
p_N	0.81	1.47	1.55	1.64	1.81
			$M=5.0$		
E_N	0.011589	0.011489	0.007467	0.002870	0.000942
p_N	0.01	0.62	1.38	1.61	1.81

Table 4: Maximum pointwise errors E_N and computed rates of convergence p_N for $A=-0.14, B=0.12$ and $\delta_1=-0.1, \delta_2=0.15$ for several values of M.

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N	32	64	128	256	512
$M = 0.5$					
E_N	0.180983	0.143453	0.110390	0.080335	0.052783
p_N	0.34	0.38	0.46	0.61	0.88
$M = 1.0$					
E_N	0.085538	0.061915	0.043604	0.028841	0.016853
p_N	0.47	0.51	0.60	0.78	1.11
$M = 1.5$					
E_N	0.045875	0.030817	0.020145	0.012320	0.006583
p_N	0.57	0.61	0.71	0.90	1.27
$M = 1.75$					
E_N	0.034398	0.022204	0.013931	0.008164	0.004170
p_N	0.63	0.67	0.77	0.97	1.34
$M = 2.0$					
E_N	0.026008	0.016078	0.009638	0.005387	0.002621
p_N	0.69	0.74	0.84	1.04	1.41
$M = 2.5$					
E_N	0.023249	0.013408	0.007045	0.003548	0.001730
p_N	0.79	0.93	0.99	1.04	1.15
$M = 5.0$					
E_N	0.038897	0.026724	0.015550	0.007945	0.003770
p_N	0.54	0.78	0.97	1.08	1.19

Table 5: Maximum errors E_N and computed rates of convergence p_N for $A = B = 0$ and $\delta_1 = -0.3849, \delta_2 = 0.15$ for several values of M.

N	32	64	128	256	512
$M = 0.5$					
E_N	0.044614	0.031580	0.022031	0.014935	0.009546
p_N	0.50	0.52	0.56	0.65	0.83
$M = 1.0$					
E_N	0.008456	0.004289	0.002150	0.001052	0.000490
p_N	0.98	1.00	1.03	1.10	1.26
$M = 2.5$					
E_N	0.007947	0.003913	0.001323	0.000441	0.000170
p_N	1.02	1.56	1.58	1.37	1.22
$M = 5.0$					
E_N	0.009141	0.008955	0.005072	0.001631	0.000543
p_N	0.03	0.82	1.64	1.59	1.83

Table 6: Maximum errors E_N and computed rates of convergence p_N for $A = 0.0, B = -0.3849$ and $\delta_1 = -0.1, \delta_2 = 0.15$ for several values of M.

	Number of Mesh Points N					
ϵ	32	64	128	256	512	1024
2^{-2}	3	3	3	3	3	3
2^{-4}	4	4	4	4	4	4
2^{-6}	7	7	7	7	7	7
2^{-8}	11	11	11	11	11	11
2^{-10}	14	14	14	14	14	14
2^{-12}	17	17	17	17	17	17
2^{-14}	21	21	21	21	21	21
2^{-16}	28	27	26	26	26	26
2^{-18}	37	35	34	33	33	33
2^{-20}	50	47	44	42	42	42
2^{-22}	67	63	59	55	53	53
2^{-24}	89	83	78	73	69	67

Table 7: Number of Iterations for $A = B = 0$ and $\delta_1 = -0.3849, \delta_2 = 0.15$ and $M = 2.5$.

	Number of Mesh Points N					
ϵ	32	64	128	256	512	1024
2^{-2}	5	5	5	5	5	5
2^{-4}	6	6	6	6	6	6
2^{-6}	7	7	7	7	7	7
2^{-8}	13	13	13	13	13	13
2^{-10}	17	17	17	17	17	17
2^{-12}	20	20	20	20	20	20
2^{-14}	25	25	25	25	25	25
2^{-16}	33	32	31	31	31	31
2^{-18}	45	42	40	40	40	40
2^{-20}	62	57	53	51	51	51
2^{-22}	84	77	72	68	65	64
2^{-24}	113	104	97	91	85	82

Table 8: Number of Iterations for $A = -0.57735, B = 0.57735$ and $\delta_1 = -0.3849001794597, \delta_2 = 0.15$ and $M = 2.5$.