A parameter robust higher order numerical method for a singularly perturbed two–parameter problem

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Abstract

In this paper a second order monotone numerical method is constructed for a singularly perturbed ordinary differential equation with two small parameters affecting the convection and diffusion terms. The monotone operator is combined with a piecewise-uniform Shishkin mesh. An asymptotic error bound in the maximum norm is established theoretically whose error constants are show to be independent of both singular perturbation parameters. Numerical results are presented which support the theoretical results.

Keywords: Singular perturbation, uniform convergence, Shishkin mesh, high order.

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1 Introduction

Differential equations with a small parameter $0 < \varepsilon \leq 1$ multiplying the highest order derivatives are called singularly perturbed differential equations. Typically, the solutions of such equations have steep gradients (proportional to $\varepsilon^{-p}$, $p > 0$) in narrow layer regions of the domain. Classical numerical methods are inappropriate for singularly perturbed problems [5]. For any numerical method applied to a singularly perturbed problem, it is informative to establish an asymptotic pointwise error bound of the form

$$\|u - U^N\|_{\Omega^N} \leq C_p N^{-p}, \quad p > 0,$$

where $U^N$ are the numerical approximations, $N$ is the number of mesh elements used in each coordinate direction, $u$ is the solution of the continuous problem and $\|v\|_D = \max_{x \in D} |v(x)|$ is the maximum pointwise norm. A numerical method is said to be parameter-uniform if the error constant $C_p$ is independent of any perturbation parameters and the mesh parameter $N$.

A common approach to constructing a parameter-uniform numerical method is to combine a standard monotone finite difference operator with an appropriate piecewise-uniform mesh (Shishkin mesh [5, 11]) where half the mesh points are concentrated in the layer regions. In the case of the singularly perturbed reaction-diffusion problem

$$\varepsilon u'' - bu = f(x), \quad b(x) \geq \beta > 0, \quad u(0), u(1) \text{ given},$$

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the standard central difference operator on an appropriate Shishkin mesh (transition points between coarse and fine mesh are taken to be \( \sigma = \min\{0.25, 2\sqrt{e/\beta} \ln N\} \)) produces a second order (up to logarithmic factor) parameter-uniform error bound [7] of the form

\[ \|u - U^N\|_{\Omega^N} \leq C(N^{-1} \ln N)^2. \]

In [6] the authors analyze a monotone finite difference scheme of fourth order (up to logarithmic factor) using a classical higher order compact finite difference operator on a Shishkin mesh.

In this paper, we construct a monotone numerical method for the two parameter problem

\[ \varepsilon u'' + a u' = f(x), \quad a(x) \geq \alpha > 0, \quad u(0), u(1) \text{ given}, \]

the standard upwind difference operator on an appropriate Shishkin mesh (transition point of the form \( \sigma = \min\{0.5, \frac{2}{\alpha} \ln N\} \)) produces a first order (up to logarithmic factor) parameter-uniform error bound [5]

\[ \|u - U^N\|_{\Omega^N} \leq C N^{-1} \ln N. \]

For the above convection-diffusion problem, Stynes and Roos [13] established that a numerical method composed of the central difference operator in the layer region \((0, \sigma)\) combined with the mid-point scheme [1] outside the layer region \([\sigma, 1]\) on a Shishkin mesh with \( \sigma = \min\{0.5, \frac{2}{\alpha} \ln N\} \) is a monotone numerical method; and when \( \sigma < 0.5 \), it satisfies a parameter-uniform error bound of the form

\[ \|u - U\|_{\Omega^N} \leq \begin{cases} C N^{-1}(\varepsilon + N^{-1}), & \text{if } x_i \in [\sigma, 1], \\ C(N^{-1} \ln N)^2, & \text{if } x_i \in [0, \sigma]. \end{cases} \]

This is essentially second order in the layer and first order outside the layer. Other high order parameter-uniform methods for the convection-diffusion problem, include a weighted monotone finite difference scheme [3], the non-monotone centered difference scheme [2] and the HODIE [4] on appropriate Shishkin meshes.

In this paper, we construct a monotone numerical method for the two parameter problem

\[ \varepsilon u'' + \mu a u' - b u = f(x), \quad a(x) \geq \alpha > 0, \quad b(x) \geq \beta > 0, \quad u(0), u(1) \text{ given}, \]

where \( 0 < \varepsilon \leq 1, \ 0 \leq \mu \leq 1 \). This problem encompasses both the reaction-diffusion problem when \( \mu = 0 \) and the convection-diffusion problem when \( \mu = 1 \). The nature of the continuous solution of this two parameter problem was examined by O’Malley [8], where the ratio of \( \mu^2 \) to \( \varepsilon \) was identified as significant. Parameter-uniform methods on a uniform mesh were constructed in [12]. In [9] and [10], the standard upwind finite difference operator on two different choices of Shishkin mesh was shown to be parameter-uniform of first order. In this paper, we construct a second order parameter-uniform numerical method on the Shishkin mesh given in [10] for this two parameter problem.

The finite difference operator is a combination of the central difference, midpoint and standard upwinded difference operators. These three finite difference operators are monotone in various subdomains of the parameter space \( P = \{ (\varepsilon, \mu, N) | 0 < \varepsilon \leq 1, \ 0 \leq \mu \leq 1, N \geq N_0 \} \). When analysing the errors in the regular component in the coarse mesh region, the following interplay between the requirements of monotonicity and high order truncation error are worth noting. The standard upwinded operator is always monotone and has a second order truncation error when both \( \varepsilon \) and \( \mu \) are relatively small so that \( \varepsilon, \mu \leq C N^{-1} \). The central difference operator is monotone if \( \varepsilon \) is relatively large and \( \mu \) is relatively small so that \( N\varepsilon \geq C_1 \mu \) and it has second order truncation error away from the transition points. The midpoint scheme is monotone for all \( \varepsilon \) and for \( \mu \) relatively large so that \( \mu N \geq C_2 \). For \( \varepsilon \leq C N^{-1} \), the midpoint scheme has second order truncation error at all the mesh points. All of these ingredients are combined in this paper to produce a numerical scheme which is monotone and is uniformly second order (up to logarithmic factors) for all values of the parameters \( \varepsilon \) and \( \mu \).
2 The continuous problem

Consider the singularly perturbed boundary value problem

\[
\begin{cases}
L_{\varepsilon,\mu} u \equiv \varepsilon u'' + \mu au' - bu = f, & x \in \Omega = (0,1), \\
u(0) = u_0, & u(1) = u_1,
\end{cases}
\] (2.1)

where \(0 < \varepsilon, \mu \leq 1\) are two small parameters. The coefficients \(a, b\) and \(f\) are sufficiently smooth and

\[a(x) \geq \alpha > 0, \quad b(x) \geq \beta > 0, \quad \forall x \in \Omega, \quad \gamma = \min_{\Omega} \frac{b}{a}.
\]

The proof of the following comparison principle for the differential operator is standard.

**Lemma 1 (Comparison Principle).** Let \(v \in C^2(\Omega)\). If \(v(0) \geq 0, v(1) \geq 0\) and \(L_{\varepsilon,\mu} v(x) \leq 0, \forall x \in \Omega\), then \(v(x) \geq 0, \forall x \in \Omega\).

An immediate consequence of this comparison principle is the following parameter uniform bound on the solution \(u\).

**Lemma 2.** If \(u\) is the solution of the boundary value problem (2.1), then

\[||u|| \leq \left[\max\{|u_0|, |u_1|\} + \frac{1}{\beta}||f||\right], \quad \forall x \in \Omega,
\]

where \(\| \cdot \|\) denotes the pointwise maximum norm.

The derivatives of the solution of this problem satisfy the following parameter-explicit bounds.

**Theorem 3.** Assuming that \(a, b, f \in C^2(\Omega)\), the derivatives of the solution \(u\) of (2.1) satisfy the following bounds

\[
\begin{align*}
||u^{(k)}|| & \leq C \left(\frac{\varepsilon}{\sqrt{\varepsilon}}\right)^k \left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right) \max\{||u||, ||f||\}, & k = 1, 2, \\
||u^{(3)}|| & \leq C \left(\frac{\varepsilon}{\sqrt{\varepsilon}}\right)^3 \left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right) \max\{||u||, ||f||\} + \frac{1}{\varepsilon}||f'||, \\
||u^{(4)}|| & \leq C \left(\frac{\varepsilon}{\sqrt{\varepsilon}}\right)^4 \left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^4\right) \max\{||u||, ||f||\} + \frac{1}{\varepsilon^2}||f''|| + \frac{1}{\varepsilon}||f''||,
\end{align*}
\]

where the constant \(C\) depends only on \(\|a^{(i)}\|, \|b^{(i)}\|\), for \(i = 0, 1, 2\) and is independent of \(\varepsilon\) and \(\mu\).

**Proof.** Obtain the bounds on the first two derivatives using the argument in Lemma 2.2 of [9]. The bounds on the third and fourth derivatives come from differentiating the differential equation in (2.1) and using the inequality

\[
\sum_{i=0}^{n} t^i \leq (n + 1)(1 + t^n), \quad t \geq 0.
\]
3 Decomposition of the continuous solution

The solution of the boundary value problem (2.1) can be decomposed into the following sum

\[ u = v + w_L + w_R, \quad (3.2a) \]

where

\[ L v = f, \quad v(0), v(1) \text{ suitably chosen,} \quad (3.2b) \]
\[ L w_L = 0, \quad w_L(0) = u(0) - v(0) - w_R(0), \quad w_L(1) = 0, \quad (3.2c) \]
\[ L w_R = 0, \quad w_R(0) \text{ suitably chosen,} \quad w_R(1) = u(1) - v(1). \quad (3.2d) \]

The function \( v \) is called the regular component and \( w_L \) and \( w_R \) are the left and right, respectively, singular components of the solution \( u \). A major part of this section will involve establishing that the first three derivatives of the regular component are bounded independently of the perturbation parameters, which is a natural extension of the decomposition given in [10] for the solution of the time-dependent version of problem (2.1). We first consider the case of \( \alpha \mu^2 \leq \gamma \varepsilon \). We have the following secondary decomposition of the regular component

\[ v(x; \varepsilon, \mu) = v_0(x) + \sqrt{\varepsilon}v_1(x; \varepsilon, \mu) + (\sqrt{\varepsilon})^2v_2(x; \varepsilon, \mu) + (\sqrt{\varepsilon})^3v_3(x; \varepsilon, \mu), \quad (3.3a) \]

where

\[ -bv_0 = f, \quad bv_1 = \sqrt{\varepsilon}v_0'' + \frac{\mu}{\sqrt{\varepsilon}}av_0', \quad bv_2 = \sqrt{\varepsilon}v_1'' + \frac{\mu}{\sqrt{\varepsilon}}av_1', \quad (3.3b) \]
\[ L_{\varepsilon, \mu}v_3 = -\sqrt{\varepsilon}v_0'' - \frac{\mu}{\sqrt{\varepsilon}}av_0', \quad v_3(0; \varepsilon, \mu) = v_3(1; \varepsilon, \mu) = 0. \quad (3.3c) \]

We see that \( v_0(x; \varepsilon, \mu) = v_0(0) + \sqrt{\varepsilon}v_1(0; \varepsilon, \mu) + \varepsilon v_2(0; \varepsilon, \mu) \) and \( v(1; \varepsilon, \mu) = v_0(1) + \sqrt{\varepsilon}v_1(1; \varepsilon, \mu) + \varepsilon v_2(1; \varepsilon, \mu) \). Assuming sufficient smoothness on the coefficients \( \alpha \in C^6(\Omega), b, f \in C^8(\Omega) \) and noting that \( \alpha \mu^2 \leq \gamma \varepsilon \), we see that \( v_0 \) and its derivatives up to order eight, \( v_1 \) and its derivatives up to order six and \( v_2 \) and its derivatives up to order four are bounded independently of \( \varepsilon \) and \( \mu \).

We proceed to analyse the final term \( v_3(x; \varepsilon, \mu) \). Using the minimum principle for \( L_{\varepsilon, \mu} \) and a suitable barrier function we obtain (see [9] Lemma 2.1),

\[ \|v_3\| \leq \max \{ |v_3(0)|, |v_3(1)| \} + \frac{1}{\beta} (\|v_2''\| + \|v_2''\|). \]

Applying the bounds on \( v_2 \) we therefore have

\[ \|v_3\| \leq C. \]

Using the differential equation (3.3c) and the Mean Value Theorem on an interval of width \( \sqrt{\varepsilon} \) and noting that \( \alpha \mu^2 \leq \gamma \varepsilon \), we obtain (see [9] Lemma 2.2),

\[ \|v_3^{(k)}\| \leq \frac{C}{(\sqrt{\varepsilon})^k} \max \{ \|v_3\|, \|v_2''\|, \|v_2''\| \} \leq \frac{C}{(\sqrt{\varepsilon})^k}, \quad k = 1, 2. \]

Differentiating the equation (3.3c) and using the above bounds we also obtain

\[ \|v_3^{(k)}\| \leq \frac{C}{(\sqrt{\varepsilon})^k}, \quad k = 3, 4. \]

Substituting all of these bounds for \( v_0(x; \mu), v_1(x; \varepsilon, \mu), v_2(x; \varepsilon, \mu) \) and \( v_3(x; \varepsilon, \mu) \) into the equation for \( v(x; \varepsilon, \mu) \) gives us

\[ \|v^{(k)}\| \leq C \left( 1 + \frac{1}{(\sqrt{\varepsilon})^{k-3}} \right), \quad \text{for} \quad 0 \leq k \leq 4, \quad \text{when} \quad \alpha \mu^2 \leq \gamma \varepsilon. \quad (3.4) \]
When $\alpha \mu^2 \geq \gamma \varepsilon$ we consider the following alternative secondary decomposition of the regular component
\[ v(x; \varepsilon, \mu) = v_0(x; \mu) + \varepsilon v_1(x; \mu) + \varepsilon^2 v_2(x; \mu) + \varepsilon^3 v_3(x; \varepsilon, \mu), \quad (3.5a) \]
where
\[
\begin{align*}
L_{\mu} v_0 &= f(x), \quad v_0(1; \mu) \text{ chosen in (3.7)}, \quad (3.5b) \\
L_{\mu} v_1 &= -v_0''(x; \mu), \quad v_1(1; \mu) \text{ chosen in (3.8)}, \quad (3.5c) \\
L_{\mu} v_2 &= -v_0''(x; \mu), \quad v_2(1; \mu) = 0, \quad (3.5d) \\
L_{\varepsilon, \mu} v_3(x; \varepsilon, \mu) &= -v_0''(x; \mu), \quad v_3(0; \varepsilon, \mu) = v_3(1; \varepsilon, \mu) = 0, \quad (3.5e)
\end{align*}
\]
with $L_{\mu} z \equiv \mu a z' - b z$. We see that $v(0; \varepsilon, \mu) = v_0(0; \mu) + \varepsilon v_1(0; \mu) + \varepsilon^2 v_2(0, \mu)$. The following lemmas establish that when $v_0(1; \mu)$ and $v_1(1; \mu)$ are chosen correctly the first three derivatives of $v_0(x; \mu)$ and the first derivative of $v_1(x; \mu)$ are bounded independent of $\mu$.

**Lemma 4.** If $v_0$ satisfies the first order differential equation (3.5b) then there exists a value for $v_0(1; \mu)$ such that the following bounds hold
\[ \|v_0^{(i)}\| \leq C \left( 1 + \frac{1}{\mu^{1-3}} \right), \quad \text{for } 0 \leq i \leq 7. \]

**Proof.** We start by noting that since $a > 0$ and $b > 0$ we can establish the following
\[ L_{\mu} \left. z \right|_{[0,1]} \leq 0 \quad \text{and} \quad z(1) \geq 0, \quad \text{then} \quad \left. z \right|_{[0,1]} \geq 0, \quad (3.6) \]
using a simple proof by contradiction argument. We further decompose $v_0(x; \mu)$ as follows
\[ v_0(x; \mu) = s_0(x) + \mu s_1(x) + \mu^2 s_2(x) + \mu^3 s_3(x; \mu), \quad (3.7a) \]
where
\[
\begin{align*}
s_0(x) &= -\frac{f}{b}, \quad s_1(x) = \frac{a s_0'(x)}{b}, \quad s_2(x) = \frac{a s_1'(x)}{b}, \quad (3.7b) \\
L_{\mu} s_3(x; \mu) &= -a s_2'(x), \quad s_3(1; \mu) = 0. \quad (3.7c)
\end{align*}
\]
We see that $v_0(1; \mu) = s_0(1) + \mu s_1(1) + \mu^2 s_2(1)$ and assuming sufficient smoothness of the coefficients, we have
\[ \|s_0^{(i)}\| \leq C, \quad \|s_1^{(i)}\| \leq C \quad \text{and} \quad \|s_2^{(i)}\| \leq C \quad \text{for } 0 \leq i \leq 3. \]
Using (3.6) we deduce that $\|s_3\| \leq (1/\beta) \|a s_2''\| \leq C$ and from (3.7c) we obtain
\[ \|s_3^{(i)}\| \leq \frac{C}{\mu^i}, \quad \text{for} \quad 1 \leq i \leq 3. \]
We use these bounds for $s_0(x)$, $s_1(x)$, $s_2(x)$ and $s_3(x; \mu)$ to obtain $\|v_0^{(i)}\| \leq C$ for $0 \leq i \leq 3$ and then (3.5b) to obtain the required result on the higher derivatives of $v_0$.

**Lemma 5.** If $v_1$ satisfies the first order differential equation (3.5c) then there exists a value for $v_1(1; \mu)$ such that the following bounds hold
\[ \|v_1^{(i)}\| \leq C \left( 1 + \frac{1}{\mu^{1-4}} \right), \quad \text{for } 0 \leq i \leq 5. \]

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Proof. We decompose \( v_1(x; \mu) \) as follows

\[
v_1(x; \mu) = \rho_0(x) + \mu \rho_1(x) + \mu^2 \rho_2(x; \mu)
\]

where

\[
\rho_0(x) = \frac{v_0''}{b}, \quad \rho_1(x) = \frac{a \rho_0'(x)}{b}, \quad L_\mu \rho_2(x; \mu) = -a \rho_1'(x), \quad \rho_2(1; \mu) = 0.
\]

We see that \( v_1(1; \mu) = \rho_0(1) + \mu \rho_1(1) \) and assuming sufficient smoothness of the coefficients, we have

\[
\|\rho_0^{(i)}\| \leq C \left( 1 + \frac{1}{\mu^{i-1}} \right), \quad \|\rho_1^{(i)}\| \leq \frac{C}{\mu^i}, \quad \text{for} \quad 0 \leq i \leq 2.
\]

Using (3.6) and (3.8c) we can also obtain

\[
\|\rho_2^{(i)}\| \leq \frac{C}{\mu^{i+1}}, \quad \text{for} \quad 0 \leq i \leq 2.
\]

We use these bounds for \( \rho_0(x) \), \( \rho_1(x) \), and \( \rho_2(x; \mu) \) and their derivatives to obtain \( \|v_1^{(i)}\| \leq C(1 + \mu^{1-i}) \), for \( i = 0, 1, 2 \). The required result for \( 0 \leq i \leq 5 \) follows by differentiating the differential equation (3.5c) for \( v_1 \).

Lemma 6. If \( v_2 \) satisfies the first order differential equation (3.5d) then the following bounds hold

\[
\|v_2^{(i)}\| \leq \frac{C}{\mu^{i+1}}, \quad \text{for} \quad 0 \leq i \leq 4.
\]

Proof. The proof follows using (3.6), the differential equation (3.5d) and the bounds in Lemma 5.

Lemma 7. If \( v_3 \) satisfies the first order differential equation (3.5e) then the following bounds hold

\[
\|v_3^{(i)}\| \leq C \left( \frac{\mu}{\varepsilon} \right)^i \left( \frac{1}{\mu^3} \right), \quad \text{for} \quad 0 \leq i \leq 4.
\]

Proof. Using the minimum principle for \( L_{\varepsilon, \mu} \) and a suitable barrier function we obtain (see [9] Lemma 2.1),

\[
\|v_3\| \leq \max \{ |v_3(0)|, |v_3(1)| \} + \frac{1}{\beta} \|v_2''\|.
\]

Applying the bounds in Lemma 6 we therefore have

\[
\|v_3\| \leq \frac{C}{\mu^3}.
\]

Using the differential equation (3.5e) and the Mean Value Theorem on an interval of width \( \sqrt{\varepsilon} \) we obtain (see [9] Lemma 2.2),

\[
\|v_3^{(k)}\| \leq \frac{C}{(\sqrt{\varepsilon})^k} \left( 1 + \left( \frac{\mu}{\sqrt{\varepsilon}} \right)^k \right) \max \{ \|v_3\|, \|v_2''\| \}, \quad k = 1, 2.
\]

Simplifying this expression using the previous Lemma and (3.9)

\[
\|v_3^{(k)}\| \leq \frac{C}{(\sqrt{\varepsilon})^k} \left( 1 + \left( \frac{\mu}{\sqrt{\varepsilon}} \right)^k \right) \frac{1}{\mu^3}, \quad k = 1, 2.
\]
Differentiating the differential equation for $v_3$ and applying these bounds gives us
\[
\|v_3^{(3)}\| \leq \frac{C}{\varepsilon^3} \quad \text{and} \quad \|v_3^{(4)}\| \leq \frac{C\mu}{\varepsilon^4}.
\] (3.10)

Substituting all of these bounds for $v_0(x; \mu), v_1(x; \varepsilon, \mu), v_2(x; \varepsilon, \mu)$ and $v_3(x; \varepsilon, \mu)$ into the equation for $v(x; \varepsilon, \mu)$ and noting that $\mu \geq C\sqrt{\varepsilon}$ gives us
\[
\|v^{(i)}\| \leq C \left(1 + \left(\frac{\varepsilon}{\mu}\right)^{(3-i)}\right), \quad \text{for} \quad 0 \leq i \leq 4.
\] (3.11)

In order to fully specify the decomposition given in (3.2), it is required to specify the values of $v(0), v(1)$ and $w_R(0)$. In the case of $\alpha \mu^2 \leq \gamma \varepsilon$, $v(0), v(1)$ are specified in (3.5) and $w_R(0) = 0$. In the case of $\alpha \mu^2 \geq \gamma \varepsilon$, $v(0), v(1)$ are specified in (3.3) and (3.7) and (3.8) and $w_R(0)$ is specified below in (3.12).

**Lemma 8.** When $\mu^2 \geq \frac{\varepsilon}{\alpha}$, the value $w_R(0)$ can be specified so that $w_R$ satisfies the following bounds
\[
\|w_R^{(i)}\| \leq \frac{C}{\mu^i}, \quad 0 \leq i \leq 3.
\]

**Proof.** Consider the following decomposition
\[
w_\mu(x; \varepsilon, \mu) = w_0(x; \mu) + \varepsilon w_1(x; \mu) + \varepsilon^2 w_2(x; \mu) + \varepsilon^3 w_3(x; \varepsilon, \mu)
\] (3.12a)

where $v(1) = v_0(1) + \varepsilon v_1(1)$ is given in (3.7) and
\[
\begin{align*}
L_\mu w_0 &= 0, \quad w_0(1; \mu) = u(1) - v(1), \\
\varepsilon L_\mu w_1 &= (L_\mu - L_{\varepsilon \mu}) w_0, \quad w_1(1; \mu) = 0, \\
\varepsilon L_\mu w_2 &= (L_\mu - L_{\varepsilon \mu}) w_1, \quad w_2(1; \mu) = 0, \\
\varepsilon L_{\varepsilon \mu} w_3 &= (L_{\varepsilon \mu} - L_{\varepsilon \mu}) w_2, \quad w_3(0; \varepsilon, \mu) = w_3(1; \varepsilon, \mu) = 0.
\end{align*}
\] (3.12b-d)

We start by analysing $w_0(x)$. Using (3.6) applied to $w_0(x) = w_0(1) \psi(x)$, where $L_\mu \psi = 0$, $\psi(1) = 1$, we obtain the following bounds
\[
\|w_0^{(i)}\| \leq \frac{C}{\mu^i}, \quad 0 \leq i \leq 5.
\] (3.13)

Using this method again for $w_1(x)$ and $w_2(x)$ we obtain
\[
\|w_1^{(i)}\| \leq \frac{C}{\mu^{i+2}}, \quad 0 \leq i \leq 4, \quad \text{and} \quad \|w_2^{(i)}\| \leq \frac{C}{\mu^{i+4}}, \quad 0 \leq i \leq 3.
\] (3.14)

Finally we consider $w_3$, we can apply Lemma 2 to obtain
\[
\|w_3\| \leq \frac{C}{\mu^6}.
\]

From Theorem 3 we have the following bounds for $1 \leq i \leq 2$
\[
\|w_3^{(i)}\| \leq \frac{C}{(\sqrt{\varepsilon})^i} \left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^i\right) \frac{1}{\mu^6}.
\]

Finally we obtain
\[
\|w_3^{(3)}\| \leq \frac{C}{(\sqrt{\varepsilon})^3} \left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right) \frac{1}{\mu^6} + \frac{1}{\varepsilon} \frac{1}{\mu^4}.
\]

The required bounds follow using (3.12) and the inequality $\mu^2 \geq \frac{\varepsilon}{\alpha}$. \qed
Lemma 9. [10] The singular components \( w_L \) and \( w_R \) satisfy the following bounds
\[
|w_L(x)| \leq Ce^{-\theta_1 x}, \quad |w_R(x)| \leq Ce^{-\theta_2 (1-x)},
\]
where
\[
\theta_1 = \begin{cases} \frac{\sqrt{2\pi}}{\alpha}, & \mu^2 \leq \frac{2\pi}{\alpha}, \\ \frac{\alpha}{\sqrt{2\pi}}, & \mu^2 \geq \frac{2\pi}{\alpha} \end{cases}, \quad \theta_2 = \begin{cases} \frac{\sqrt{2\pi}}{\alpha}, & \mu^2 \leq \frac{2\pi}{\alpha}, \\ \frac{\alpha}{\sqrt{2\pi}}, & \mu^2 \geq \frac{2\pi}{\alpha}. \end{cases}
\]

Note that the derivatives of the singular components \( w_L \) and \( w_R \) satisfy the bounds given in Theorem 3.

4 Discrete Problem

To approximate the solution of problem (2.1), we employ a finite difference scheme defined on a piecewise uniform Shishkin mesh [5]. This mesh is defined as follows. Let \( N \) be a positive integer and a multiple of 4. Divide the interval \( \Omega = [0, 1] \) into three subintervals \([0, \sigma_1], [\sigma_1, 1 - \sigma_2]\) and \([1 - \sigma_2, 1]\), where the transition parameters are given by
\[
\sigma_1 = \begin{cases} \min \left\{ \frac{1}{4}, \frac{4\pi}{\sqrt{\alpha}} \ln N \right\}, & \text{if } \mu^2 \leq \frac{2\pi}{\alpha}, \\ \frac{1}{4}, & \text{if } \mu^2 \geq \frac{2\pi}{\alpha}, \end{cases}
\]
\[
\sigma_2 = \begin{cases} \min \left\{ \frac{1}{4}, \frac{4\pi}{\sqrt{\alpha}} \ln N \right\}, & \text{if } \mu^2 \leq \frac{2\pi}{\alpha}, \\ \frac{1}{4}, & \text{if } \mu^2 \geq \frac{2\pi}{\alpha}. \end{cases}
\]

Place \( N/4 + 1, N/2 + 1 \) and \( N/4 + 1 \) mesh points respectively in \([0, \sigma_1], [\sigma_1, 1 - \sigma_2]\) and \([1 - \sigma_2, 1]\). Denote the step sizes in each subinterval by \( H_1 = 4\sigma_1/N, H_2 = 2(1 - \sigma_1 - \sigma_2)/N \) and \( H_3 = 4\sigma_2/N \), respectively. The mesh points are given by
\[
x_i = \begin{cases} 0 = x_0 < x_1 < \cdots < x_i < x_{i+1} < \cdots < x_N = 1, \\ iH_1, & \text{if } 0 \leq i \leq N/4, \\ \sigma_1 + (i - N/4)H_2, & \text{if } N/4 \leq i \leq 3N/4, \\ 1 - \sigma_2 + (i - 3N/4)H_3, & \text{if } 3N/4 \leq i \leq N. \end{cases}
\]

(4.16)

We set \( h_{j+1} = x_{j+1} - x_j \) for \( j = 0, \ldots, N - 1 \), and \( h_j = (h_j + h_{j+1})/2 \) for \( j = 1, \ldots, N - 1 \). On this mesh, we define a finite difference scheme that uses the central difference, upwind and midpoint schemes. This scheme is given by
\[
L_N^N U_j \equiv r_j^- U_{j-1} + r_j^+ U_{j+1} + r_j^* U_j = Q^N(f_j),
\]
(4.17)

where
\[
L_N \equiv L_N^{cd}, \quad \text{if } 1 \leq j < N/4, \\
L_N \equiv L_N^{cd}, \quad \text{if } N/4 < j < 3N/4, \quad \text{and } \mu H_2 \|a\| < 2\varepsilon, \\
L_N \equiv L_N^{cd}, \quad \text{if } N/4 < j < 3N/4, \quad \text{and } \mu H_2 \|a\| \geq 2\varepsilon, \quad \|b\| H_2 < 2\mu \alpha, \\
L_N \equiv L_N^{cd}, \quad \text{if } N/4 < j < 3N/4, \quad \text{and } \mu H_2 \|a\| \geq 2\varepsilon, \quad \|b\| H_2 \geq 2\mu \alpha, \\
L_N \equiv L_N^{cd}, \quad \text{if } 3N/4 < j \leq N - 1, \quad \text{and } \mu H_3 \|a\| < 2\varepsilon, \\
L_N \equiv L_N^{cd}, \quad \text{if } 3N/4 < j \leq N - 1, \quad \text{and } \mu H_3 \|a\| \geq 2\varepsilon.
\]

At the left transition point \( \sigma_1 \) we use
\[
L_N \equiv L_N^{cd}, \quad \text{if } x_j = \sigma_1 = 0.25, \\
L_N \equiv L_N^{cd}, \quad \text{if } x_j = \sigma_1 < 0.25, \quad \text{and } \|b\| H_2 < 2\mu \alpha, \\
L_N \equiv L_N^{cd}, \quad \text{otherwise.}
\]

At the right transition point \( 1 - \sigma_2 \) we use
\[
L_N \equiv L_N^{cd}, \quad \text{if } x_j = 1 - \sigma_2 = 0.75, \quad \mu H_3 \|a\| < 2\varepsilon, \\
L_N \equiv L_N^{cd}, \quad \text{if } x_j = 1 - \sigma_2 = 0.75, \quad \mu H_3 \|a\| \geq 2\varepsilon, \\
L_N \equiv L_N^{cd}, \quad \text{if } x_j = 1 - \sigma_2 > 0.75, \quad \text{and } \|b\| H_3 < 2\mu \alpha, \\
L_N \equiv L_N^{cd}, \quad \text{otherwise.}
\]
Associated with each of these finite difference operators, we have the following finite difference scheme

\[ L_c^N U_j = \varepsilon \delta^2 U_j + \mu a_j D^0 U_j - b_j U_j = f_j, \]
\[ L_{up}^N U_j = \varepsilon \delta^2 U_j + \mu a_j D^+ U_j - b_j U_j = f_j, \]
\[ L_{mp}^N U_j = \varepsilon \delta^2 U_j + \mu a_j D^- U_j - b_j U_j = f_j, \]

with \( \varepsilon_j = (z_j + z_{j+1})/2, \) and

\[ \delta^2 U_j = \frac{1}{h_j} \left( \frac{U_{j+1} - U_j}{h_{j+1}} - \frac{U_j - U_{j-1}}{h_j} \right), \quad D^+ U_j = \frac{U_{j+1} - U_j}{h_{j+1}}, \quad D^0 U_j = \frac{U_{j+1} - U_{j-1}}{h_j + h_{j+1}}. \]

Thus the discrete problem is:

\[ L^N U = Q^N(f_j), \quad x_i \in \Omega^N, \quad (4.18a) \]
\[ U(0) = u(0), \quad U(1) = u(1). \quad (4.18b) \]

Note that the elements in the system matrix \( L^N \) are as follows

\[ r_j^- = \frac{\varepsilon}{h_j h_j} - \frac{\mu a_j}{2h_j^2}, \quad r_j^+ = \frac{\varepsilon}{h_j h_{j+1}} + \frac{\mu a_j}{2h_j h_{j+1}}, \quad r_j^c = -r_j^+ - r_j^- - b_j, \quad \text{if} \ L^N \equiv L_{cd}^N, \]
\[ r_j^- = \frac{\varepsilon}{h_j h_j}, \quad r_j^+ = \frac{\varepsilon}{h_j h_{j+1}} + \frac{\mu a_j}{h_{j+1}}, \quad r_j^c = -r_j^+ - r_j^- - b_j, \quad \text{if} \ L^N \equiv L_{up}^N, \]
\[ r_j^- = \frac{\varepsilon}{h_j h_{j+1}}, \quad r_j^+ = \frac{\varepsilon}{h_{j+1} h_j} + \frac{\mu a_j}{h_{j+1}} - \frac{b_{j+1}}{2}, \quad r_j^c = -r_j^+ - r_j^- - b_j, \quad \text{if} \ L^N \equiv L_{mp}^N. \]

and

\[ Q^N(f_j) = f_j, \quad \text{if} \ L^N \equiv L_{cd}^N \text{ or } L_{up}^N, \]
\[ Q^N(f_j) = \bar{f}_j, \quad \text{if} \ L^N \equiv L_{mp}^N. \]

In the left layer region \((0, \sigma_1),\) note that

\[ \frac{\mu \|a\| H_1}{2\varepsilon} = \frac{2\mu \|a\| \sigma_1}{\varepsilon N} \leq \frac{8\|a\| \ln N}{\alpha N}. \]

In the right-hand layer region \((1 - \sigma_2, 1),\) note that

\[ \frac{\mu \|a\| H_3}{2\varepsilon} = \frac{2\mu \|a\| \sigma_2}{\varepsilon N} \leq \frac{8\|a\| \ln N}{\alpha N}, \quad \text{for} \ \alpha \mu^2 \leq \gamma \varepsilon. \]
\[ \frac{\|b\| H_3}{2\alpha \mu} = \frac{2\|b\| \sigma_2}{\alpha \mu N} \leq \frac{8\|b\| \ln N}{\alpha \gamma N}, \quad \text{for} \ \alpha \mu^2 \geq \gamma \varepsilon. \]

To guarantee a monotone difference operator \( L^N, \) we impose the following mild assumption on the minimum number of mesh points

\[ N(\ln N)^{-1} > 8 \max \left\{ \frac{\|a\|}{\alpha}, \frac{\|b\|}{\alpha \gamma} \right\}. \quad (4.19) \]

**Lemma 10.** Assume (4.19). The finite difference scheme (4.17) satisfies a discrete minimum principle of the form: For any mesh function \( Z: \)

\[ \text{If } Z(0) \geq 0, Z(1) \geq 0, \text{ and } L^N Z \leq 0, \text{ then } Z(x_i) \geq 0, \forall x_i. \]

**Also, for any mesh function \( Z \)**

\[ \|Z\| \leq \frac{1}{\beta} \|L^N Z\| + \max\{|Z(0)|, |Z(1)|\}. \quad (4.20) \]
Proof. We will simply check that the choice of the different discretizations used in the definition of the difference scheme (4.17) allow us to establish the following inequalities on the entries in the system matrix

\[ r^{-}_j > 0, \quad r^{+}_j > 0, \quad r^{-}_j + r^{+}_j + r^{+}_j < 0. \] (4.21)

From these sign patterns it follows that the matrix \( L^N \) is the negative of an M-matrix. In the case of the upwind operator \( L_{up}^N \), note that the conditions (4.21) are automatically satisfied. In the case of the central difference operator \( L_{cd}^N \), the conditions (4.21) are satisfied if

\[ h_j r^{-}_j = \frac{\varepsilon}{h_j} - \frac{\mu |a|}{2} > 0. \]

For the mesh points outside the left layer region \( x_i > \sigma_1 \), the sign pattern of \( r^{-}_j > 0 \) is guaranteed as \( L_{cd}^N \) is only used when \( \mu H_2 \|a\| < 2\varepsilon \) or \( \mu H_3 \|a\| < 2\varepsilon \). Using the assumption (4.19), one can check that \( r^{-}_j > 0 \) for \( x_i \leq \sigma_1 \). In the case of the mid-point operator \( L_{mp}^N \), the conditions (4.21) are satisfied if

\[ h_{j+1} r^{+}_j > \mu a_j - \frac{b(x_{j+1})h_{j+1}}{2} > 0. \]

The condition \( \|b\| H_2 < 2\mu a \) (when the mid-point operator is used outside the layers) ensures that \( r^{+}_j > 0 \) for \( x_j \in [\sigma_1, 1 - \sigma_2] \). In the right-hand layer region \([1 - \sigma_2, 1]\), using (4.19), we can establish that \( r^{+}_j > 0 \) for \( x_j \in [1 - \sigma_2, 1] \). \( \square \)

We can deduce the following truncation error bounds for two of the three different difference operators employed in \( L^N \): On an arbitrary mesh, we have that

\[ \| (L_{cd}^N - L)u \| \leq \varepsilon h_i \|u^{(3)}\| + \mu h_i \|a\| \|u^{(2)}\|, \]

\[ \| (L_{up}^N - L)u \| \leq \varepsilon h_i \|u^{(3)}\| + \mu h_{i+1} \|a\| \|u^{(2)}\|, \]

and on a uniform mesh with step size \( h \)

\[ \| (L_{cd}^N - L)u \| \leq \varepsilon h^2 \|u^{(4)}\| + \mu h^2 \|a\| \|u^{(3)}\|, \]

\[ \| (L_{up}^N - L)u \| \leq \varepsilon h^2 \|u^{(4)}\| + \mu h \|a\| \|u^{(2)}\|. \]

Note the increase in the order of the truncation error in the case of a uniform mesh. Let us finally look at the mid-point scheme

\[ L_{mp}^N (U - u) = f_j - L_{mp}^N u = \varepsilon \left( u''(x_{j+1}) - u''(x_j) \right) + u''(x_j) - \delta^2 u(x_j) + 0.5 \mu a(x_j) u'(x_j) - a(x_j) u'(x_{j+1}) - a(x_{j+1}) u'(x_j) + a(x_{j+1}) u'(x_{j+1}) - a(x_{j+1}) D^+ u_j. \]

Hence, we have the following bound on a non-uniform mesh

\[ \| (L_{mp}^N - L)u \| \leq \varepsilon h_i \|u^{(3)}\| + C_{(\|a\|, \|a'| \|)} \mu h_{i+1}^2 (\|u^{(3)}\| + \|u^{(2)}\|), \]

and on a uniform mesh there is no increase in the order as

\[ \| (L_{mp}^N - L)u \| \leq \varepsilon h \|u^{(3)}\| + C_{(\|a\|, \|a'| \|)} \mu h^2 (\|u^{(3)}\| + \|u^{(2)}\|), \]

with \( C_{(\|a\|, \|a'| \|)} \) a positive constant that depends on \( \|a\| \) and \( \|a'| \|. Note that the mid-point scheme has a higher order of truncation error in the convective \((\mu a u')\) term than the centered difference operator on a non-uniform mesh. This is why the mid-point scheme is chosen at the transition points.
5 Error analysis

The solution of the discrete problem (4.18) can be decomposed into the following sum
\[ U = V + W_L + W_R, \]  
(5.22a)

where

\[ L^N V = L^N U, \quad V(0) = v(0), \quad V(1) = v(1), \]  
(5.22b)
\[ L^N W_L = 0, \quad W_L(0) = w_L(0), \quad W_L(1) = 0, \]  
(5.22c)
\[ L^N W_R = 0, \quad W_R(0) = w_R(0), \quad W_R(1) = w_R(1). \]  
(5.22d)

Lemma 11. Assume (4.19). The regular component of the error satisfies the following error bound
\[ \|v - V\| \leq CN^{-2}, \]  
(5.23)

where \( v \) is the solution of (3.2b) and \( V \) is the solution of (5.22b).

Proof. If the mesh is uniform \((16 \sigma_1 \sigma_2 = 1)\), then
\[ |L^N(v - V)(x_j)| \leq CN^{-2}, \]
and the result follows from the uniform stability given in (4.20). In the case when the mesh is not uniform \((16 \sigma_1 \sigma_2 < 1)\), away from the transition points the mesh is uniform and in all cases one can check that
\[ |L^N(v - V)(x_j)| \leq CN^{-2}, \quad x_j \neq \sigma_1, 1 - \sigma_2. \]
If the mesh is nonuniform, then at the transition points we either employ the mid-point scheme or if \(2\mu \alpha \leq \|b\| h_{j+1} \) we use the upwind scheme. Either way, we have the following truncation error bound
\[ |L^N(v - V)(x_j)| \leq \begin{cases} CN^{-2}, & \text{if } j \neq N/4, 3N/4, \\ CN^{-1}(\epsilon + N^{-1}), & \text{otherwise.} \end{cases} \]

Define the barrier function
\[ \psi_j = CN^{-2}(\theta(z_j) + 1), \]
where
\[ \theta(z_j) = \begin{cases} 1, & \text{if } 0 \leq z \leq \sigma_1, \\ 1 - \frac{z - \sigma_1}{2(1 - \sigma_1 - \sigma_2)}, & \text{if } \sigma_1 \leq z \leq 1 - \sigma_2, \\ \frac{1 - z}{2\sigma_2}, & \text{if } 1 - \sigma_2 \leq z \leq 1. \end{cases} \]

Noting that \(1/\sigma_2 \geq 4\), we have that
\[ \epsilon \delta^2 \psi_j = \begin{cases} 0, & \text{if } j \neq N/4, 3N/4, \\ O(-N^{-1} \epsilon), & \text{otherwise,} \end{cases} \quad D^0 \psi_j \leq 0, \quad D^+ \psi_j \leq 0. \]

Combining this with (4.20) completes the proof.

Motivated by the pointwise bounds on the continuous singular components \(w_L, w_R\), consider the following barrier functions
\[ \Psi_{L,j} = \begin{cases} \prod_{i=1}^{j} (1 + \theta_L h_i)^{-1}, & 1 \leq j \leq N, \\ 1, & j = 0, \end{cases} \quad \Psi_{R,j} = \begin{cases} \prod_{i=j+1}^{N} (1 + \theta_R h_i)^{-1}, & 0 \leq j < N, \\ 1, & j = N, \end{cases} \]
where
\[
\theta_L = \begin{cases} 
\frac{\sqrt{\gamma \alpha}}{2 \sqrt{\epsilon}}, & \text{if } \mu^2 \leq \frac{\gamma \epsilon}{\alpha}, \\
\frac{\mu a_j}{2 \epsilon}, & \text{if } \mu^2 \geq \frac{\gamma \epsilon}{\alpha}.
\end{cases}
\]
\[
\theta_R = \begin{cases} 
\frac{\sqrt{\gamma \alpha}}{2 \sqrt{\epsilon}}, & \text{if } \mu^2 \leq \frac{\gamma \epsilon}{\alpha}, \\
\frac{1}{2 \mu}, & \text{if } \mu^2 \geq \frac{\gamma \epsilon}{\alpha}.
\end{cases}
\]

First we prove the following technical result.

**Lemma 12.** The barrier functions \( \Psi_{L,j}, \Psi_{R,j} \) satisfy the inequalities
\[
L^N \Psi_{L,j} \leq 0, \quad L^N \Psi_{R,j} \leq 0.
\]

**Proof.** We begin by applying the operator to the left-hand barrier function \( \Psi_{L,j} \)
\[
L^N \Psi_{L,j} = \Psi_{L,j} \left( (1 + \theta_L h_j) r_j^- + r_j^c + \frac{1}{1 + \theta_L h_{j+1}} r_j^+ \right) = \\
= \Psi_{L,j} \left( r_j^- + r_j^c + r_j^+ + \theta_L \left( h_j r_j^- - \frac{h_{j+1} r_j^+}{1 + \theta_L h_{j+1}} \right) \right).
\]

Now we analyze each of the different discretizations used in the definition of the operator \( L^N \). First, in the case of the central difference operator we have
\[
L^N_{cd} \Psi_{L,j} \leq \Psi_{L,j+1} (2 \epsilon \theta_L^2 - \mu a_j \theta_L - b_j).
\]
If \( \mu^2 \leq \gamma \epsilon / \alpha \), let \( \theta_L = \frac{\sqrt{\gamma \alpha}}{2 \sqrt{\epsilon}} \) to prove that
\[
L^N_{cd} \Psi_{L,j} \leq \Psi_{L,j+1} \left( \frac{\gamma \alpha}{2} - b_j - \mu a_j \theta_L \right) \leq \Psi_{L,j+1} \left( \frac{\gamma \alpha}{2} - b_j \right) \leq 0.
\]
If \( \mu^2 \geq \gamma \epsilon / \alpha \), use \( \theta_L = \frac{\mu a_j}{2 \epsilon} \) to get
\[
L^N_{cd} \Psi_{L,j} \leq \Psi_{L,j+1} \left( \frac{\mu^2 a_j}{2 \epsilon} \left( \alpha - a_j \right) - b_j \right) \leq 0.
\]

Now consider the upwind scheme. Then we obtain
\[
L^N_{up} \Psi_{L,j} \leq \Psi_{L,j+1} (2 \epsilon \theta_L^2 - \mu a_j \theta_L - b_j),
\]
and we argue as for \( L^N_{cd} \). Finally for the midpoint scheme we can deduce that
\[
L^N_{mp} \Psi_{L,j} \leq \Psi_{L,j+1} (2 \epsilon \theta_L^2 - \tilde{\mu} a_j \theta_L - b_j/2),
\]
and we finish as for \( L^N_{cd} \).

Applying the operator to the right hand barrier function we obtain
\[
L^N \Psi_{R,j} = \Psi_{R,j} \left( r_j^- + r_j^c + r_j^+ + \theta_R \left( h_{j+1} r_j^- - \frac{h_j r_j^+}{1 + \theta_R h_j} \right) \right).
\]
If we use the central difference or the upwind scheme, we have
\[
L^N \Psi_{R,j} \leq \Psi_{R,j} \left( 2 \epsilon \theta_R^2 + \mu a_j \theta_R - b_j \right).
\]
Now it is not necessary to consider two cases depending on the relation between \( \epsilon \) and \( \mu \). In both cases, we obtain
\[
L^N \Psi_{R,j} \leq \Psi_{R,j} \left( \frac{\gamma \alpha}{2} + \frac{a_j \gamma}{2} - b_j \right) \leq \Psi_{R,j} (\gamma a_j - b_j) \leq \Psi_{R,j} a_j \left( \gamma - \frac{b_j}{a_j} \right) \leq 0,
\]
since $\gamma = \min (b/a)$. If we use the midpoint scheme, then

$$L^N \Psi_{R,j} \leq \Psi_{R,j} \left( 2\varepsilon \theta_R + \mu \tilde{a}_j \theta_R - \tilde{b}_j \right) \leq \Psi_{R,j} \left( \frac{a_j}{2} \left( \gamma - \frac{\tilde{b}_j}{a_j} \right) + \frac{a_J+1}{2} \left( \gamma - \frac{\tilde{b}_J+1}{a_J+1} \right) \right) \leq 0,$$

and this completes the proof. \hfill \Box

An immediate consequence of the previous lemma and the minimum principle is the following result, which implies that the discrete singular components $W_L, W_R$ are computationally small ($\leq CN^{-2}$) outside their respective layer regions.

**Corollary 13.** The barrier functions $\Psi_{L,j}$ and $\Psi_{R,j}$ and the singular components $W_L$ and $W_R$ satisfy

$$|W_{L,j}| \leq |W_{L,0}| \Psi_{L,j}, \quad |W_{R,j}| \leq |W_{R,N}| \Psi_{R,j}.$$  

Moreover, it holds

$$\Psi_{L,j} \leq CN^{-2}, \quad \text{if } N/4 \leq j \leq N,$$

$$\Psi_{R,j} \leq CN^{-2}, \quad \text{if } 0 \leq j \leq 3N/4.$$  \hfill (5.24)

**Proof.** The bounds of (5.24) are obtained in a standard way. For $j \geq N/4$, the component $W_L$ satisfies

$$\Psi_{L,j} \leq \Psi_{L,N/4} = \left( 1 + \theta_L \frac{4\sigma_1}{N} \right)^{-N/4} = (1 + 8N^{-1} \ln N)^{-N/8} \leq CN^{-2},$$

where we have used the inequality $\ln(1+t) > t(1-t)/2$ to deduce $(1 + 8N^{-1} \ln N)^{-N/8} \leq 8N^{-1}$. We use the same argument to bound the barrier function $\Psi_{R,j}$. \hfill \Box

**Lemma 14.** Assume (4.19). The left singular component of the error satisfies the following estimate

$$\|w_L - W_L\| \leq \begin{cases} CN^{-2} \ln^3 N, & \text{if } \gamma \varepsilon \leq \alpha \mu^2, \\ C(N^{-1} \ln N)^2, & \text{if } \gamma \varepsilon \geq \alpha \mu^2, \end{cases} \quad (5.25)$$

where $w_L$ is the solution of (3.2c) and $W_L$ is the solution of (5.22c).

**Proof.** First consider the uniform mesh case of $\sigma_1 = 1/4$. If $\mu^2 \leq \gamma \varepsilon / \alpha$, using $\|w_L^{(k)}\| \leq C \varepsilon^{-k/2}$ and $1/\sqrt{\varepsilon} \leq C \ln N$, we deduce that

$$|L^N (w_L - W_L)(x_j)| \leq C N^{-2} (\varepsilon \|w_L^{(4)}\| + \mu \|w_L^{(3)}\|) \leq C N^{-2} / \varepsilon \leq C (N^{-1} \ln N)^2.$$

If $\mu^2 \geq \gamma \varepsilon / \alpha$, using $\|w_L^{(k)}\| \leq C (\mu / \varepsilon)^k$ and $\mu / \varepsilon \leq C \ln N$, we deduce that

$$|L^N (w_L - W_L)(x_j)| \leq C N^{-2} (\varepsilon \|w_L^{(4)}\| + \mu \|w_L^{(3)}\|) \leq C N^{-2} \mu^4 / \varepsilon^3 \leq C \mu N^{-2} \ln^3 N.$$

To complete the proof in the case of a non-uniform mesh, we split the argument into two cases depending on the localization of the mesh point. In the first case $x_j \in [\sigma_1, 1)$. The triangular inequality, lemma 9 and (5.24) yield

$$|(w_L - W_L)(x_j)| \leq |w_L(x_j)| + |W_L(x_j)| \leq C (e^{-\theta_1 \sigma_1 x_j} + N^{-2}) \leq C (e^{-\theta_1 \sigma_1} + N^{-2}).$$

Using the fact that $\theta_1 \sigma_1 = 4 \ln N$, we conclude that

$$|(w_L - W_L)(x_j)| \leq CN^{-2}, \quad \text{if } x_j \in [\sigma_1, 1).$$

If $x_j \in (0, \sigma_1)$, we calculate a bound on the truncation error of the form

$$|L^N (w_L - W_L)(x_j)| = |L_{cd}^N (w_L - W_L)(x_j)| \leq C (\varepsilon H_1^2 \|w_L^{(4)}\| + \mu H_1^2 \|w_L^{(3)}\|) \leq CN^{-2} (\varepsilon \sigma_1^2 \|w_L^{(4)}\| + \mu \sigma_1^2 \|w_L^{(3)}\|).$$
Note that this is a second order truncation error bound because the scheme is the central difference operator on a uniform mesh. If $\mu^2 \leq \gamma \varepsilon / \alpha$, from $\|w_L^{(k)}\| \leq C \varepsilon^{-k/2}$ and $\sigma_1 = \mathcal{O}(\sqrt{\varepsilon} \ln N)$, it follows that

$$|L_{cd}^{N}(w_L - W_L)(x_j)| \leq C(N^{-1} \ln N)^2 \left(1 + \frac{\mu}{\sqrt{\varepsilon}}\right) \leq C(N^{-1} \ln N)^2.$$ 

If $\mu^2 \geq \gamma \varepsilon / \alpha$, from $\|w_L^{(k)}\| \leq C(\frac{\mu}{\varepsilon})^k$ and $\sigma_1 = \mathcal{O}(\varepsilon / \mu \ln N)$, we conclude that

$$|L_{cd}^{N}(w_L - W_L)(x_j)| \leq C \frac{\mu^2}{\varepsilon} (N^{-1} \ln N)^2.$$ 

We use this bound to obtain an appropriate bound on the error in the layer region $(0, \sigma_1)$. Consider the barrier function

$$\psi_j = C \left(N^{-2} + (N^{-1} \ln N)^2 (\sigma_1 - x_i) \frac{\mu}{\varepsilon}\right).$$

Taking into account that $\delta^2 \psi_j = 0$ and $D^0 \psi_j < 0$, it follows that

$$L_{cd}^{N}\psi_j = - \left(b_j \psi_j + C \frac{\mu^2}{\varepsilon} (N^{-1} \ln N)^2\right) \leq -|L_{cd}^{N}(w_L - W_L)(x_j)|.$$

Therefore the discrete minimum principle on $[0, \sigma_1]$, yields

$$|(w_L - W_L)(x_j)| \leq \psi_j \leq C \left(N^{-2} + (N^{-1} \ln N)^2 \sigma_1 \frac{\mu}{\varepsilon}\right) \leq C N^{-2} \ln^3 N.$$

\[\square\]

**Lemma 15.** Assume (4.19). The right singular component of the error satisfies the following estimate

$$\|w_R - W_R\| \leq C(N^{-1} \ln N)^2,$$  \hspace{1cm} (5.26)

where $w_R$ is the solution of (3.2d) and $W_R$ is the solution of (5.22d).

**Proof.** The proof is similar to the proof given for the left boundary layer component. If $x_j \in (0, 1 - \sigma_2]$, again we have

$$|(w_R - W_R)(x_j)| \leq |w_R(x_j)| + |W_R(x_j)| \leq C(e^{-\theta_2 x_2} + N^{-2}) \leq C N^{-2}.$$

If $x_j \in (1 - \sigma_2, 1)$, the truncation error associated with the central differences satisfies

$$|L_{cd}^{N}(w_R - W_R)(x_j)| \leq C(\varepsilon H_3^2 \|w_R^{(4)}\| + \mu H_3^2 \|w_R^{(3)}\|) \leq C N^{-2} \sigma_2^2 (\varepsilon \|w_R^{(4)}\| + \mu \|w_R^{(3)}\|),$$

and the truncation error associated with the midpoint scheme satisfies

$$|L_{mp}^{N}(w_R - W_R)(x_j)| \leq C(\varepsilon H_3^2 \|w_R^{(3)}\| + \mu H_3^2 (\|w_R^{(3)}\| + \|w_R^{(2)}\|)) \leq C (\mu H_3^2 \|w_R^{(3)}\| + \mu H_3^2 \|w_R^{(2)}\|) \leq C N^{-2} \mu \sigma_2^2 (\|w_R^{(3)}\| + \|w_R^{(2)}\|),$$

since the midpoint scheme is activated when $\mu H_3 \|a\| \geq 2 \varepsilon$.

We begin the proof with the uniform mesh case, where $\sigma_2 = 1/4$. If $\mu^2 \leq \gamma \varepsilon / \alpha$, then using $1/\sqrt{\varepsilon} \leq C \ln N$ and the bounds $\|w_L^{(k)}\| \leq C \varepsilon^{-k/2}$, we have that

$$|L_{cd}^{N}(w_R - W_R)(x_j)| \leq C N^{-2}(\varepsilon \|w_R^{(4)}\| + \mu \|w_R^{(3)}\|) \leq C N^{-2} / \varepsilon \leq C(N^{-1} \ln N)^2,$$

$$|L_{mp}^{N}(w_R - W_R)(x_j)| \leq C N^{-2} \mu (\|w_R^{(3)}\| + \|w_R^{(2)}\|) \leq C N^{-2} \mu / \varepsilon^{3/2} \leq C(N^{-1} \ln N)^2.$$
If \( \mu^2 \geq \gamma \varepsilon / \alpha \), using the facts that \( \| w_R^{(k)} \| \leq C \mu^{-k} \), \( k \leq 3 \), \( \varepsilon w_{R}^{(4)} = (-\mu w_{R}^\prime + bw_{R})'' \) and \( 1/\mu \leq C \ln N \), we deduce that

\[
L_{cd}^N(w_R - W_R)(x_j) \leq C N^{-2} \varepsilon \| w_{R}^{(4)} \| + \mu \| w_{R}^{(3)} \| \leq C N^{-2} / \mu^2 \leq C (N^{-1} \ln N)^2,
\]

\[
L_{mp}^N(w_R - W_R)(x_j) \leq C N^{-2} \mu (\| w_{R}^{(3)} \| + \| w_{R}^{(2)} \|) \leq C N^{-2} / \mu^2 \leq C (N^{-1} \ln N)^2.
\]

Now consider the non-uniform mesh case of \( \sigma_2 < 1/4 \). We begin with the central difference scheme. If \( \mu^2 \leq \gamma \varepsilon / \alpha \), from \( \| u_{R}^{(k)} \| \leq C \varepsilon^{-k/2} \) and \( \sigma_2 = O(\sqrt{\varepsilon \ln N}) \), it follows that

\[
L_{cd}^N(w_R - W_R)(x_j) \leq C (N^{-1} \ln N)^2 \left( 1 + \frac{\mu}{\sqrt{\varepsilon}} \right) \leq C (N^{-1} \ln N)^2.
\]

If \( \mu^2 \geq \gamma \varepsilon / \alpha \), from \( \| u_{R}^{(k)} \| \leq C \mu^{-k} \) and \( \sigma_2 = O(\mu \ln N) \), we have that

\[
L_{cd}^N(w_R - W_R)(x_j) \leq C (N^{-1} \ln N)^2 \left( 1 + \frac{\varepsilon}{\mu^2} \right) \leq C (N^{-1} \ln N)^2.
\]

To finish the proof we consider the midpoint scheme. The midpoint scheme is not used in the case \( \mu^2 \leq \frac{2\varepsilon}{\alpha} \) under the assumption (4.19). Hence, we consider only the case of \( \mu^2 \geq \frac{2\varepsilon}{\alpha} \). From \( \| w_{R}^{(k)} \| \leq C \mu^{-k} \), \( k \leq 3 \), \( \varepsilon w_{R}^{(4)} = (-\mu w_{R}^\prime + bw_{R})'' \) and \( \sigma_2 = O(\mu \ln N) \) we get that

\[
L_{mp}^N(w_R - W_R)(x_j) \leq C N^{-2} \mu^2 \sigma_2^2 (\| w_{R}^{(3)} \| + \| w_{R}^{(2)} \|) \leq C (N^{-1} \ln N)^2.
\]

Thus, in all cases,

\[
L^N(w_R - W_R)(x_j) \leq C (N^{-1} \ln N)^2.
\]

Complete the proof using (4.20).

**Theorem 16.** Assume that \( N \) is sufficiently large so that (4.19) is satisfied. The maximum pointwise error satisfies the following parameter-uniform error bound

\[
\| u - U \| \leq \begin{cases} 
C N^{-2} \ln^3 N, & \text{if } \gamma \varepsilon \leq \alpha \mu^2, \\
C (N^{-1} \ln N)^2, & \text{if } \gamma \varepsilon \geq \alpha \mu^2,
\end{cases}
\]

(5.27)

where \( u \) is the solution of the continuous problem (2.1) and \( U \) is the solution of the discrete problem (4.18).

**Proof.** The result follows from the triangular inequality, Lemmas 11, 14 and 15.

\( \square \)

### 6 Numerical experiments

In this final section, we apply the numerical method (4.18) to the following constant coefficients problem

\[
\varepsilon u''(x) + \mu u'(x) - u(x) = -x, \quad x \in \Omega = (0, 1),
\]

(6.28)

whose exact solution and boundary conditions are given by

\[
u(x) = (x + \mu) + \frac{(1 + \mu)e^{m_2/(2\varepsilon)} + 1 - \mu e^{-m_1/(2\varepsilon)}}{1 - e^{-m_1/\varepsilon}} - \frac{1 + \mu + (1 - \mu)e^{-m_1/(2\varepsilon)}}{1 - e^{-m_1/\varepsilon}} e^{(1-x)m_2/(2\varepsilon)},
\]

where

\[m_{1,2} = \mu^2 \pm \sqrt{\mu^2 + 4\varepsilon}.
\]

Figures 1–2 display solutions of this problem (6.28) for several values of the singular perturbation parameters \( \varepsilon \) and \( \mu \).
For any value of $N$, the maximum pointwise errors $E_{\varepsilon,\mu}^N$, the $\varepsilon$–uniform errors $E_{\mu}^N$ and the $(\varepsilon, \mu)$–uniform errors $E^N$ are calculated using

$$E_{\varepsilon,\mu}^N = \|u - U\|_{\Omega^N}, \quad E_{\mu}^N = \max_{\varepsilon} E_{\varepsilon,\mu}^N, \quad E^N = \max_{\varepsilon,\mu} E_{\varepsilon,\mu}^N$$

where $u$ is the exact solution of (6.28) and $U$ is the numerical solution of the finite difference scheme (4.18).

From these values the local orders of convergence $p_{\varepsilon,\mu}^N$, the local order of $\varepsilon$–uniform convergence $p_{\mu}^N$ and the local order of $(\varepsilon, \mu)$–uniform convergence $p^N$ are calculated using

$$p_{\varepsilon,\mu}^N = \log_2 \frac{E_{\varepsilon,\mu}^N}{E_{\varepsilon,\mu}^{2N}}, \quad p_{\mu}^N = \log_2 \frac{E_{\mu}^N}{E_{\mu}^{2N}}, \quad p^N = \log_2 \frac{E^N}{E^{2N}}.$$  

(6.30)

In Tables 1 we display the $(\varepsilon, \mu)$–uniform errors $E^N$ and the corresponding orders $p^N$ for $N = 2^6, 2^8, \ldots, 2^{15}$ and

$$\max_{\mu \in A} \max_{\varepsilon \in C_{\lambda}} E_{\varepsilon,\mu}^N$$

with $A = \{2^0, 2^{-1}, \ldots, 2^{-20}\}$, $B_{\lambda} = \lambda \times \{2^0, 2^{-1}, \ldots, 2^{-60}\}$ and $C_{\lambda} = \{\varepsilon \in B_{\lambda} | 2^{-20} \mu^2 \leq \varepsilon\}$.

We consider this set of values in order to elude the precision error of the computer (all calculations are in double precision) and to ensure that for each value of $\mu$ the $E_{\varepsilon,\mu}$ stabilize (smaller values of $\varepsilon$.
do not change $E^N_{\epsilon,\mu}$ for all values of $N$. For example, we observed that for $\mu = 1$ it is necessary to consider values of $\epsilon \leq 2^{-17}$ and for $\mu = 2^{-20}$ it is necessary to consider $\epsilon \leq 2^{-60}$ before stabilization with respect to $\epsilon$ is observed. If $\epsilon$ is too small (e.g., $\epsilon = 2^{-60}, \mu = 1$) then rounding errors pollute the calculation of the order of parameter-uniform convergence.

Also, a surface plot of the maximum pointwise errors $E^N_{\epsilon,\mu}$ is displayed in Figure 3 for $\epsilon = 2^{-26}, 2^{-28}, \ldots, 2^{-46}, \mu = 2^{-13}, 2^{-14}, \ldots, 2^{-25}$ and $N = 64$. Observe that as $\epsilon \to 0$ the errors $E^N_{\epsilon,\mu}$ begin increasing, then decrease and finally stabilize at a constant value.

![Surface plot of the maximum pointwise errors](image)

Figure 3: Surface plot of the maximum pointwise errors $E^N_{\epsilon,\mu}$ for $N = 64$ within the range $0 < \mu < 1.2 \times 10^{-4}$ and $0 < \epsilon < 1.4 \times 10^{-9}$

Table 1: The $(\epsilon, \mu)$–uniform errors $E^N$ and the $(\epsilon, \mu)$–uniform orders $p^N$ for various values of $N$ computed over the parameter range $\mu = 2^{-p}, \epsilon = 2^{-q}$ and $2^{-20} \leq \mu \leq 1, 2^{-20} \mu^2 \leq \epsilon \leq 1$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$E^N$</th>
<th>$p^N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>4.577E-2</td>
<td>1.180</td>
</tr>
<tr>
<td>128</td>
<td>2.021E-2</td>
<td>1.270</td>
</tr>
<tr>
<td>256</td>
<td>7.477E-3</td>
<td>1.323</td>
</tr>
<tr>
<td>512</td>
<td>2.719E-3</td>
<td>1.393</td>
</tr>
<tr>
<td>1024</td>
<td>8.728E-4</td>
<td>1.723</td>
</tr>
<tr>
<td>2048</td>
<td>2.643E-4</td>
<td>1.159</td>
</tr>
<tr>
<td>4096</td>
<td>1.184E-4</td>
<td>1.409</td>
</tr>
<tr>
<td>8192</td>
<td>3.380E-5</td>
<td>1.031</td>
</tr>
<tr>
<td>16384</td>
<td>9.499E-6</td>
<td>2.865</td>
</tr>
<tr>
<td>32768</td>
<td>2.607E-6</td>
<td></td>
</tr>
</tbody>
</table>

The results are in line with the theoretical error bound given in Theorem 16, although we observe that the $(\epsilon, \mu)$–uniform orders of convergence are not always increasing. To understand this anomalous behaviour, we display in Table 2 for $\mu = 2^{-10}$ the $\epsilon$–uniform errors for $\epsilon \leq \mu^2$.

In Table 2, the maximum error $E^N_{\epsilon,\mu=2^{-10}}$ lies along two distinct diagonals. The corresponding orders of convergence $p^N_{\epsilon,\mu=2^{-10}}$ for these errors are bigger than two due to the fact that the midpoint scheme is being replaced by the central difference scheme. Below the diagonals the operator is the midpoint (the orders of convergence are decreasing because there are related to the bound $N^{-1}\epsilon$ if $\epsilon/\mu^2 \leq CN^{-1}$) and central difference scheme above the diagonal (the orders of convergence are seen to be increasing).

Nevertheless, we observe a jump in the $\epsilon$–uniform orders of convergence for $N = 2048$. The reason for this jump is that we can only consider a discrete set of values for the parameters $\epsilon$ and $\mu$. In Table 3 we show the results when we compute over a more detailed range of $\epsilon$, $\epsilon = \{0.6, 0.7, 0.8, 0.9, 1\} \times \{2^{-21}, \ldots, 2^{-30}\}$ and $\mu = 2^{-10}$. The jump in the $\epsilon$–uniform orders has been moved to $N = 8192$ and damped for this larger set of parameter values. In Table 4 we display the
As in [5] we have estimated the order of convergence corresponding to the theoretical error bounds \((N^{-1}\ln N)^2\) and \(N^{-2}\ln^3 N\). These values are given in Table 5. On comparing these rates with the computed rates given in Table 4, the computed rates appear to be in close agreement with the theoretical rates established in Theorem 16.
Table 5: Orders of convergence corresponding to different theoretical error bounds.

<table>
<thead>
<tr>
<th>((N - 1 \ln N)^2)</th>
<th>N=64</th>
<th>N=128</th>
<th>N=256</th>
<th>N=512</th>
<th>N=1024</th>
<th>N=2048</th>
<th>N=4096</th>
<th>N=8192</th>
<th>N=16384</th>
</tr>
</thead>
<tbody>
<tr>
<td>N - 2 \ln^3 N</td>
<td>1.277</td>
<td>1.383</td>
<td>1.461</td>
<td>1.522</td>
<td>1.570</td>
<td>1.609</td>
<td>1.642</td>
<td>1.669</td>
<td>1.796</td>
</tr>
<tr>
<td>N = (N - 1 \ln N)^2</td>
<td>1.526</td>
<td>1.593</td>
<td>1.644</td>
<td>1.683</td>
<td>1.715</td>
<td>1.741</td>
<td>1.762</td>
<td>1.780</td>
<td>1.796</td>
</tr>
</tbody>
</table>

The second test problem that we consider is the variable coefficients problem

\[
\begin{cases}
\varepsilon u'' + \mu (1 + x) u' - u = (1 + x)^2, & x \in (0, 1), \\
u(0) = u(1) = 0.
\end{cases}
\]  

To estimate the errors we use the following variant of the double mesh principle. Compute

\[ D_{N}^{\varepsilon, \mu} = \| U_j^{N} - \tilde{U}_{2j}^{2N} \|_{\infty, \Omega^N}, \quad D_{N} = \max_{\varepsilon, \mu} D_{N}^{\varepsilon, \mu}, \]

where \{\tilde{U}_{2j}^{2N}\}_{j=0}^{2N} is the solution obtained on a mesh containing the mesh points of the original Shishkin mesh and its midpoints \(x_{j+1/2} = (x_{j+1} + x_j)/2, \ j = 0, \ldots, N - 1.\) We note that we use the same difference operators to calculate \{U_j^{N}\}_{j=0}^{N} and \{\tilde{U}_{2j}^{2N}\}_{j=0}^{2N}.

To estimate the errors we compare the values of both computed solutions on the coarser mesh \(\Omega^N\) without using polynomial interpolation. From these differences we estimate the \((\varepsilon, \mu)\)-uniform errors using \(D_N\) and the corresponding \((\varepsilon, \mu)\)-uniform orders of convergence \(q_N\) are estimated using

\[ q_N = \log_2 \frac{D_N}{D_{2N}}. \]

In Table 6 we show the estimated \((\varepsilon, \mu)\)-uniform errors when the singular perturbation parameters take the values \(\mu \in A\) and \(\varepsilon \in \bigcup_{\lambda=0.6,0.7,0.8,0.9,1} C_{\lambda}\). We observe the same behaviour as in the previous test problem.

Table 6: The \((\varepsilon, \mu)\)-uniform differences \(D_{N}^{\varepsilon, \mu}\) and the computed orders of \((\varepsilon, \mu)\)-uniform convergence \(q_N\) for problem (6.32).

<table>
<thead>
<tr>
<th>N=64</th>
<th>N=128</th>
<th>N=256</th>
<th>N=512</th>
<th>N=1024</th>
<th>N=2048</th>
<th>N=4096</th>
<th>N=8192</th>
<th>N=16384</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_N^{\varepsilon, \mu})</td>
<td>1.90E-1</td>
<td>4.84E-2</td>
<td>2.35E-2</td>
<td>1.49E-2</td>
<td>1.63E-2</td>
<td>8.46E-4</td>
<td>7.67E-5</td>
<td>2.41E-5</td>
</tr>
<tr>
<td>(q_N)</td>
<td>1.303</td>
<td>1.343</td>
<td>1.405</td>
<td>1.409</td>
<td>1.552</td>
<td>1.667</td>
<td>1.798</td>
<td>1.665</td>
</tr>
</tbody>
</table>

References


