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Mathematics Education:
Crossing Boundaries

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On behalf of the organising committee of the Fifth Conference on Research in Mathematics Education in Ireland, MEI 5, I wish to thank the keynote speakers, presenters, reviewers, session chairs and participants. MEI, a series of biennial conferences held in St. Patrick’s College, Dublin, builds on the knowledge and collegial networks established in previous MEI conferences. It is dedicated to creating a stage for sharing ideas and best practices in the field of maths teaching, and exchanging findings of current research in Mathematics Education.

The diversity of contributors to the conference is a real strength and provides an opportunity for consultation, discussion, and co-operation. Our conference theme – Crossing Boundaries – is reflected in the variety of papers and posters that we have received. We hope that this conference will also allow for a renewal of friendships and the establishment of contacts within the Mathematics Education community.

We also express our sincere gratitude to all those who supported the conference, including Dr Daire Keogh, President of St. Patrick’s College, Mr Ruairí Quinn, TD, Minister for Education and Skills, Dr Sarah Brady and Dr Deirdre Mc Cabe, CASTel, DCU and our sponsors – Irish National Teachers’ Organisation, Carroll Education, and the College Research Committee for their support over the years.

We look forward to your participation in what promises to be another memorable conference.

Ronan Ward (Chairperson, MEI 5 Organising Committee)

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DEVELOPMENT AND RESEARCH IN STUDENTS’ MODELLING PROJECTS FOR THE TEACHING AND LEARNING OF MATHEMATICS AND MODELLING

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We will discuss how we, at upper secondary and university level, 1) use modelling projects to develop students’ modelling competency, including how to foster their internal and external reflections in their modelling activities, 2) use modelling as a didactical activity for enhancing students’ conceptual learning of mathematics, and 3) integrate theories on modelling and on the learning of mathematics so as to be helpful for teachers’ development of their own practice. Our discussions will be illustrated and concretized by three examples: 1) a project in mathematical modelling in biology conducted by a group of first year university science students, 2) a project designed to support and challenge students’ understanding and images of central features of the concept of the integral, and 3) a project designed by upper secondary teachers in our in-service course that exemplify how theories from mathematics education can be used to analyse the potentials for learning mathematical concepts – in this particular case, the linear function and the exponential function.

INTRODUCTION

For the past 12 years we have collaborated in on-going research and developmental work on the role and function of mathematical modelling for the teaching and learning of mathematics at university and upper secondary level. In our research and developmental work we have focused on:

1. how to develop students’ modelling competency including how to foster their external and internal reflections in their modelling activities;

2. how to use modelling as a didactical activity for enhancing students’ conceptual learning of mathematics; and

3. how to integrate theory in mathematics education and teaching practice, i.e., how to establish a research practice that integrates the process of research in mathematics education with the process of development of teaching practice.

The pedagogical vehicle for our research and developmental work has been various forms of students’ problem oriented project work in mathematical modelling. The institutional frameworks have been students’ project work and a modelling course at the two year introductory study programme at the interdisciplinary Bachelor of Science programme at Roskilde University, Denmark, as well as an in-service course for upper secondary teachers in which they experiment with their teaching by developing their own modelling projects and implementing them in their class rooms.
More specifically, our work has been guided by research questions such as:

- How can we challenge and support students to work with the entire modelling process in a way that supports their autonomy?
- How can students’ internal and external reflections in their modelling activities be developed?
- How can we give sufficient support to students’ work with the processes of mathematization and mathematical analysis as integrated elements of teaching mathematical modelling?
- How can mathematical modelling activities support and strengthen students’ learning of mathematical concepts?
- How can modelling activities challenge and support overcoming specific learning difficulties related to important mathematical concepts?
- How can we integrate theories on modelling and on the learning of mathematics in an in-service course for teachers at upper secondary level so as to be helpful for the teachers’ development of their own practice?

In the following we will briefly discuss the question of how to develop students’ modelling competency within the problem oriented project work at the interdisciplinary Bachelor of Science programme at Roskilde University, Denmark, in order to introduce the theoretical construct of the modelling cycle, the definition of modelling competency and the pedagogical practice of problem oriented project work.

In the main part of the paper we will focus on how to establish the interplay between processes of research in mathematics education and processes of development of teaching practice, and, in particular, on how we use theories from research in mathematics education in our own teaching practice, and how we make such theories available for upper secondary teachers in their development and experimentation of their teaching practice. We will contextualize the discussions with respect to our own teaching of the modelling course in the interdisciplinary Bachelor of Science programme and our in-service course for upper secondary teachers. We will focus on how to plan for, notice and encourage progress in students’ conceptual understanding of mathematics involved in modelling activities. We will concretize the interaction between theory and practice with examples from our own teaching practice relating to students’ learning of the integral concept and from our in-service course on students’ learning of linear and exponential functions.

### MODELLING COMPETENCY AND THE MODELLING CYCLE

Mathematical modelling in itself, and as a vehicle for learning and teaching mathematics, is becoming part of mathematics education, and theories about teaching and learning mathematical modelling have developed in mathematics education research. In the Danish mathematical competence project (Niss & Højgaard, 2011), mathematical modelling competency is identified as one of eight main mathematical competencies that, together with three types of secondary competencies, are seen to span mathematical competence. The
The project was initiated by the Danish ministry of education with the purpose of describing mathematics curricula on all levels in the Danish school system based on mathematical competencies instead of on a catalogue of subjects, notions, and results (Niss, 2004). Within the competence framework, to possess mathematical modelling competency can be defined as:

A person’s insightful readiness to autonomously carrying through all aspects of a mathematical modelling process in a certain context and to reflect on the modelling process and the use of the model (Blomhøj & Jensen, 2003, p.127).

The definition refers to a modelling process, as well as students’ autonomy and reflections.

**The modelling cycle**

The modelling process is the process during which a mathematical model is built. One way of conceptualising the modelling process is to describe it as a cyclic process consisting of six sub-processes that in principle are part of any model construction, see figure 1. The processes can be more or less visible or articulated in a specific model construction.

![Figure 1: A visual representation of a mathematical modelling process](Blomhøj, 2004, p. 148)
As formulated in (Blomhøj & Kjeldsen, 2010a), the six processes can be described as follows:

a) “Formulation of a task (more or less explicit) that is related to a perceived reality and influenced by the modeller’s interests. Through this process the object of the modelling process is constructed. The object can be reconstructed as a result of the modelling process. However, it is the object and the formulated task that guides the identification and construction of a domain of inquiry.

b) Selection and construction of the relevant objects, relations etc. from the domain of inquiry, and idealisation of these, in order to make a mathematical representation possible.

c) Transformation and translation of selected objects and relations from their initial mode of appearance to mathematics by further abstraction and idealisation.

d) Using mathematical methods to achieve mathematical results and conclusions.

e) Interpretation of these as results and conclusions regarding the system or the initiating domain of inquiry.

f) Evaluating the validity of the model by comparison with data (observed or predicted) and/or with already established knowledge (theoretically based or shared/personal experience based).”

The modelling cycle is an analytic model of the modelling process. In real-life practice, modelling does not necessarily follow the order in the modelling cycle. The modelling cycle is a tool that can be used to analyse the structure of a modelling process.

DEVELOPING STUDENTS’ MODELLING COMPETENCY THROUGH PROJECT WORK

In order to develop modelling competency, students need to be challenged to work autonomously with the entire modelling process in a more holistic approach where modelling is not atomized into, and students are not only trained within, the separate sub-processes. On the other hand, there is also a need to pay special attention to the learning of the inner parts of the modelling process. In the interdisciplinary Bachelor of Science programme at Roskilde University, Denmark, students get to work with the entire modelling process in problem oriented, student directed project work that runs throughout an entire semester; in addition, special attention to the learning of those parts of the modelling process ((c) and (d)) which are mathematical demanding (Niss, 1989) are taken care of in our modelling course, where modelling activities also serve as a vehicle for the learning of mathematics.

The ability to reflect upon the modelling process and the use of the model is an essential part of mathematical modelling competency. We distinguish between internal and external reflections (Blomhøj & Kjeldsen, 2011). By internal reflections we understand reflections that are connected to the sub-processes of the modelling process in order to become aware of what is going on in the modelling process. The modelling cycle can help students structure their internal reflections. The reflections can be promoted by questions such as: “Why did we (the modeller) formulate the problem like we did?”; “What is the purpose of our modelling process?”; “Which elements did we include in our system – and which did we leave out – and why?”; “Why did we mathematize the system in this particular way?”; “How did we estimate the parameters – and on what grounds?”; “Why do we think the model is valid in relation to
our problem?” These kinds of reflections are related to the documentation of the modelling process. They can lead to critique of the modelling process by considering questions such as “Could we have made other choices?”, “What consequences would a change in the modelling process have?”

While internal reflections are connected to the construction of the model, external reflections are related to the use or possible uses of the model. The objects of these reflections are the role and function of the model in a given societal, technical or scientific context. In applications of models there are interests involved – interests that can be related to issues of power, politics, economics, ethics, the modeller’s underlying conception of theory of science, etc. Such issues are often, whether consciously or not, built into the model in the process of its construction. As it has been pointed out by Ole Skovsmose (1990, p. 129-130), the process of applying a mathematical model tends to cause:

a) a reformulation of the problem so as to be adequate for investigation by means of a model;

b) changes in the discourse about the problem towards pro and contra the model and possible adjustments;

c) a limitation of the possible actions taken into consideration to those that can be evaluated in the model; and

d) a delimitation of the group of people that can take part in the discussion and act as a basis of critique.

Developing students’ mathematical modelling competency is part of the Bachelor of Science programme. Pedagogically, it is done through student directed project work, which ensures that the students get experiences with all phases of the modelling process as a unifying whole in a way that supports their autonomy. The two types of reflections related to modelling are explicitly included in the learning outcomes for the students’ project work.

The students’ project work is organised in the following way. In the beginning of each semester, the students form project groups of three to eight students. Each group decide on a problem they want to investigate and (hopefully) solve during the semester. The students use half of their study time on this problem oriented project work. The other half is used on more traditional course work. The students’ project work is guided by the problem they come to agree on in the beginning of the semester. There is no fixed curriculum. The students choose their problem sometimes all by themselves but often in collaboration with some of the professors who are assigned to supervise the students’ project work. The students’ free choice of problem is constrained by a theme and they need a professor’s approval that project work guided by the problem will be suitable for the students as far as the regulations of the study programme, progression and the level of the subject matter of the chosen problem are concerned. In the first three semesters of the Bachelor of Science programme, the students should work with a problem that is exemplary with respect to (1) the use of science (including mathematics) in society, (2) the function of and relation between models, experiments and theories in the production of scientific knowledge, and (3) science as a cultural phenomenon. The students, who choose to major in mathematics, will have to conduct a mathematical
modelling project either in their fourth or their fifth semester. In principle, the students can work with mathematical modelling projects in all of the three themes (see (Blomhøj & Kjeldsen, 2010a) and (Kjeldsen & Blomhøj, 2012)).

To illustrate how students’ modelling competency is developed in the project work we will discuss a project work that was done under the second semester theme by a group of six students in the spring of 2008. The project is called “The dynamics of the HPA-axis: from biology to mathematics” (Hansen, Hammer, Hermann, Nielsen, Tawfik & Thurah, 2008). The HPA-axis stands for the hypothalamic-pituitary-adrenal axis. In the project, the students wanted to investigate a mathematical model of the dynamics of the HPA-axis that is presented in a paper by Jelic, Cupic and Kolar-Anic (2005). The HPA-axis controls the secretion of cortisol, which is a hormone that is related to stress. There is an interest in gaining a better understanding of the dynamics of the HPA-axis because it is related to illnesses such as depression. The dynamics of this HPA-axis is not fully understood. The corticotropin-releasing hormone CRH is released from the hypothalamus in response to stress. The adrenocorticotropic hormone ACTH is secreted from the pituitary in response to the release of CRH. Due to the release of ACTH, cortisol is secreted from the adrenal cortex. The hormone aldosterone is also secreted from the adrenal cortex. The compartment diagram in figure 2 is from the students’ project report, and it shows their representation of Jelic, Cupic and Kolar-Anic’s theoretical model.

![Figure 2: A compartment model of the HPA-axis](Hansen et al., 2008, p.12)

The students discussed the biological processes of the HPA-axis in detail, and how they correspond to the various parts of the model as it is represented in figure 2. They realized that the model is based on the assumption that the system can be described by nine biochemical reactions, leading to a system of four ordinary differential equations, which, as explained by the students, Jelic et al (2005) reduced to a system of two differential equations. The students evaluated the model and realized that in the model, the concentrations of cortisol and ACTH are of the same order of magnitude. This is in contradiction to experimental results, where they differ by a factor of five. Hence, as discussed by the students, the model could not be validated by experimental results.
This observation led the students to simulate the modelling process in order to explain the deviation between the model and the experimental results. They used the modelling cycle in figure 1 as a tool to structure their analysis of the model. They criticized and evaluated each hypothesis, assumption, and implementation in the model. Due to their analysis, they became aware that the simulation of hormone release is implemented in the model as if the hormones are transformed into one another. This is not in accordance with what really happens, so here the model deviates from what is known about the biological processes. This insight arises from the students’ considerations and discussions that belong to what we have called internal reflections.

The students analysed the function and the status of the model in a scientific investigation as a tool to gain knowledge of the dynamics of the HPA-axis. They realized that in the application process the object changed from understanding the dynamics of the HPA-axis to the problem of modelling the dynamics of the concentration of cortisol and ACTH. They also experienced a change in discourse. The internal reflections and critique of the modelling process led to a change from discussions about the dynamics of the HPA-axis to a discussion about whether it is a good or bad model. As a result of their internal and external reflections the students attempted to find alternative ideas about the HPA-axis. In this process they became aware that such ideas were reduced to changes that can be implemented in the model. These are examples of what we have called external reflections.

The students worked with the problem of how to model the biochemical dynamics of the HPA-axis for one semester, using half of their study time on the project, supervised by a mathematics professor and a biology professor who was also connected to the project group. They read the paper by Jelic, Cupic and Kolar-Anic (2005), and analysed it with respect to how they constructed their mathematical model, what assumptions they made both in relation to the interpretation of the biological processes and to how the individual elements in the model are constructed. The students’ goal was, based on their analysis and simulation of the modelling process underneath Jelic, Cupic and Kolar-Anic’s construction, to build a more, as they wrote in their report, correct model of the HPA-axis.

The elements, issues and critique from the students’ project work presented above give a flavour of how the students came to work with the whole modelling process in a holistic way that supported their autonomy and their ability to reflect upon the modelling process.

THE USE OF MODELLING TO ENHANCE STUDENTS’ CONCEPTUAL LEARNING OF MATHEMATICS

As mentioned in the introduction, there is a balance to strike between working with the entire modelling process and the inner parts of the modelling process which are mathematical demanding and related to the learning of mathematics (Niss, 1989). In this section we describe and analyse how we have used theory from research in mathematics education in an undergraduate modelling course in order to “strike this balance” and to use modelling as a vehicle for learning mathematics.

During the past 12 years we have collaborated in the development, implementation and teaching of a course in mathematical modelling for first year science students in the Bachelor
of Science programme. The course has a dual purpose: on the one hand it focuses on training
students’ ability to mathematize and perform mathematical analyses, and on the other hand
uses modelling as a means for learning mathematics. A central element of the course is the
students’ work with what we call mini-projects (to distinguish them from the semester
projects described above) where they solve a modelling problem. During a semester, the
students complete three mini-projects in groups and write a group report of each project. On
the one hand, these projects support the students’ work with several of the processes in the
modelling cycle and, on the other hand, they are also designed deliberately to support the
students’ learning of mathematical concepts, as will be illustrated below for the case of the
concept of the integral.

In the design of our modelling course and the mini-projects, we have made use of theories on
problem oriented project work (Blomhøj and Kjeldsen, 2009), on mathematical modelling,
e.g. justifications for modelling, modelling competency and the modelling cycle (Niss et al.,
2007), (Blomhøj and Kjeldsen, 2010a), and on the important role of representations for the
learning of mathematical concepts (Steinbring, 1987), the process-object duality in concept
formation (Sfard, 1991), and concept images (Vinner and Dreyfus, 1989). As we have
explained in Blomhøj and Kjeldsen (2013), in using modelling activities as a vehicle for
students’ learning of mathematical concepts, principal learning difficulties caused by the
epistemology and abstract nature of mathematics have to be concretized in particular
modelling activities. Over the years, we have analysed students’ modelling activities to single
out aspects of mathematical concepts which are causing difficulties for the students. These
difficulties seem to be rather stable across cohorts of students (Blomhøj and Kjeldsen, 2010b),
and they can, to a large extent, be related to learning difficulties that are described in the
research literature. Especially, we have found it useful to work with Tall and Vinner’s (1981,
p. 152) concept of students’ concept images, which signifies a student’s total cognitive
structure associated with a particular concept. We also use Sfard’s (1991) model for concept
formation where she emphasizes the duality between object and process aspects of concept
formation. Finally, we explore explicitly the interplay between different representations of a
concept within modelling activities: natural language, types of diagrams, numerical tables,
computer algorithms, symbolic/analytic representations and graphical representations. We use
these theories to analyse and explain empirical findings from our observations of students’
concept formation in modelling activities, and as tools to design and re-design modelling
activities which can serve as a vehicle for students’ learning of mathematical concepts. Our
observations and analyses of students’ difficulties with concept formation, and our design and
re-design of modelling mini-projects addressing these difficulties, form a basis for our
dialogical interactions with the students during their group work with the mini-projects.
Below we will give an example, where we use modelling to support the students’ learning of
the concept of the integral.
Learning of the integral concept in a mathematical modelling course

The mini-projects mentioned above play a key role in our modelling course for first year science students at Roskilde University. They are developed so as to create opportunities for broadening and challenging the students’ understanding of central methods and concepts. These opportunities do not, however, come through automatically. In the teaching and learning situation, we (teachers) need to pay close attention to the students’ discussions in their group work with the mini-projects to notice when the learning potentials of a specific mini-project can be pursued in the collective insights of a group of students, and find ways to challenge students to deepen their engagement in the relevant reflections. During dialogues with project groups of students in the classroom, we have recorded pedagogical observations that pointed out specific difficulties that students faced, when they needed to draw on a not well-developed image of their mathematical concept. In such situations, the students change focus from the modelling task they are working on, towards their understanding of the mathematical concept involved. They experience a so-called cognitive conflict in their evoked concept image (Tall & Vinner, 1981). Below, we will illustrate how modelling activities in a project can create incidences of cognitive conflicts for students, and how such potential conflicts can be revealed through dialogues between the group of students and a teacher, who, in the situation, is attentive to possible cognitive conflicts.

The example, we will present, is from a mini-project that is called “The CO2-balance of a lake”. It is designed to support and challenge the students’ understanding and images of central features of the concept of the definite integral. Before the students begin the work on the mini-project, we have introduced the integral concept in the classroom. We have focused on its interpretation in various modelling contexts, on numerical integration using MatLab, and having the students integrate by ‘counting rectangles’ and adding them up. We have found that almost all of the students enter into our course with a concept image from high school that is limited to the operation of finding an anti-derivative, insert the endpoints of the interval of integration in the anti-derivative, and subtract the two magnitudes. As a consequence, the students will not be able to evaluate the definite integral of a function that is not given by an analytic expression, or a function to which an analytical anti-derivative cannot be determined. Prior to our course, this has not caused problems for the students who seem to have a concept image close to that of Leonhard Euler from the 18th century, where functions were given by analytical expressions. The students are also confused about the arbitrary constant C in the general formula for the anti-derivative. The mini-project “The CO2-balance of a lake” is designed to challenge students’ understanding of the concept of the anti-derivative, the concept of the definite integral, the significance of the constant, as well as interpretations of these concepts in different problem situations.
The mini-project begin with a set of data that shows the rate of change of CO2 in a lake over a 24 hour period (see table 1)

<table>
<thead>
<tr>
<th>Hours after dawn</th>
<th>CO₂ mmol/l/hour</th>
<th>Hours after dawn</th>
<th>CO₂ mmol/l/hour</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.666</td>
<td>-0.027</td>
<td>12.666</td>
<td>0.028</td>
</tr>
<tr>
<td>1.333</td>
<td>-0.048</td>
<td>13.333</td>
<td>0.058</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>12.000</td>
<td>0.000</td>
<td>24.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 1: Data express the rate of change in CO₂ concentration (mmol/l/hour) over a 24-hour period. At dawn, the concentration was 2.600 mmol CO₂.

We set the scene for the students’ work

1. through the set of data;
2. through three groups of questions to guide their modelling; and
3. by means of challenges given to the groups in dialogues with the teacher during their modelling work.

In the first group of questions, the students are asked to describe and interpret the plot of data (see figure 3).

Figure 3: A plot of the data of table 1
That the sign of the rate of change is negative during the day and positive during the night is straightforward for all students. However, they have difficulties interpreting the consequences of this for the CO2 content in the lake. First of all, the students are not sure about the relationship between the CO2 rate of change and the CO2 content, and secondly they cannot interpret the significance of the CO2 rate for the system. They often confuse the graphical picture of the CO2 rate of change with the CO2 content in the lake. To reason about a function from a graphical representation of its derivative is not strongly represented in the students’ conceptual framework. They tend to think of the CO2 rate of change as the derivative of the CO2 content, and not the CO2 content as an anti-derivative to the CO2 rate of change. The students’ concept images of these two different ways of thinking about the relationship between the CO2 content and the CO2 rate of change is not integrated at this point. The mini-project offers opportunities for the students to develop links between these relations.

The second group of questions is designed to force the students to interpret the CO2 content as an anti-derivative of the CO2 rate of change. Typical questions include: When will the CO2 content in the water be at its lowest, and how much CO2 will be in the water when that happens? Is the lake in equilibrium with regard to the CO2 content? How can this question be decided graphically? How much CO2 was released from the water during the 12 daytime hours and how much was transferred to it during the 12 hours of night? Only a few of the students have a concept image that is so well developed that it does not matter to them whether they are interpreting data from a plot of a function that is the derivative of another function, or a plot of function about which they seek information in relation to its anti-derivative. The students are forced to integrate numerically by ‘counting rectangles’, and, in so doing, they discover a valuable method providing answers to questions about the CO2 content in the lake.

The function of the third group of questions is to support the students’ perception of the definite integral as an accumulated sum and to address the problem they have with understanding the role of the constant C. The students are asked to produce a graphical representation of the CO2 content in the lake as a function of time. This task turned out to be a crucial task from a didactical point of view. For most of the students it initiates either a conflict with their concept image or reveals a huge gap in their knowledge. Over the years, we have used pedagogical observations in the sense of dialogues while teaching the course as a tool to get insights into the students’ learning processes and problems with learning various concepts. This tool functions very well in modelling activities because the modelling context encourages students to discuss mathematics, formulate mathematical questions and problems, and test their understanding of the mathematics involved in group discussions with their fellow students. During pedagogical observations in the form of dialogues between the teacher and groups of students working with the task of determining the CO2 content in the lake, we have observed that the students realize that they have to integrate the function represented by their plot of data, i.e. the CO2 rate of change. However, they are not able to complete the task due to vague or missing connections between their concept definition and their concept image of the definite integral, and because many of them are unable to distinguish between the definite integral and anti-derivative functions. These insights have
become clear to us through dialogues with students. Below we present such dialogues which we have reconstructed from our observations:

The teacher (T) has been summoned by the group; the students (S) are stumped:

T: What are you being asked to calculate?
S: The CO$_2$ content in the lake.
T: Correct, how can you determine that?
S: Well, we know the rate of change …
S: That means we know $f'$ …
S: Is it $f$ we have to find? (Everyone looks at the teacher)
T: I don’t know. It is the CO$_2$ content you are asked to find.
S: We know the velocity so it is $f$ we have to find.
S: We must integrate.
T: Yes, what is it you need to integrate?
S: Our function (points at the plot of data).

Then the teacher leaves. After a while the teacher is called back:

T: How is it going? Have you integrated your function?
S: No. We can’t.
S: We don’t have an expression for the function.
T: OK – but aren’t there other ways to integrate?
S: Yes, numerically, and we did that, but we got -0.025 which cannot be true.
T: How did you arrive at this number?
S: We integrated the function by counting rectangles. The ones below the ordinate axe are negative.

Here the students clearly have problems understanding how the integral concept is connected with the problem of determining an anti-derivative:

T: Let’s try to focus on what it is you have to find.
S: We have to find the integral.
T: Yes, but why is it that you want to integrate?
S: … ??? …
T: What does it say in the text?
S: Oh yes, we have to find the CO$_2$ content.
T: Yes, when?
S: … ??? … What do you mean?
T: Is it at midnight … or … ?
S: Yes … no … it is at dawn.
T: Where does it say that?
S: Eh h .. it doesn’t.
T: No, what does it say?

Rereading the text they realise they must find the concentration as function of time:

S: But then we have to calculate many numbers!
T: Yes.
S: But the integral is just one number … right? How can it then become a whole function?
T: How will you estimate the CO$_2$ content at time 0.666?
S: Integrate.
T: Yes, but what?
S: The function – the rate of change.
T: Yes, but where from and where to?
S: Ohhh – from 0 to 0.666.
T: Yes.
The teacher leaves again, but is called back after a while:
S: We still don’t get it. Do we need to count rectangles all over?
T: What do you mean by “all over”?
S: We have done it for the first data point from 0 to 0.666. And we also did it for the second data point from 0.666 to 1.333. Is that then the CO₂ content at time 1.333?
T: That is a good question.

After some discussion the group realises that they must add the first integral from 0 to 0.666 to the second to get the CO₂ content at time 1.333. First at this point, do the students really understand that they can tabulate an anti-derivative (the CO₂ content) by successively adding “the next column” of rectangles. They realise why there is no conflict between the definite integral being one number and the possibility of tabulating an entire anti-derivative function by integration.

S: That is a huge calculation …
T: Yes, but the function to be integrated is tabulated in the data set, so maybe you can use one of your MatLab programs.
S: Oh yes, numerical integration.

At this point, most of the groups of students are able to continue by themselves and produce a graph of the CO₂ content in the lake and use it to answer biological questions about the lake. Some of the groups used Excel to integrate. They were then able to keep track of what was added to what and when. All of these groups realized on their own, that they had to add the initial amount of the CO₂ content in the lake and that this is represented by the constant C. The groups of students who didn’t realize this found negative amount of CO₂ in the water. This gave rise to discussions about the feasibility of such results in relation to the modelling context.

The modelling project demystified the constant C in the formula for the anti-derivative. The modelling context allowed the students to read meaning into the general formula for anti-derivatives. They realized that without the constant they calculated increasing and decreasing amounts of CO₂ and ended up with negative amount of CO₂ in the lake. In the modelling context, this makes no sense, and this triggered the students’ reflections. The students’ understanding of the concept of the integral became connected to the definition of the concept. The students’ reflections are contextualized and hence, one cannot presuppose that their understanding of the integral concept can be transferred to situations outside this particular modelling situation. However, we do have some soft evidence that students afterwards referred back to this particular modelling projects in other situations where they worked with the integral concept.
SUPPORTING TEACHERS’ USE OF THEORY IN DEVELOPING MODELLING PROJECTS IN AN IN-SERVICE COURSE

In our in-service course we introduce the theories from mathematics education mentioned above to the teachers. However, in order to be helpful to them in developing their own teaching practice, the theories need to be concretized and re-contextualized in relation to the particular modelling project developed by the teacher. To facilitate the interplay between theory and practice, that is to bring the theories into the teachers’ development of practice; we use three different forms of intermediate representations to bridge theory and development of practice. The three forms are

1. the detailed description of the modelling process (see above);
2. schemes, spanning all the different presentations of particular mathematical concepts and their interpretations in a given modelling context (see below);
3. construction of anticipated dialogues between the teacher and a group of students facing some particular modelling challenges or learning difficulties.

We present these forms for the teachers, and discuss the potentials of the various theoretical ideas for the development of the teachers’ practice of teaching (with) modelling (Blomhøj & Kjeldsen, forthcoming)

The first example given above, from our modelling course, illustrates how we use dialogues in supporting students’ learning in modelling activities without taking over the students’ tasks. In the example below, we illustrate how we, in our in-service course for high school teachers, are using schemes spanning the different representations of particular mathematical concepts and their interpretations in a given modelling context as a tool for exploring the mathematical learning potentials in a modelling activity and for analysing the students’ activities in a modelling project. We use such schemes in our discussions with the teachers about the potentials of the various theoretical ideas that we have introduced in the course, for the learning of mathematical concepts within their particular modelling project. It is through our interplay with the teachers during the various seminars throughout the course that it becomes possible to bring the theories into play in relation to the teachers’ modelling projects.

Constructing schemes to support teachers’ use of theory in design of modelling projects

The particular project we will use in this illustration was developed in our in-service course by a group of teachers from three different high schools. They designed a modelling project on the decay of alcohol and THC (tetrahydrocannabinol), which is the active drug in hashish. The teachers’ choice was motivated on the one hand by a wish to have the students reflect upon alcohol behaviour and experimentation with drugs, and on the other hand because the decay of alcohol and THC is essential linear and exponential, respectively.
The teachers formulated seven learning goals for the students in the project work (translated from the teachers’ report), that is, to

1. provide the students with a positive experience on using their mathematical skills to answer interesting and relevant questions from their life world;
2. support the students’ conception of modelling and applications of mathematics;
3. teach students to have a critical outlook on mathematical models;
4. support the students’ learning of linear and exponential functions;
5. develop the students’ understanding of the parameters in the two models;
6. train the students to communicate mathematics; and
7. support the students’ IT competences.

These learning outcomes were inspired by the theories we had introduced in the in-service course. They can be divided into three groups: (I) aspects of developing the students’ modelling competency (1-3); II) aspects of developing the students’ concept images and their mathematical understanding of linear and exponential functions (4-5); (III) aspects of developing the students’ IT and communication skills (6-7).

The teachers set the scene for the students’ project work by giving them a set of four exercises followed by an open writing task:

Write an article for students of your own age about the decay of alcohol and THC in the human body. In the article you should also explain the mathematics you have used to complete the exercises. Your answers to the exercises and your graphs should be integrated into your article. (Our translation from the teachers’ report)

In the following, we will focus our discussion on the second set of learning goals, i.e. about developing students’ concept images and their mathematical understanding of linear and exponential functions. In the exercises, the students were given a set of realistic data for the decay of alcohol and THC. The students drew graphs using either Excel or T-Inspire, they described the graphs using their everyday language, determined the time of decrease for half of the amount of alcohol and THC, respectively, twice (so as to experience a fundamental mathematical difference between the two cases), determined the mathematical expression for the functions represented in their graphs, interpreted the significance of the parameters of the functions in the two contexts, and finally, compared the decay of hashish and alcohol.

Our – and the teachers’ – analyses of whether, and if so in what sense, the goals for the students’ learning were fulfilled or not rely on discussions that took place at the in-service course, the teachers’ reports, their design of the project, the tasks given to the students and the articles written by two groups from each of the three classes. As the following quote from one of the teachers’ report show, it is unclear whether the students actually realized the significance of the parameters and the fundamental difference between the exponential and the linear function.
The idea was that the students should realize that the half-life was a constant in the exponential case and not in the linear case. Many students didn’t realize that because they used the graphs [to determine when half of the amount had decayed and when half of that half of the amount had decayed] and they reached two different approximations [for the exponential function].

Surprisingly many of the students had problems de-mathematizing the parameters $a$ [in] $y=a\cdot x+b$ and $a$ [in] $y=b\cdot e^{ax}$, and interpreting the significance of these parameters for the decay of alcohol and THC, respectively. The main problem was the understanding of $a$ indicating the [absolute] amount of decrease of alcohol per hour and $a$ being a number determining the relative decrease of THC per hour as $e^a$ … So next time I will use different names.

Anna Sfard’s (1991) model for concept formation and the fundamental idea that we gain access to mathematical concepts through the meaning of their representations and their relations (Vinner and Dreyfus 1989; Steinbring, 1987) can be combined to form a scheme that can capture the potentials for learning mathematics in modelling activities. In figure 4 we have constructed such a scheme for a linear function and various forms of representations within the modelling context of decay of alcohol. The concrete model of the decay of alcohol and of the general linear model is presented in each cell in each form of representation (both the process aspect and the object aspect of each form of representation are represented).

Figure 4: Process and object aspects of representations of the alcohol model ((Blomhøj & Kjeldsen, forthcoming), modified from Tall (1996))
Every one of these representation can be interpreted within the modelling context. The mathematical properties can be related to the properties of the general linear model, and hereby the modelling context can help students read meaning into the mathematical concepts involved. Such schemes can function as a transition tool to support students to move from a process to an object understanding of a mathematical concept, which, according to Sfard (1991), is the crucial step of reification of a process into an object in students’ formation of mathematical concepts. In the particular modelling project of decay of alcohol and THC, as it was carried out in the classroom, the students were not systematically challenged to work with the different representations. The graphical and algebraic representations were dominant in the students’ work with the modelling project. During the final seminar of our in-service course, suggestions for improvements of the teachers’ modelling projects were discussed, and here the scheme functioned as a tool that helped the teachers to pinpoint and reflect upon the unfulfilled learning potentials in the project. Suggestions were also discussed for how to guide the students (in a future implementation of the project) so they would come to work with all the different representations.

CONCLUDING REMARKS AND FURTHER CHALLENGES

As we hope to have illustrated above, the modelling context provides a window through which we, by deliberately using dialogues, can get some kind of access to students’ understanding and their images of mathematical concepts and their understanding of mathematical modelling. The students use the modelling context to read meaning into the representations of mathematical concepts and to interpret them. This, in turn, allows the teacher to get insights into the thinking of the students when the students in their project groups discuss and interpret various representations of a mathematical concept. In this process, learning difficulties emerge that can be explained by the literature in mathematics education. Especially, we have found Sfard’s (1991) model of concept formation and Tall and Vinner’s (1981) distinction between concept image and concept definition to have high explanatory power in this respect.

In order to make theories from didactics of mathematics available for teachers who want to develop their own teaching practice, we use three kinds of intermediate forms: the modelling cycle, the deliberate use of dialogues, and schemes of representation. Our experiences from teaching an in-service course for upper secondary high school teachers are that it is possible to integrate theories about the teaching and learning of mathematics with teachers’ development of their teaching practice, provided the theories are contextualized and concretized; indeed, we find the modelling context to be an excellent arena for integrating theory and development of practice.

A challenge that we have not really addressed in this paper is the question of how to conceptualize progress in students’ understanding of mathematical concepts in and through modelling activities. Research into this aspect of using mathematical modelling to teach mathematics still needs to be undertaken.
REFERENCES


IDENTIFYING BOUNDARIES IN MATHEMATICS EDUCATION

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Sometimes the practices of learning and teaching of mathematics in classrooms in distant countries can be similar whilst the practices in classrooms in nearby schools (or even within one school) can differ widely. I will present some examples of these similarities and differences. I will explore phenomena presented in these examples by a consideration of people and artefacts. I will then use these considerations to frame an operational definition of mathematics classrooms and use this to reconsider boundaries in mathematics education.

INTRODUCTION

I was asked to talk on ‘crossing boundaries in mathematics education’. This was problematic as I was not sure what boundaries in mathematics education were, so I sought to identify what they might be. This is not a criticism of our conference title since the important things in education are often just the things that are hard to make precise and mathematics educators are faced with difficult questions: why do students in country X do so much better in TIMSS/PISA than students in my country?; why does an approach work with this class but not that class? These questions, I assume, are about boundaries (or differences or transfer).

In seeking to identify what boundaries are I thought of the ‘betweens’ in boundaries in mathematics education – if there are metaphorical boundaries, then there may be metaphorical things on either side of the boundaries and the boundaries might be ‘between’ these things. I thought of boundaries between countries and between schools and between classrooms; my primary focus is on boundaries between schools and between classrooms.

I begin my journey with four ‘between schools and classrooms’ vignettes. I then discuss these with regard to the idea of boundaries as different levels of context. This leads me to a consideration of artefacts, mediation and agency. I then present an operational definition of mathematics classrooms and reconsider the vignettes with a view to identifying boundaries in mathematics education.

FOUR BETWEEN SCHOOLS AND CLASSROOMS VIGNETTES

The following four vignettes impinged on my thoughts as I considered boundaries. They are all quite recent in my memory (from experience and/or from reading).

Vignette 1 (V1)

In 2010 I observed two lessons (each with a different teacher) in a school in the outskirts of Paris as part of a European project working towards integrating digital technology in secondary mathematics lessons. Outwardly the school had some differences to English schools: the language used; the children did not wear a uniform and the teachers wore casual clothes; social relationships between teachers and student, though also founded on mutual respect and a good work ethic, were also casual and the teachers had no obvious pastoral role. But, these surface features aside, I was struck with a sense of déjà vu, these two French classrooms were similar to some English classrooms in a past teacher-researcher project.
(reported in Monaghan, 2004). The positioning of the computers in the class, the actions of the pairs of students around a desktop computer, the approximate ratio of student ‘ICT: mathematics time’ in executing the task, the actions of the teachers circulating around the pairs of students (staying about two minutes with talk and gestures centred on what was on the computer screen), the teachers stopping the class occasionally to make a point to the whole class … were all things I had seen in some English classrooms.

**Vignette 2 (V2)**

Noyes (2012) examines differences in students’ performance and attitudes between schools and between classes within schools. The results reported on in the paper arose from a mixed method study of post-14 mathematics and data collected included that “from 11-year-olds in order to build up a clearer picture of the departmental curricula and pedagogy and the general attitudes of students to mathematics” (ibid., p.276). Noyes adopts what he calls a ‘multi-scale approach’, focusing out on whole-school measures and focusing in on classes within specific schools. Noyes presents measures of mathematics performance in a high stakes examination for 16 project schools; there is wide variation (30% - 75%) in the percentage of A*-C GCSE grades. He then presents student reports (not differentiated by school) of their favourite subjects, mathematics is low (8% of boys and 6% of girls), and of their least favourite subjects, mathematics is the highest (17% of boys and 23% of girls). Noyes develops a tool to report on student views on whether their mathematics lessons are student/teacher-centred (or neither); teacher-centred lessons were the mode. He presents three measures for each of the 16 schools’ ‘mean student-centred score’ and the percentages for mathematics as the favourite and least favourite subject. The mean student-centred scores are similar (ranging from 19 to 23.2) but there is great variation in the favourite measures. Noyes focuses in on one school (with 45% A*-C grades) and its 10 Year 7 (pupils aged 11 years) classes. This school had: the lowest mean student-centred score; the highest (13%) for “mathematics is favourite subject”; and the second highest (26%) for “mathematics is least favourite subject”. Noyes then presents student reports of their favourite and least favourite subjects and there is great variation in the 10 Year 7 classes, from 48% favourite (3% least favourite) to 0% favourite (53% least favourite), and a statistically significant (at the 0.001 level) correlation “between students’ identification of student-centred teaching activity and their likelihood of being positive about mathematics” (ibid., p.282).

**Vignette 3 (V3)**

Gresalfi, Barnes & Cross (2012) focuses on two USA middle school schools/classrooms involved in a project. The two classrooms used an identical set of three inquiry-based tasks designed to address misconceptions about measures of centre. The paper focuses on the role of the teacher in supporting student engagement with regard to the ‘opportunities to learn’ presented by the teacher and the realisation of these opportunities by the students. Four types of engagement are considered: (i) procedural – using procedures accurately; (ii) conceptual – understanding why; (iii) consequential – recognising the use of disciplinary tools and connecting solutions to outcomes; (iv) critical – concerning agency in problem solving, choosing tools and assessing their impact in attaining desired ends. The tasks were set in a context that required students to act as statistical analysts and all tasks afforded all forms of
engagement. The opportunities for engagement presented by the teachers were similar. These were mainly procedural and conceptual and generally resulted in student engagement in the same form. Opportunities (by teachers) to engage consequentially and critically, however, did not always result in student engagement at these levels. Gresalfi et al. (ibid., p.261) suggest:

that prompts that invited particular forms of engagement were insufficient on their own for supporting consequential engagement; in addition, teachers had to give students insight into the nature of the response that they were looking for. This additional information served to set clear expectations for the kind of response the teacher was targeting …

Vignette 4 (V4)

Abdul Hussain, Monaghan & Threlfall (2012) focuses on one Bahraini upper primary school/classroom involved in a one-year intervention study to establish a classroom inquiry community (with interrelated foci on learners, teachers and the school). The paper examines interrelated changes, over the course of the year, in students’ and one teacher’s zones of free movement (ZFM) and zones of promoted actions1 (ZPA); these zones interact and, to emphasise this, Valsiner calls them ‘ZFM/ZPA complexes’. Classroom observations focus on: teacher-student discourse (interactive-non-interactive and dialogic-authoritative; and discourse analysis); teacher interventions (e.g. shaping students’ ideas); social and socio-mathematical norms. At the outset of the study: (i) the dominant teaching style is explanation-drill and the most prevalent form of teacher-student discourse is ‘initiation-response-evaluation’ (I-R-E); (ii) the teacher is the source of the knowledge; and (iii) students work on their own. The paper traces gradual changes in classroom work and attributes these to the joint development of both the classroom teacher and the students’ ZFM/ZPA complexes:

The teacher, over time, perceives the kind of student actions (ZPA) needed for inquiry classrooms. In the transformed ZFM, students need to … take a central role, to articulate their views/reasons and to listen to the views/arguments of their peers. (ibid., p.300)

The paper also argues that the development of the classroom teacher was influenced by his collaboration with a senior teacher and the support of the school management support. In the next section, however, I consider wider influences on the classroom and locate problems with the idea that there are simple boundaries between schools and classrooms.

ARE BOUNDARIES DIFFERENT LEVELS OF CONTEXT?

In a consideration of “context” Cole (1996, p.132) writes: “The notion of context as ‘that which surrounds’ is often represented as a set of concentric circles representing different ‘levels of context’.” He then produces Figure 1 in order to discuss this view of context (including its limitations). I use this figure to explore the boundaries it presents.
I’d first like to say that this figure is not, in my opinion, without merit. For example, the two sets of arrows convey interrelationships; teachers influence what learners do and learners’ actions influence what teachers do, and this accords with one of the main results of V4. But I recalled this diagram when I was thinking of boundaries and it seems that something like boundaries was present when it was first drawn, as Cole (ibid., p.133) writes:

This image is probably best known in connection with Urie Bronfenbrenner’s (1979) book on the ecology of human development, starting with the microsystem at the core and proceeding outward through mesosystems and exosystems.

I interpreted the perimeters of the circles as representing boundaries. But what kind of boundaries might they be? The only clear criteria for a boundary that I can detect is a joint physical and social geographical one: inner circles (with the people-doing-things who inhabit their spaces) are physically contained within the peopled-spaces of the outer circles. This doesn’t seem to say a lot about boundaries with regard to learning. I now try to locate the people and activities in my vignettes in this figure.

In presenting V1 I was interested in the similarities (with regard to teachers’ actions and the arrangement of the computers amongst other things) between some English and two French
ICT-mathematics lessons. These similarities, in terms of Figure 1, are in the Classroom and the Lesson circles, but note that these similarities are there despite differences in the School circle. Note that I said “some English classrooms”; in the project I mention I did record differences between classes. In terms of boundaries I feel these observations suggest that: boundaries between School and Classroom/Lesson may be present nationally but may be much less pronounced (in some cases) across nations. I do not think this is particularly surprising but I do feel it is worth noting in a consideration of boundaries. What I think is important, however, is that my ‘similarity observation’ was based on teachers’ actions and the arrangement of artefacts (computers) in the rooms. The artefacts, including the arrangement of the artefacts, I feel, are important and they do not appear in Figure 1. I return to this later in this paper.

Noyes’ study, V2, concerns the School and Classroom/Lesson circles. Although it does not provide details of what goes on in classroom/lessons it does suggest that in a single school there can be different boundaries between School and Classroom/Lesson. As with V1, I do not think this is particularly surprising but it is worth noting (and it would be very interesting to conduct research on what actually happens in these classrooms).

Gresalfi et al.’s study, V3, like V2, concerns the School and Classroom/Lesson circles, though in V3 the two classrooms are in different schools. In V3 we are provided with details about what goes on in the classrooms and these details focus on the Lesson and Task/Concept circles because the two teachers were using the same three tasks; these details reveal that the teacher (who features in the Classroom and the Lesson circles) is very important in what the learner does with the task (the inner circle). In V2 attention towards the teacher concerns students’ opinions as to whether their mathematics lessons are student or teacher-centred but in V3 attention to the teacher is drawn to the teachers’ prompts and the tools (artefacts):

Ms. Gibson’s prompt is also consequential, as it asks the students to consider whether their choice makes a difference … offering a resource for students to engage consequentially by making explicit for students how they might decide if … In this way, students are not only invited to engage consequentially, but are also offered tools that would be required to actually take up that consequential engagement. (Gresalfi et al., 2011, p.262)

V3 suggests that identifying boundaries in mathematics education should include a consideration of teacher and artefact mediation, both of which I return to later.

V4, the ZFM/ZPA study, has links with Figure 1 in that it examined interrelated development between three levels, “the mathematics classroom level … mathematics teaching level … whole school level” (Hussain et al., 2012, p.285). But it also presents things not represented in Figure 1: it focuses on the ZFM and ZPA, both of which can be construed of as boundaries. Like V3 it focuses on the Classroom-Lesson- Task/Concept circles. However, it is unlike V3 in that the teacher remained constant (though one could argue that he did change in the course of his development) but the kinds of tasks given to the learners changed over time, over the course of a one year intervention; we could say that a Lesson-Task/Concept boundary emerged over time. The changes in this intervention were manifest in the forms of
interactions between people (student-student and student-teacher) and the arrangement of resources (artefacts).

Figure 1 presents some boundaries in mathematics education but I now leave it and move on to consider matters raised by the vignettes that are not represented in Figure 1. The two matters I raise a number of times above are the teacher and artefacts. I shall consider these with regard to mediation. In considering these things I shall bring in two further constructs – affordances and agency.

MEDIATION, ARTEFACTS AND RELATED CONSTRUCTS

What do we mean by “mediation”? My basic response is that it is something that comes between a person (or group of people) and the object/goal of their activity/action. Some people, e.g. Wertsch (1998), speak of “mediational means” and in mathematics education we tend to think of these as tools: rulers, compasses, calculators, etc. Vygotsky was interested in language, signs and mediation:

the basic analogy between sign and tool rests on their mediating function … The tool’s function is to serve as the conductor of human influence on the object of activity; it is externally oriented; it must lead to a change in objects … The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself; the sign is internally oriented. (Vygotsky, 1978, pp.54-55)

Then there is this term “the mediator” and it is usually regarded as a person (or group of people). Maracci & Mariotti (in press), for example, consider that:

the mediator is not the artefact but it is the person who intentionally takes the initiative and the responsibility for the use of the artefact to mediate a specific content. Hence in a teaching-learning context the mediator is the teacher, who introduces the artefact to mediate students’ appropriation of mathematical knowledge.

This is a basic tenet of the theory of semiotic mediation and it is not alien to my view of teaching, it is consistent with “The object of a teaching activity from the point of view of the teacher is a didactical objective, namely the students’ acquisition of a specific knowledge or skill” (Monaghan, in press). I do, however, view it as somewhat anthropocentric; “the mediator is not the artefact” suggests that there are never non-human mediators and I’d like to challenge this. To do this I need to consider artefacts in some depth.

An artefact is a material object, usually something that is made by humans for a specific purpose, e.g. a pencil. An artefact becomes a tool when it is used by an agent, usually a person, to do something. The compass becomes a (mathematical) tool when it is used to draw a circle (which is an intended purpose); the same artefact becomes a different tool when it is used to stab someone. This establishes, for me, an irreducible bond between agent, purpose and tool. After being used as a tool (for whatever purpose), the compass returns to being an artefact.

Many artefacts in mathematics classroom are things we can touch such as rulers and calculators. But the materiality of an artefact is not just that open to touch. An algorithm, e.g. for adding two natural numbers, is an artefact and it is material in as much as it written and
can be programmed into a computer. The I-R-E (or I-R-F; ‘F’ for ‘follow up’ or ‘feedback depending on the author) form of teacher-student classroom interaction is material in as much as an audio recorder can record the spoken words and a video recorder can record gestures accompanying the spoken words.

When an artefact is used (as a tool) it also has another, non-material, form; Cole (1996) calls this the “ideal” form. This is the form of use of the artefact/tool in the mind of the user prior to use. The ideal form has at least two implications for tool use in mathematics education: (i) the distinction between a tool and ways of using the tool; (ii) interaction between material and ideal forms of the tool. To illustrate (i) consider the place value algorithm for adding natural numbers. There are a number of ways of enacting the algorithm, two common forms in the UK are the traditional algorithm and the grid method (which keeps, say, units, tens, etc. quite separate). Behind these mathematically isomorphic forms of the algorithm are intentions, understandings and routines with regard to ways of using the algorithm. In this vein Wells (1993) shows that the I-R-F form of teacher-student interaction serves various classroom functions; although the ‘F’ can function to test students’ understanding, as in What is \( x+x \) – \( 2x \) – Correct, it can also function “as an opportunity to extend the student’s answer, to draw out its significance, or to make connections” (ibid., p.30). With regard to (ii) consider an agent using, say, GeoGebra. To carry out material actions in GeoGebra the agent needs an idea, which may be quite crude, of how to act with GeoGebra, but actions in GeoGebra provide feedback to the user which may change the agent’s idea (ideal form) of how to use GeoGebra. This feedback is a form of mediation and GeoGebra is the mediator.

Further to this, artefacts in mathematics classrooms rarely, if ever, function as tools in isolation from other artefacts. Consider a child in a class using a compass or a calculator. Quite apart from everyday artefacts (the table the child is working on) and school-level artefacts (the length of a lesson), the child is working on a task that probably appears in written form in a worksheet or textbook, and the teacher has a set of artefacts for whole class display (e.g. an interactive whiteboard) and communication artefacts (e.g. the I-R-F sequence).

I now return to mediation from my detour on artefacts. There many forms of mediation, general forms (to many types of activity) include: artefact-mediation; person-mediation; sign (semiotic)-mediation; and language-mediation. To this we can add forms specific to (mathematics) classrooms: the arrangement of desks in the classroom; the task on which learners are engaged. For example, in small group problem solving work it may be useful to put learners’ desks together (which is a problem in countries where the desks are riveted to the floor!) and for learners to be engaged with an open task. Putting learners’ desks together mediates learner-learner discourse on this open task. Wertsch (1991, p.12) considers mediation so important that he states “the relationship between action and mediational means is so fundamental that it is more appropriate, when referring to the agent involved, to speak of ‘individual(s)-acting-with-mediational-means’ than to speak simply of ‘individual(s)’”.

As with artefacts, forms of mediation rarely act in isolation. In the mathematics classroom the following (at least) jointly mediate student learning: the teacher, mathematical signs,
mathematical tools, forms of discourse, the structure of the desks and the task on which students are engaged.

I now introduce the “related constructs” in the section heading: ‘affordances and ‘agency’. The dual constructs of affordances and constraints were developed by E. and J. Gibson over three decades. The following late formulation encapsulates affordances (constraints can be viewed as that which the environment does not offer to the animal):

The affordances of the environment are what it offers the animal, what it provides or furnishes, either for good or ill … It implies the complementarity of the animal and the environment. … If a terrestrial surface is nearly horizontal … nearly flat … and sufficiently extended (relative to the size of the animal) and if its substance is rigid (relative to the weight of the animal), then the surface affords support. (Gibson, 1979, p.127)

Affordances and constraints are widely used in academic design, education and psychology discourse though often in ways that the Gibsons would not recognise (see Norman, 1999); for example, the Gibson’s viewed an affordance of an environment to an animal as being present independent of whether or not the animal perceived this affordance. Affordances and constraints are ever present in mathematics classrooms: putting desks together affords face-to-face discussion; black/white boards afford writing large letters which can be seen from a distance; calculators afford arithmetic calculations. As with artefacts, how these affordances are used in mathematics education is of crucial importance. For example, the affordances provided to a student by a calculator in the task ‘357 x 78’ may be used to obtain a correct answer but in the task ‘37 x 7 = 27846’, the affordances of the calculator may be used for conceptual understanding.

Agency is, historically, concerned with an individual’s free will but clearly other people impinge on our free will; in mathematics classrooms teachers and students’ agencies, what they can and cannot do, interact and both are subject to institutional agencies. In the 1980s actor-network-theory (ANT) (e.g., Latour, 2005) introduced artefacts as agentful actors and soon after this Pickering (1995), using a framework with many similarities to ANT, added “disciplinary agency” to the list of agencies (e.g. the discipline of mathematics ‘forces’ certain actions, such as \(a+a=2a\)). These are, to me, meaningful extensions of agency as something that permits or constricts what we are free to do in a given situation.

Agency is related to affordances and constraints but it is not identical. When teachers or students use an artefact they are free to utilise (or not) the affordances of the artefact they perceive but there are often also features of the artefact that can have unintended (emergent) influence on mathematical and/or pedagogical practice. An example in mathematical practice is that we might use the affordances offered by, say, GeoGebra to connect four equal line segments to form a square, but this square might be destroyed by subsequently dragging an object connected to the square (i.e. we wanted to draw a square but the GeoGebra ‘foiled’ our efforts). A pedagogic example is presented in Monaghan (2004) where a single printer in a computer room afforded the printing of student work in Excel but forced a queue of students at the printer and thereby thwarted (exerted agency) the teacher’s intentions for the outcomes
of the lesson. Examples like these are reasons that I cannot fully go along with Maracci & Mariotti (in press) anthropocentric view of mediation in the mathematics classroom. I now return to the four vignettes in the light of these considerations on mediation.

IDENTIFYING BOUNDARIES RELATED TO LEARNING IN MATHEMATICS CLASSROOMS

I focus on things (which can be viewed in terms of boundaries) in mathematics lessons which enable two classes in, say, different countries to be similar and two classes in the same school to be different, with regard to the learning that goes on in lessons. I present an operational definition of a mathematics classroom lesson and then review vignettes 1, 3 and 4 in the light of this definition.

My operational definition of a mathematics classroom lesson is: “a physical or virtual space in which a teacher (or teachers) organises a mediated network of agentful artefacts towards learners’ acquisition of specific knowledge”. This definition is operational for research on learning in classrooms because it can be used to frame research questions and methodological approaches in explorations of classroom learning activity. This definition focuses attention on: the space, what Lave (1988) calls the ‘arena’; the people; the learning objective; and the mediated network of agentful artefacts. I expect that the fourth component is the one that readers will ponder over, so I explore this in further detail. ‘Network of artefacts’ implies two things: there are a number of artefacts; these artefacts are related (in practice, not necessarily in essence). I suggest that classroom research starts by listing the set of artefacts and noting:

- their interrelations;
- the affordances and constraints of the artefacts (and the space);
- how the people use the artefacts in activity/action;
- who uses them and the object of this use.

Expanding ‘network of artefacts’ to ‘mediated network of agentful artefacts’ has, I feel, been anticipated in my discussion in the previous section in that I regard that agency and mediational roles can extend beyond people, to artefacts. I thus suggest that classroom research includes noting the agency (in practice) and the mediational roles of the actors (people and artefacts).

This definition is now used to reconsider vignettes 1, 3 and 4 with a view to identifying boundaries, and the absence of boundaries, in mathematics classrooms/lessons.

Vignettes 1, 3 and 4 in the Light of this Definition

In V1 the similarities of the space and the arrangement of the desktop computers in this space were similar in the two French classrooms and some English classrooms. This spatial arrangement afforded two students to work together on a single computer and afforded the teacher to ‘circulate the classroom’ and ‘view at a glance’ the computer work of these pairs of students. Another similarity was that the computer work was used to solve a task set out on a teacher-designed worksheet and a flat surface beside the computer afforded student pencil-and-paper work to support or record their computer work. The worksheets (in the similar classrooms) were semi-structured in that they directed, but did not over direct, the student work; it could be said that the worksheets exerted agency but allowed some student agency
(and this ‘distribution of agency’ was designed by the teacher). With students (by design) engaged in actions s/he felt were directed to the learning objectives, the teacher had time to circulate the class and attend to (mediate the actions of) individual pairs of students. On a small number of occasions the teacher, prompted by a number of pairs of students requiring similar forms of assistance, stopped the class and explained an ICT-mathematics aspect of the work (in both French lessons this consisted of introducing a variable and then setting up the task-problem to make use of that variable). This form of teacher mediation was afforded by an interactive white board.

In V1 I focused on similarities. In V3 I first note similarities but focus on differences. Similarities between the two classes include: same age students in nearby schools, both of which are involved in the same project; their classroom work is centred on the same three inquiry-based tasks; the same statistical tools for obtaining measures of centre are available for each class to use. I cannot comment on the affordances and constraints of the artefacts the space or how the people use the artefacts as the paper does not provide these details but it does provide details on “how particular teacher moves might become affordances for particular forms of engagement for students” (Gresalfi et al., 2012, p.250). The paper notes, however, that the opportunities to engage consequentially and critically (both of which, according to Gresalfi et al., concern learner agency in problem solving) afforded by the way each teacher mediated the task to the student was different and resulted in differential student engagement at these levels. This difference in the teacher mediation of the task was related to the mediational means (the tools) suggested (or not) in the teacher prompts. Commenting on “consequential engagement”, Gresalfi et al. present two questions, one posed by each teacher, to his/her class. The first question refers to context (the storyline) behind the task but not to the mathematical tools available. The second question refers to both the context of the task and the mathematical tools available. The second mediational form of questioning affords student consequential engagement: “In this way, students are not only invited to engage consequentially, but are also offered tools that would be required to actually take up that consequential engagement.” (ibid., p.262)

In V4 (boundary) differences developed in a single classroom over the course of one year. Prior to the intervention the classroom space was organised with each student at desk which was facing the teacher and a board the teacher wrote on. The teacher explained the knowledge set out in the textbook and the students’ tasks were textbook exercises. Shortly after the intervention started there were changes in the network of artefacts, desks were rearranged into small groups and teacher-designed tasks replaced textbook exercises. But some aspects of the old system (which were not deemed ‘positive’ for inquiry classrooms) remained, such as I-R-E pattern of teacher-student interaction (a mediational tool). Further to this, although the new tasks afford inquiry learning, this affordance is not realised because the students have limited agency with regard to methods of task-solution: “The teacher determines what counts as an acceptable answer; teacher is not obliged to accept students’ ideas (and mistakes); students are not obliged to express their non-understanding or to negotiate their solutions with each other” (Abdul Hussain et al., 2012, p.291).
As the intervention proceeds new artefacts are introduced (e.g. thread to wind round a circle
to measure the circumference) and new relationships between people and artefacts appear, for
example, students are ‘free’ agents to come to the board to present their solutions. As the
intervention comes to the end of the year, there are many changes in the mediated network of
agentful artefacts: the teacher remains silent for long periods of time during which students
challenge other students’ solution; I-R-E patterns of discourse are no longer dominant and are
replaced by I-R-F-R-F-… chains.

CONCLUSION: CROSSING BOUNDARIES

There are many geographical and metaphorical boundaries in mathematics education. I have
not attended to important ‘high stakes boundaries’ such as curriculum and assessment, nor
‘affective boundaries’ (that almost certainly influence outcomes reported in Noyes, 2012) but
I have focused on some metaphorical boundaries at the classroom level. I have described
these classroom level boundaries in term of artefacts, affordances, agency and forms of
mediation (and interrelations between these). I have presented vignettes in an attempt to show
that these classroom level boundaries can transcend geographical boundaries in terms of
learning, that mathematics classrooms in distant countries can be similar whilst the learning in
classrooms in nearby schools can differ widely.

In terms of crossing boundaries I have no easy answers but I do feel that the identification of
boundaries is an important prelude to any attempt to cross boundaries. I hope my paper/
presentation provides food for thought on the identification of the role of artefacts towards the
aim of crossing boundaries in the learning of mathematics. Although there is much work on
specific artefacts (tests, computers, forms of discourse, etc.), there is relatively little work on
artefacts per se in mathematics education. Artefacts exist both as things and in the modes of
use of these things. They have affordances and constraints and are ‘agentful’. Artefacts are
mediational means and they are, along with humans, ‘actors’ who contribute to the learning
which occurs in classrooms mathematics. Identifying these general aspects of artefacts can
focus researchers’ attention on the role of specific artefacts used in mathematics classrooms
and learning boundaries in mathematics classrooms.

NOTES

1 The ZFM characterises the child-environment relationship, “the child’s freedom of choice of action (and
thinking) is limited by a set of constraints” (Valsiner, 1987, p.97). The ZPA refers to the “set of activities,
objects, or areas in the environment, in respect of which the child’s actions are promoted” (ibid., pp.99-100).

2 Wartofsky (1979) regarded the distinction between an artefact/tool and ways of using the artefact/tool as
sufficiently important that he distinguished between them as artefacts. Primary artefacts are things like hammers
and calculators whilst “Secondary artifacts are therefore representations of such modes of action” (ibid., p.202)
in the use of primary artefacts.

3 V2 is not reviewed as it does not provide the detail required for such a review.
REFERENCES


This presentation shares some research from a study in which young children were introduced to multiplication and division problems during their first or second year of school. The focus was on building children’s conceptual understanding of the ‘repeated groups’ idea as a fundamental aspect of multiplication (and division). The lessons began with simple word problems involving groups of two using familiar contexts such as pairs of shoes and socks. Children were able to work with all four operations, often using addition (and subtraction) as they solved multiplication (and division) problems. The findings are interpreted in relation to the New Zealand Number Framework describing the increasingly sophisticated ways that children use mental strategies to solve problems, including the transition from using counting strategies to utilising number properties. This work is consistent with literature arguing for children’s understanding of number to extend beyond units of one to encompass composite units.

New Zealand, like many Western countries, responded to its poor results on the international comparisons of mathematics achievement by making mathematics a high priority. It introduced a major initiative, the Numeracy Development Projects [NDP], aimed at raising expectations for student progress and achievement in mathematics, and enhancing the professional capability of teachers (Bobis, Clarke, Clarke, Thomas, Wright, Young-Loveridge & Gould 2005; Ministry of Education, 2001). The NDP was implemented on a large scale from 2001 to 2009 as part of mathematics education reform. Recent years have focused on in-depth sustainability of the initiative in targeted schools.

Several key features characterize numeracy initiatives such as the NDP, including the use of research-based frameworks to describe progressions in mathematics learning (Bobis et al, 2005). At the core of the NDP is the Number Framework, consisting of a sequence of stages describing the mental processes students use to solve problems with numbers (Strategy), as well as the key pieces of knowledge that students need to learn in order to be able to use strategies effectively (Knowledge). The Number Framework has been informed by research showing that there are identifiable progressions in how children develop number concepts (Young-Loveridge & Wright, 2002). According to the Ministry of Education (2008a, p. 1), Strategy “creates new knowledge through use”, while Knowledge “provides the foundations for strategies.” Each Strategy stage varies slightly according to three operational domains: Addition and Subtraction, Multiplication and Division, and Proportions and Ratios. Knowledge stages also vary by domain, and include: Basic Facts, Grouping/Place Value, Number Sequence, and Numeral Identification.

The Strategy section of the Framework consists of nine stages (see Figure 1). The first five stages (0 to 4) focus on counting, with each stage involving increasingly sophisticated
counting skills. The Framework begins with the Emergent Stage (Stage 0), at which there is little or no counting, and progresses through (Stage 1) counting or producing a single collection, (Stage 2) counting from one to join or separate two collections of tangible objects, (Stage 3) counting from one mentally to solve addition/subtraction problems, to (Stage 4) counting on or back to solve addition/subtraction, or skip counting in multiples to solve multiplication/division. The four upper stages of the Framework involve the use of increasingly complex part-whole strategies. These strategies are based on using knowledge of number properties to split numbers apart (partitioning) and recombine them in ways that make the problem solution easier. The first of the part-whole stages is Early Additive part-whole thinking (Stage 5) involving a limited number of partitioning and recombinining strategies for solving problems. In the case of addition and subtraction problems, known addition/subtraction facts are used to derive answers to unfamiliar problems. For multiplication and division problems, known multiplication facts (e.g. x2, x5, x10 tables) can be combined with repeated addition or repeated subtraction to solve problems (e.g. to solve 8 x 5, knowing that 5 + 5 = 10 can be used to work out that 10 + 10 + 10 + 10 = 40).

<table>
<thead>
<tr>
<th>Stage</th>
<th>Addition/Subtraction [Multiplication/Division]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td><strong>Emergent - One-to-One Counting</strong>&lt;br&gt;Cannot count or form a group of ten objects - Can count a small collection up to 10, but cannot form groups of objects or use counting to solve addition/subtraction [multiplication/division] problems.</td>
</tr>
<tr>
<td>2-3</td>
<td><strong>Counts from One on Materials - Counts from One by Imaging</strong>&lt;br&gt;Solves addition/subtraction [multiplication/division] problems by counting all objects in the groups - Solves addition/subtraction [multiplication/division] problems by counting all mentally using imaging of the objects.</td>
</tr>
<tr>
<td>4</td>
<td><strong>Counting On [Skip Counting]</strong>&lt;br&gt;Recognises that the last number in a counting sequence stands for all the objects in the collection, so solves addition problems by counting on the second addend, and subtraction problems by counting back [multiplication/division problems by skip counting equal groups], and may keep track of the counts using materials or imaging.</td>
</tr>
<tr>
<td>5</td>
<td><strong>Early Additive Part-Whole Thinking [Multiplication by Repeated Addition or Division by Repeated Subtraction]</strong>&lt;br&gt;Recognises that numbers are abstract units that can be partitioned and recombinined. Solves simple addition/subtraction problems using known facts to derive answers [multiplication problems by combining known multiplication facts and repeated addition, or division problems by repeated subtraction or repeated addition].</td>
</tr>
<tr>
<td>6</td>
<td><strong>Advanced Additive Part-Whole Thinking [Early Multiplicative Part-Whole Thinking]</strong>&lt;br&gt;Solves addition/subtraction problems by choosing from a broad range of different part-whole strategies [multiplication/division by using a combination of known multiplication and division facts and mental strategies to derive answers]. Strategies may include place-value partitioning, doubling &amp; halving, rounding &amp; compensating, or reversibility.</td>
</tr>
<tr>
<td>7</td>
<td><strong>Advanced Multiplicative Part-Whole Thinking [Early Proportional Part-Whole Thinking]</strong>&lt;br&gt;Solves addition/subtraction problems with fractional quantities [multiplication/division problems with whole numbers] by choosing from a broad range of different part-whole strategies. Partitioning may be additive or multiplicative.</td>
</tr>
<tr>
<td>8</td>
<td><strong>[Advanced Proportional Part-Whole Thinking]</strong>&lt;br&gt;[Solves multiplication/division problems with fractional quantities by choosing from a broad range of different part-whole strategies to estimate and calculate answers.]</td>
</tr>
</tbody>
</table>

Figure 1: New Zealand’s number framework for the strategy domain of addition/subtraction [and multiplication/division]
Advanced Additive part-whole thinking (Stage 6) involves choosing from a wide range of strategies to solve addition and subtraction problems with multi-digit whole numbers, or using a combination of known facts and mental strategies to derive answers to multiplication and division problems. Stage 7 (Advanced Multiplicative part-whole thinking) involves choosing from a broad range of strategies to solve addition and subtraction with fractional quantities, or multiplication and division with whole numbers. Stage 8 (Advanced Proportional part-whole thinking) involves choosing from a broad range of strategies to solve proportion and ratio problems.

Another key aspect of the NDP is the individual diagnostic interview, which aligns with the Number Framework (Ministry of Education, 2008b). The tasks are designed to assess the mathematical thinking of students and help teachers make decisions about the learning experiences necessary for students, both individually and in groups. Several equivalent versions of the assessment tasks have subsequently been developed to reduce possible practice effects (Ministry of Education, n.d.). The initial version of the diagnostic interview used the strategy domain of Addition and Subtraction as a means of deciding which difficulty level should be used for the remaining strategy and knowledge domains. The easiest form (Form A), does not include the domains of either Multiplication and Division or Proportion and Ratio. If children show no evidence of counting on to solve a single-digit addition problem, then Form A is used for the remainder of the assessment. As most children in their initial school years are assessed using Form A tasks, an implicit message is being conveyed to teachers working at this level that their classroom mathematics programme should focus on addition and subtraction, not multiplication, division, or fractional quantities.

Although Multiplication (grouping) and division (equal-sharing) are not mentioned in the New Zealand mathematics curriculum (Ministry of Education, 2007) at Level One (Years 1-2, ages 5-7), there are implicitly referred to in objectives where:

[Students] will solve problems and model situations that require them to:

- use a range of counting, grouping, and equal-sharing strategies with whole numbers and fractions;
- know groupings with five, within ten, and with ten;
- communicate and explain counting, grouping, and equal-sharing strategies, using words, numbers, and pictures.

Despite the implicit presence of multiplication and division in New Zealand’s mathematics curriculum at Level One, evidence suggests that few of our teachers use multiplication and division contexts with young children in mathematics lessons.

Level Two (Years 3-4, ages 7-9) objectives also makes no explicit reference to multiplication and division, although students are expected to understand place value, knowing “how many ones, tens, and hundreds are in whole numbers to at least 1000” and “know simple fractions in everyday use” (Ministry of Education, 2007). This is somewhat paradoxical given the fact that place value is inherently multiplicative in its use of groups and powers of ten, and fractions are based on division processes. It is not until Level Three (Years 5-6, ages 9-11)
that there is explicit reference to multiplication and division strategies, facts, and properties. This is similar to other education systems (e.g., Ireland: NCCA, 1999; US: CCSSI, 2010). For example, children in the US are expected to understand place value in Grade 1 (age 6), despite the fact that the foundations for multiplication are not introduced until Grade 2 (age 7). Multiplication only becomes a serious focus at Grade 3 (age 8). Similarly in Ireland, place value is mentioned for First and Second classes (ages 7-8) while multiplication and division are not mentioned until Third and Fourth classes (ages 9-10). In Australia and England, place value is part of the label “Number and place value” for a domain that begins at Year 1 (England: Department for Education, 2013) or prior to Year 1 (Australia: ACARA, 2011) and continues through all years of the primary school system.

New Zealand’s Mathematics Standards has embedded the Number Framework within the expected outcomes for all New Zealand students (Ministry of Education, 2009), beginning at the end of one year at school, when children should use “count all” to join two collections. Evidence from a large and representative sample shows that at the end of Year 1 most children (86%) met the Stage 2-3 expectation (Young-Loveridge, 2010). One year later, children should join two collections by counting on. However, just over half the children (57%) met the Stage 4 expectation. After three years at school, it is expected that children should use so-called “part-whole strategies” (deriving answers from knowledge of basic facts). Just over one-third of children (40%) at the end of Year 3 met the early Stage 5 expectation, and one year later, less than two-thirds of the Year 4 children (62%) met the late Stage 5 expectation. The shortfall was even greater at the end of Years 5 and 6, where less than a quarter (23%) and just over one third (36%) of the students met the Stage 6 expectations, respectively.

Once children are able to use known basic facts to derive answers to unfamiliar problems, a wide range of strategies becomes possible. Instead of working by ones, they are able to split up a number into units of varying sizes and recombine them in order to find solutions to problems (Hunting, 2003). It is not until children have reached Early Additive Part-whole thinking (Stage 5), that multiple solution strategies become a possibility. According to Baroody (2004, p. 200), “The construction of a part-whole concept is an enormously important achievement.” Baroody views a part-whole concept as the foundation for understanding several more advanced concepts of number, including place value, fractions, ratios, multiplication and division. Baroody (2004) argues that missing addend problems (e.g., \(8 + \square = 17\)) provide evidence that children have a part-whole concept and conversely, the inability to solve these kinds of problems is evidence that a part-whole concept is lacking. Baroody, Bajwa and Eiland (2009, p. 70) argue that mastery of basic facts depends on “the development of meaningful and well-connected knowledge about numbers” that grows first from counting, and then from (part-whole) reasoning strategies.

The concept of unit is a fundamental idea underpinning all of mathematics learning (Behr, Harel, Post, & Lesh, 1994; Confrey & Harel, 1994; Lamon, 1994; Langrall, Mooney, Nesbit, & Jones, 2008; Shipley & Shepperson, 1990; Sophian, 2007; Sophian & Kaililiwa, 1998). Behr and colleagues (1994) argue that there is a hidden assumption underpinning primary school mathematics, which is that “all quantities are represented in terms of units of one” (p.
123). Learning to count provides an example of a focus on singleton units. Iteration, the repetition of the unit, is an important concept connected to the concept of unit. According to Langrall, et al, (2008), children develop an understanding of number through their experience of counting processes. As they construct number sequences of increasing length and consistency, this results in the development of “increasingly more abstract unit types,” including iterable units, composite units (units greater than one), and iterable composite units (Langrall et al, 2008).

It is important for students to develop both counting-based and collections-based approaches to working with numbers (Yackel, 2001). Yang and Cobb (1995, p. 10) have highlighted “an inherent contradiction” in the way that Western children are initially encouraged to count by ones and thus construct unitary counting-based number concepts, but are then expected to reorganise these into collections-based concepts involving units of ten and one when place-value instruction begins. Yang and Cobb contrast the Western counting-based view with the collections-based approach of Chinese mothers and teachers, who emphasize groups (units) of ten. The difference in emphasis on counting versus grouping by tens helps to explain Yang and Cobb’s (1995) finding of more advanced mathematical understanding by the Chinese children relative to that of the American children.

A consistent message from the literature is that young children’s mathematics learning could be assisted in the long term by providing them with experiences of units other than one, from the beginning of primary schooling (e.g., Thomas, 1996). Behr, et al. (1994, pp. 123-124) state emphatically:

We assert that giving children situations of whole number arithmetic that involve a variety of unit types and units of units and experience in representing and manipulating quantities that can be represented in these unit types will provide a more adequate foundation for learning and understanding whole number arithmetic and a cognitive bridge to learning and understanding rational number concepts and operations.

Sophian (2007) argues that overlooking the concept of the unit in the early school years makes it harder for children to understand ideas such as place value, multiplication, division, and proportions/ratios, concepts that depend on the use of units other than one. Multiplication and division involve working with higher-order units (composite units). According to Sophian (2007, p. 104):

All numerical representations are representations of the ratio between the quantity they describe and the unit of quantification used to measure it. In this sense, the seeds of multiplicative reasoning are already contained in the first act of enumeration. But the importance of the idea does not become clear until we consider alternative units of quantification, and as soon as we do that, it becomes necessary to deal with multiplicative relations. In instruction, it is natural to want to sidestep this complexity initially and familiarize children with addition and subtraction before introducing multiplication and division. But the danger when we do so is that we obscure the very ideas that children will later need to grasp when we do teach multiplication and division.
A clear implication of this research is that children’s mathematics learning can be supported by introducing multiplication and division in the early years of school. Evidence shows clearly that children prior to school age can work with equal-group multiplication and fair-shares division (e.g., Baroody, Lai, & Mix, 2006; Blote, Lieffering, & Ouwehand, 2006; Matalliotaki, 2012; Nunes & Bryant, 1996; Park & Nunes, 2001; Pepper & Hunting, 1998; Squire & Bryant, 2003). Hence, it makes sense to capitalize on that prior knowledge in the mathematics classroom. We might also expect that experience of working with units greater than one could help children develop part-whole thinking sooner than otherwise.

Several researchers with an interest in the early years have focused on ways that everyday experiences of composite units, such as pairs of eyes, shoes, gloves, and wheels on bicycles (twos), legs on animals and tables (fours), fingers on hands, toes on feet, and petals on flowers (fives) can support learning of multiplication and division (e.g., Pepperell, Hopkins, Gifford & Tallant, 2009; Wright, Stanger, Stafford & Martland, 2006). The introduction of multiplication and division word problems provide the opportunity to work with units other than one, but still leave open the possibility that children can solve these problems using their preferred strategy, whether it be counting all, counting on, repeated addition, or some form of multiplication. Even those using less sophisticated strategies may still learn something important about units greater than one. In multiplication and division, because the groups (parts) are repeated, the idea of the parts within the whole may be more salient than occurs with addition and subtraction.

This paper presents selected aspects of a collaborative study involving two teachers and two researchers who worked together as a team to design assessment and teaching tasks focused on using multiplication and division contexts. The researchers worked alongside each of the teachers during the classroom mathematics lessons. We focus on tasks designed to assess children’s use of strategies to solve addition, subtraction and multiplication problems, and knowledge tasks utilising groups or multiples of ten. A brief description of the initial teaching phase of the study is also included.

**METHOD**

**Participants**

The participants in the study were 38 five- and six-year-olds (20 girls & 18 boys) in two classrooms (average age = 6.2 years) at a decile 5 school [1]. The majority of the children (71%) were in their second year at school, and the remainder were nearing the end of their first year. The children came from a diverse range of ethnic backgrounds, with approximately one third (32%) of European ancestry, one quarter (26%) Māori (the indigenous people of New Zealand), and other ethnicities, that included Asian (16%), African (13%), and Pasifika (11%). One third of the children had been identified as English Language Learners [ELL] (i.e., English was not their first language). Approximately half of the children (n=18) had previously participated in a pilot project designed as an introduction to multiplication and division contexts [2].
**Procedure**

During the initial assessment phase, children were assessed individually using a diagnostic task-based interview designed to explore their number knowledge and problem-solving strategies. The interview took approximately 20 to 30 minutes and included card material and manipulatives. Two soft toys, a dog and a rabbit, were used to facilitate the interview process. Children were encouraged to help the dog and rabbit learn about numbers.

**RESULTS**

Children’s performance on the tasks was examined to look for patterns and progressions. The tasks designed to assess children’s mental strategies were analysed according to the level of sophistication used to solve the problem. For example, strategies were categorised according to whether the child counted all the objects (or images of the objects), or used counting on or back by ones (for addition/subtraction), or skip counting by units larger than one (for multiplication). Recall of basic facts (BF) was distinguished from the use of a derived-fact (DF) strategy or repeated addition (RA). Responses to knowledge tasks designed to assess basic facts, place value, and forwards number-word sequence (FNWS) and involving groups or multiples of ten were also analysed.

**Strategy tasks: Addition and subtraction**

Two tasks assessed students’ use of strategies for addition. The first task involved three plastic beans inside one small opaque bag and four beans of another colour inside a second bag. The researcher tipped the contents of one bag into her hand to show the child, placed the beans back into the bag and laid it on the table, then the process was repeated for the second bag. She said: “There are three beans in this bag, and four beans in this bag. How many beans are there altogether?” Three-quarters (76%) of the children successfully solved the task. The most popular strategy (34%) was to count by ones, either using fingers to represent the two addends, or imagining the beans inside the bags. A quarter (26%) of the children counted on from one of the collections. Six (16%) children recalled the fact or used a derived-fact strategy.

For the second task, the beans representing the two addends (5 + 8) were shown briefly then screened by cards. One third (34%) of the children succeeded on the task, the majority (26%) by using a counting on strategy. Two children counted all the beans. Just one (Nisha) used a derived-fact strategy, adding one to her recalled fact of 8 + 4 = 12.

The subtraction task used 14 beans screened under a card. The child was briefly shown the 14 beans then five were removed. One fifth (21%) of the children succeeded on the task, with all but one (Nisha, who recalled the basic fact) counting back from 14 to 9.

**Strategy tasks: Multiplication**

Three tasks were used to assess students’ understandings of multiplication. The first task used small kete (baskets or kit bags woven from flax). This provided a cultural context familiar to most of the children, particularly those of Māori or Pasifika ancestry (see Figure 2). The second task, Monkeys and Bananas, is from the Junior Assessment of Mathematics [JAM]
(Ministry of Education, 2011; see Figure 3). The third task, Cupcakes, was adapted from JAM, and presented three rows of ten cupcakes (see Figure 4).

**Kete (6 x 2)**

The researcher placed 6 small kete, each containing two shells, on the table in front of the child. The child was told that: “each kete has two shells inside it.” The first kete was upended and the shells tipped out to show the child. The shells were put back and the kete returned to the row. The researcher then asked: “How many shells are there altogether?” making a circular gesture with her hand above the row of kete.

Overall, two thirds of the children (68%) solved the kete task. Only two children responded to the question by appearing to use basic facts. Nisha responded: “two times six equals twelve.” Hao also responded quickly with “twelve”. When asked to explain how he worked out his answer, he responded: “two plus two equals four and four plus four equals twelve.” It was not clear whether Hao (an English language learner) was having trouble explaining his strategy, or whether his correct answer was based on faulty reasoning. Approximately one third of the children (34%) used skip counting by twos to work out their answer. Another 19 per cent counted by ones, pointing to or touching each kete twice as they said the sequence of numbers from one to twelve (saying “one, two” and pointing to one kete, then “three, four” to the next, etc). The kete task was particularly interesting because the children could not see the shells inside the kete and had to construct an image of them in order to count them.

**Monkeys and bananas (4 x 5)**

The children were told that the four monkeys each had five bananas. They were then asked: “How many bananas are there altogether?” Half of the children correctly solved the problem. Three children used known or derived facts. Nisha’s response reflected her knowledge of basic facts: “four times five equals twenty.” Hao stated that “five and five equals ten, two times five equals ten, ten plus ten equals twenty.” Kiri used her knowledge of doubles facts for five and ten to get her answer of twenty, saying: “five plus five equals ten, and another ten makes twenty”. Three children used skip counting in fives to 20. Earl stated that “five plus five equals ten”, and then said “ten plus ten…,” but could not finish this sentence. Because he did not know $10 + 10 = 20$, he resorted to counting on by ones from 11 to 20 to work out his answer. Twelve children (32%) counted all the bananas to work out their answer.

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**Figure 2: A photograph of the kete used for the 6 x 2 task (Note: the two shells inside each kete are shown above the first kete)**

**Monkeys and bananas (4 x 5)**

The children were told that the four monkeys each had five bananas. They were then asked: “How many bananas are there altogether?” Half of the children correctly solved the problem. Three children used known or derived facts. Nisha’s response reflected her knowledge of basic facts: “four times five equals twenty.” Hao stated that “five and five equals ten, two times five equals ten, ten plus ten equals twenty.” Kiri used her knowledge of doubles facts for five and ten to get her answer of twenty, saying: “five plus five equals ten, and another ten makes twenty”. Three children used skip counting in fives to 20. Earl stated that “five plus five equals ten”, and then said “ten plus ten…,” but could not finish this sentence. Because he did not know $10 + 10 = 20$, he resorted to counting on by ones from 11 to 20 to work out his answer. Twelve children (32%) counted all the bananas to work out their answer.
Cupcakes (3 x 10)

One third of the children correctly solved the problem. Only two children (Cain & Nisha) used knowledge of basic facts to work out that there were 30 cupcakes altogether. Cain gave his answer of 30 very fast, explaining that he knew that three tens were 30. Nisha may also have known the answer to the cupcakes problem as a multiplication fact, but her explanation was based on repeated addition: “ten plus ten plus ten equals thirty.” Seven children used skip counting in tens to solve the problem. Earl once again resorted to counting on by ones from 11 to 30 because he did not know how to add up the groups of ten. Two children used counting all successfully to solve the problem.

Knowledge tasks: Basic facts involving groups/multiples of ten

Children’s performance on the tasks was examined to look for patterns and progressions. Some tasks involved adding tens or making a sum of ten, while others involved multiplying groups of ten.

Adding groups of ten

Eight of the children (21%) knew immediately that adding 10 dots to 5 dots resulted in a total of 15. When another 10 dots were added, six of those children (16%) knew instantly that the result was 25. Six children knew that adding another 10 dots would total 35, including one child (Len) who had solved the first two additions of 10 by counting on (Ata could not...
continue the pattern of adding ten). There were five children who completed all three of the incrementing by tens tasks correctly. Seven children knew immediately that $20 + 7 = 27$ and $10 + 8 = 18$, showing a reasonably good understanding of place value for their age.

**Adding to make ten: Missing addend task**

Responses to the basic facts task presented in the form of a missing addend problem was examined on the grounds that it could indicate which children were using part-whole thinking to solve problems. Only Nisha and Cain were able to recall the answer to $7 + 3 = 10$ presented as a missing addend problem: $7 + \Box = 10$.

**Multiplying groups of ten**

Although more than half (55%) of the children knew that $10 + 10 = 20$, only four (Nisha, Hao, Haki, Hemi) knew that $2 \times 10 = 20$. Only Nisha and Haki recalled the answer to $3 \times 10$ as 30, despite the fact that in the cupcakes task, Cain appeared to know this fact. Only Nisha and Haki knew $4 \times 10 = 40$. Four children (Nisha, Haki, Len, Ata) knew how many $10$ notes would be needed for an $80$ purchase.

**Forward number-word sequences**

All of the children in this group, apart from Sam, could skip count by tens to 100 (Sam got to 90). Nisha was the only child in this group who skip counted by twos to 100 (the 13 selected children could all skip count to at least 12). Skip counting by fives to 100 was shown by six of the children (Nisha, Hao, Cain, Haki, Hemi, Gay). Overall, more children could skip count by tens to 100 (39%) than by fives (24%) to 100.

**Crossing the boundary from counting to part-whole thinking**

Children who had used either recall of basic facts or derived facts for one of the strategy tasks, as well as those who were successful on knowledge tasks involving groups or multiples of ten were selected for further analysis (see Table 1). The two most capable children were Nisha who used a Stage 5 (Early Additive part-whole thinking) strategy on all six of the strategy tasks, and Hao who did this for three tasks. Another six children did this for one of the tasks (Cain, Gay, Wim, Kiri, Sam, Reta). Five other children showed that they knew basic facts for adding and multiplying by ten (Haki, Len, Hemi, Ata, Hiri). The children’s rote counting by ones and skip counting by twos, fives, and tens was also examined. Virtually all the children could rote count by ones to at least one hundred (only Hemi got confused in the upper decades). All but Sam and Reta skip counted by tens to 100. Most, if not all, of the 13 children included in Table 1 were on well the way to crossing the boundary from counting to part-whole strategies.

Those children who still had considerable progress to make before reaching the boundary between counting and part-whole thinking were those whose knowledge of number was just emerging. All of these children could construct a collection of five objects, and all but one could construct a group of nine. All but four knew that one more than nine is ten. All could subitise a dot pattern for three and most (95%) could subitise dot patterns for four and five. All children could rote count by ones to at least 12. Only two children did not know some of the basic facts presented. The easiest basic facts included $1 + 1$ (95%), $5 + 5$ (89%), $2 + 2$
(84%), 2 + 1 (68%), 3+3 (66%), and 10 + 10 (55%). It was interesting to note that considerably more children knew 5 + 5 than knew 1 + 4 (50%) or 2 + 3 (26%), despite the fact that the total was 10 rather than 5 or smaller (as specified in the progressions outlined in the Framework document: Ministry of Education, 2008a). This finding suggests that teachers need to be aware that certain key number facts that may be learned earlier than expected, despite involving sums greater than five. There is a strong case to be made for children being encouraged to learn the easiest facts, regardless of the number size.

**Teaching using multiplication and division contexts**

The children were introduced to groups of two, using familiar contexts such as pairs of socks, shoes, gumboots, jandals, and mittens (see Figure 5). Multiplication was introduced using simple word problems, such as:

Kiri, Sam, and Len each get 2 socks from the bag. How many socks do the 3 children have altogether?

Quotitive (measurement) division was introduced using a problem such as:

We have 6 gumboots. How many pairs do we have?

Children’s solution strategies were recorded in a class modelling-book (see Figure 6). Teachers endeavoured to select children in order from the simplest strategy such as counting by ones (1, 2, 3, 4, 5, 6), through skip counting (2, 4, 6), to the most sophisticated, such as repeated addition (2 + 2 + 2 = 6). Laminated cards showing pictures of the objects were displayed on the whiteboard, and drawings of objects or iconic representations of objects in the modelling book (e.g. tally marks) were also encouraged. Teachers made a point of introducing the children to the language of multiplication and wrote into the modelling book expressions such as “3 groups of 2 equal 6”, and “3 x 2 = 6”. We talked to the children about the idea of “working like a mathematician” by recording problem-solving strategies using numerals and operator signs.

![Figure 5: Objects in pairs, including baby socks, penguin feet, and mittens (used for multiplication), and a single jandal and gumboot (used for division into groups of 2)](image)

After the children had discussed the class problem together, they went to their tables to solve a parallel problem that had been pasted into their individual project books. This problem provided the same language and context but gave the children a choice of three different numbers for the multiplier. For example:

The 4 [6] [8] children each get 2 socks from the bag. How many socks do the children have altogether?
The children were able to use materials to support their learning, if required, and were encouraged to show their thinking by drawing pictures, then recording the process using number sequences and equations. Many of the children were able to work independently on solving their problem. Most children made an appropriate choice of multiplier, with the more able children progressing from 8 groups to 12 groups of two after three lessons. Some children did not want to draw pictures, preferring instead to use number sequences and/or addition equations to record their solution strategy.

A few children wanted to partition the multiplier additively to show its composition. Nisha insisted on transforming the problem into a “doubles” problem, using the commutative property. She drew two rows of objects (equal in size to the multiplier), and made lines connecting an object in the top row with an object in the bottom row. After several days of gentle encouragement to draw “groups of two” rather than “two groups” she wrote an explanation of her strategy to convince us that she understood how the multiplication process works. Her explanation showed that she was thinking about one group consisting of a member of each pair, and the second group composed of the other member of each pair. For example, on a problem involving 10 penguins each with a pair of feet, she wrote:

\[
\text{Count one foot from each penguin} = 10. \text{ Now count the rest of the penguins [feet]} = 10. \\
10 + 10 = 20 \\
2, 4, 6, 8, 10 \\
2, 4, 6, 8, 10 \\
10 + 10 = 20
\]

For some children, the drawing of the pictures became all-consuming, and the focus was on details such as patterns on the socks, rather than on the multiplication process itself. Over four weeks of lessons, with encouragement to make the drawings simple, the focus shifted away from drawing towards the mathematics process itself.
Table 1: Performance of children who used part-whole strategies, or were successful on knowledge tasks involving groups/multiples of ten

<table>
<thead>
<tr>
<th>Child</th>
<th>Nisha</th>
<th>Hao</th>
<th>Cain</th>
<th>Haki</th>
<th>Len</th>
<th>Hemi</th>
<th>Ata</th>
<th>Hiri</th>
<th>Gay</th>
<th>Wim</th>
<th>Kiri</th>
<th>Sam</th>
<th>Reta</th>
</tr>
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<tbody>
<tr>
<td>Age in Yrs</td>
<td>6.4</td>
<td>6.6</td>
<td>6.6</td>
<td>6.2</td>
<td>6.7</td>
<td>6.5</td>
<td>6.2</td>
<td>6.2</td>
<td>6.9</td>
<td>5.8</td>
<td>6.3</td>
<td>6.6</td>
<td></td>
</tr>
<tr>
<td>Gender</td>
<td>F</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>M</td>
<td>F</td>
<td>M</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>English Language Learner</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>

**Strategy Tasks**
- 3 + 4: BF, DF
- 5 + 8: DF
- 14 - 9: BF
- 6 x 2: BF, DF
- 4 x 5: BF, DF
- 3 x 10: RA, BF

**Knowledge Tasks**

**Basic Facts**
- 5+10: √, √, √, √, √, √, √
- 15+10: √, √, √, √, √
- 25+10: √, √, √, √, √
- 20+7 & 10+8: √, √, √, √, √
- 7+□=10: √
- 2x10: √, √, √
- 3x10: √, √
- 4x10: √, √, √
- $10$s in $80$: √, √, √, √

**Counting to 100**
- by ones: √, √, √, √, √, √, √, √, √, √
- by twos: √
- by fives: √, √, √, √, √, √
- by tens: √, √, √, √, √, √, √, √, √

**DISCUSSION**

The assessment tasks provided the children with familiar contexts in which to think about addition/subtraction and multiplication. The findings from the analysis of responses to strategy tasks and knowledge tasks involving groups/multiples of ten (Table 1) are consistent with research showing the hierarchical progression of strategies from counting all (Stages 2-3), through counting on/skip counting (Stage 4) to early additive part-whole thinking (Stage 5) (Young-Loveridge & Wright, 2002).

The findings of the study are consistent with the idea that Knowledge provides the foundation for Strategy, a major principle underpinning the New Zealand Number Framework (Ministry of Education, 2008a). The children in the selected group of 13, who could use either recall of
basic facts or derived facts to solve strategy tasks, or knew how to add or multiply by ten without counting, tended to have the strongest number knowledge, including skip counting by tens to 100. These findings are consistent with an analysis of the Number Framework showing that the majority of students who were Advanced Additive or Multiplicative part-whole Thinkers (Stages 6 or 7) were at Stage 5 or above on Knowledge domains, and conversely, that the majority of children who could not find tens in numbers to 100 (a Knowledge task), were at Stage 4 or below on all three Strategy domains (Young-Loveridge & Wright, 2002). The data from our study support the idea that having some minimal level of knowledge about numbers is a prerequisite for the development of part-whole strategies (Ministry of Education, 2008a; Young-Loveridge & Wright, 2002).

Most of the literature on place value is based on an assumption that children will learn about the decade-based structure of the number system through working with addition and subtraction (Baroody, 1990; CCSSI, 2010; Fuson, 1990; Fuson & Briars, 1990; Fuson, Smith, & Cicero, 1997; Hiebert & Wearne, 1992). The emphasis is on the positional property, the base-ten property, and the additive property of numbers (see Ross, 1989), but the multiplicative property seems to have been overlooked. It is possible that the introduction of multiplication and division prior to place-value instruction could be beneficial to students, not just in understanding multiplication and division, but also to consolidate place-value understanding. Yang and Cobb’s (1995) point about the sudden switch from a counting-based to a collections-based approach to number that is typical for most place-value instruction, suggests that providing children with experience of different units (e.g. twos, fives, tens) could be immensely helpful in supporting place-value understanding, as well as assisting with the transition across the boundary from counting strategies to part-whole thinking and reasoning.

We found that introducing groups of two was a good way to start talking about multiplication and division because of the familiarity children had with the idea of “groups of two” shoes, socks, gumboots etcetera, although the meaning of the word “pair” to refer to a group of two was difficult for some children to learn. This contrasts with Sullivan and McDuffie’s (2009) use of collective nouns to describe groups in multiplication and to connect their mathematical problems with real-life contexts. However, this may have been because our children were approximately two years younger than their Grade 3 students. Our teachers were enthusiastic about shifting the lessons from a focus on groups of two to groups of five. Some students were quick to see the connection between \(5 + 5 = 10\) (a basic fact known by 89%) and a possible solution strategy to problems about groups of fives that involved groups of ten (pairs of fives). We look forward to the next phase of the study when we plan to deepen children’s understanding of fives, extend that to groups of ten, and make links to place value.

The use of a whole-class teaching approach meant that all children initially solved the same problem. This gave an opportunity for the teacher to start the sharing session by choosing a child likely to offer a simple strategy such as counting all. This was then recorded in the modelling-book. The teacher then selected a child who had used skip counting, followed by someone who used repeated addition. The final most sophisticated strategy was the multiplication fact. Each of these strategies was recorded in the book. The modelling-book
was an effective tool for supporting the children’s learning and could later be used as a reference point in teacher-led discussion (Higgins, 2006). The whole-class approach meant there was heterogeneity in mathematics ability, and is consistent with research showing the advantages of maintaining mixed groups rather than separating children into ability groups (e.g. Burris, Heubert, & Levin, 2006). Another feature was using the children’s names in the problems and keeping the language simple, given the number of ELL children (Hart, 1996).

The other recording tool was the children’s individual project-books. These provided children with the opportunity to solve a problem in different ways and included pictures, number sequences, and equations. Encouraging children to act “like a mathematician” prompted the use of a variety of equations to record different ways of thinking about the problem. The children were given the choice of three multipliers as a way to differentiate the content challenge of the task (Bray, 2009). A few children made poor choices, and were influenced by others sitting near them. With encouragement, the more able children worked alone and used larger multipliers. Those who were having difficulties were gently persuaded to select a smaller multiplier. This is a strategy that Bray (2009, p. 180) suggests to help children become “better decision makers and stewards of their own learning”. However, many children responded positively to the opportunity to work with larger numbers when given a choice.

Although this was a small project, it provides interesting insights into the challenges young children face when solving problems using multiplication and division contexts. This also challenged the teachers who had previously focused only on the operations of addition and subtraction. The writing of appropriate word problems proved to be more challenging than initially anticipated. We noticed that children had difficulties in making connections between their sequence knowledge (e.g. of twos) and the operation of adding two each time. It is important not to underestimate the need for many opportunities to be reminded that the purpose of skip counting is the iterative adding of a constant group in order to determine the total. It is not easy for children to recognise the relationship between repeated addition and multiplication. If teachers of young children introduce multiplication earlier, this could help children develop a better understanding of higher-order units and provide a stronger foundation in multiplication, division, proportion/ratio, and algebra.

Like Sullivan and McDuffie’s students, our children sometimes confused the role of the number of groups (multiplier) with the size of each group (multiplicand). Considering that our children were only aged five or six, this is not surprising. Despite the added challenge of needing to distinguish between the multiplier and the multiplicand (language we used with the teachers but not the children), our children were very excited about working with multiplication and division, and showed extraordinary capability for their age. As the study progresses, we will be interested to look at the children’s recording in their project books to see whether there is a relationship to their progress on a follow-up assessment. We are particularly interested in ways in which children make the transition across the boundary from counting to part-whole strategies.
ACKNOWLEDGEMENTS

This project was made possible by funding from the Teaching and Learning Research Initiative [TLRI] through the New Zealand Council for Educational Research and the interest and support of the teachers and children involved in the project.

NOTES

1. New Zealand schools are ranked into approximately ten equal groups, from 1 (highest proportion of students from low socio-economic communities) to 10 (lowest proportion of students from low socio-economic communities). It is based on Census information about household income, occupation, and education, and is used by the Ministry of Education to target funding to schools with the greatest learning needs.

2. The children are referred to in this paper by pseudonyms.

REFERENCES


This paper summarises the main mathematics-related findings arising from Ireland’s participation in TIMSS 2011 at Fourth class. It begins by describing mathematics achievement in Ireland together with international comparisons, including the proportions of Irish Fourth class pupils achieving at various performance “benchmarks” compared to Fourth grade pupils in other countries. Changes in national performance since Ireland last participated in TIMSS, in 1995, are also considered. Broader contextual findings relating to mathematics teaching and learning are then discussed. These include pupils’ attitudes towards mathematics, the time spent teaching the subject, teachers’ confidence and participation in mathematics-related continuing professional development, teaching practices in the classroom, and the use of ICT.

In 2011, Fourth class pupils participated in the Trends in International Mathematics and Science Study (TIMSS). In contrast to Ireland’s frequent participation in the Programme for International Student Assessment (PISA) at post-primary level – with the fifth cycle of assessment since 2000 completed last year – TIMSS 2011 marked the first occasion since 1995 that Irish pupils took part in a large-scale international assessment of mathematics or science at primary level. This paper focuses exclusively on the mathematics component of the assessment, and sets out the main achievement-related and contextual findings.

WHAT IS TIMSS?

TIMSS is a project of the International Association for the Evaluation of Educational Achievement (IEA), an independent, international cooperative of national research institutions and governmental research agencies. It has been conducted in four-yearly cycles since 1995, when Ireland last participated. Following non-participation in the 1999, 2003, and 2007 studies, a decision was taken by the (then-) Department of Education and Science to take part in the 2011 study, alongside participation in PIRLS 2011, a corresponding project of the IEA that assesses reading achievement. In 2011, 63 countries took part in TIMSS at either Fourth grade (Fourth class, in Ireland), Eighth grade (Second Year), or both. Ireland took part in the Fourth grade component only.

In March/April 2011, more than 4500 Fourth class pupils in a nationally-representative sample of 151 schools completed the mathematics and science assessment. The same pupils also provided background information through a detailed questionnaire, as did their parents, their teachers, and the principals of their schools. The study thus generated a wealth of data relating to pupils’ mathematics learning and contextual characteristics.
For brevity, further details on the studies, and on sampling and methodology, are kept to a minimum here – interested readers are referred to the initial national and technical reports for Ireland (Eivers & Clerkin, 2012a, 2012b) or to the international report (Mullis, Martin, Foy & Arora, 2012), all of which were released in December 2012. The information presented in the remainder of this paper collates some of the analyses previously included in the Irish national reports on achievement and contextual findings from TIMSS 2011 (Clerkin, 2013; Clerkin & Creaven, 2013; Eivers & Clerkin, 2012a, 2013), and summarises the key mathematics-related findings arising from Ireland’s participation in TIMSS 2011.

**MATHEMATICS ACHIEVEMENT IN TIMSS 2011**

TIMSS reports mathematics performance on an international scale set to a “centrepoint” of 500, with a standard deviation of 100. The centrepoint was fixed at 500 in 1995 to allow comparisons between study cycles, and is different from the international average which may vary cycle by cycle depending on performance and on the particular set of countries taking part on each occasion. The IEA reports countries’ individual performance with exclusive reference to the scale centrepoint.

**Overall performance**

Table 1 shows the mean mathematics achievement scores for the countries that participated in the Fourth grade component of TIMSS 2011, ranked in descending order. The table is presented in three sections. On the left are those countries that achieved a mean score significantly higher than Ireland’s; in the middle are Ireland and the countries where performance was not significantly different; and, on the right, countries (and the TIMSS scale centrepoint) where mean mathematics achievement was significantly lower than in Ireland.

As shown, Fourth grade pupils in 13 countries achieved a mean score higher than Fourth class pupils in Ireland, while Irish Fourth class pupils outperformed their peers in 33 countries. Gender differences in performance were not statistically significant, either at the international average (girls: 490, and boys: 491) or in Ireland (girls: 526, and boy: 529).

On the basis of their performance on the mathematics assessment, each pupil was also classified as reaching one of four International Benchmarks. These Benchmarks facilitate a skill-based description of pupils’ mathematical proficiency. For example, pupils at the Low Benchmark (scoring 400 scale points) are described as exhibiting basic mathematical knowledge, such as adding and subtracting with whole numbers. Those at the Intermediate Benchmark (475 points) can demonstrate the skills of the Low Benchmark and, in addition, are likely to be able to apply basic mathematical knowledge to straightforward situations – for example, using charts and tables to solve simple problems. Pupils reaching the High Benchmark (550 points) would be expected to apply their mathematical knowledge and understanding to solve problems, including solving word problems and extending patterns. Finally, pupils at the Advanced Benchmark (625 points) are expected to apply mathematical knowledge in a variety of complex situations and explain their reasoning. For example, they are expected to be able to solve a variety of multi-step word problems involving whole numbers, including proportions, and to show an increasing understanding of fractions and
decimals. A more complete list of the skills that are associated with each Benchmark can be found in Eivers & Clerkin (2012a) or Mullis et al. (2012).

Table 1: Mean country scores and standard errors for Fourth grade mathematics achievement in TIMSS 2011, with comparison to Irish performance

<table>
<thead>
<tr>
<th>Significantly higher than IRL</th>
<th>Mean</th>
<th>SE</th>
<th>Similar to IRL</th>
<th>Mean</th>
<th>SE</th>
<th>Significantly lower than IRL</th>
<th>Mean</th>
<th>SE</th>
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<td>Singapore</td>
<td>606</td>
<td>3.2</td>
<td>Lithuania</td>
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<td>Serbia</td>
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<td>Portugal</td>
<td>532</td>
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<td>Australia</td>
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<td>Slovenia</td>
<td>513</td>
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<td></td>
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<td>TIMSS centrepoint</td>
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<td>–</td>
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<td></td>
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<td>1.3</td>
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<td></td>
<td></td>
<td></td>
<td>Norway</td>
<td>495</td>
<td>2.8</td>
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<td></td>
<td></td>
<td>Spain</td>
<td>482</td>
<td>2.9</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td>Bahrain</td>
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<td>3.3</td>
</tr>
<tr>
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<td>545</td>
<td>2.3</td>
<td></td>
<td></td>
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<td>UAE</td>
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<tr>
<td>England</td>
<td>542</td>
<td>3.5</td>
<td></td>
<td></td>
<td></td>
<td>Iran, Rep.</td>
<td>431</td>
<td>3.5</td>
</tr>
<tr>
<td>Russian Fed.</td>
<td>542</td>
<td>3.7</td>
<td></td>
<td></td>
<td></td>
<td>Qatar</td>
<td>413</td>
<td>3.5</td>
</tr>
<tr>
<td>United States</td>
<td>541</td>
<td>1.8</td>
<td></td>
<td></td>
<td></td>
<td>Saudi Arabia</td>
<td>410</td>
<td>5.3</td>
</tr>
<tr>
<td>Netherlands</td>
<td>540</td>
<td>1.7</td>
<td></td>
<td></td>
<td></td>
<td>Oman</td>
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<td>2.9</td>
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<td>Denmark</td>
<td>537</td>
<td>2.6</td>
<td></td>
<td></td>
<td></td>
<td>Tunisia</td>
<td>359</td>
<td>3.9</td>
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<tr>
<td>Japan</td>
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<td>1.7</td>
<td></td>
<td></td>
<td></td>
<td>Kuwait</td>
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<tr>
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<td>562</td>
<td>2.9</td>
<td></td>
<td></td>
<td></td>
<td>Morocco</td>
<td>335</td>
<td>4.0</td>
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<tr>
<td>Belgium (Flemish)</td>
<td>549</td>
<td>1.9</td>
<td></td>
<td></td>
<td></td>
<td>Yemen</td>
<td>248</td>
<td>6.0</td>
</tr>
</tbody>
</table>

As shown in Table 2, greater percentages of Fourth class pupils in Ireland than in many other countries reached each of the Benchmarks. At 9%, the percentage of Irish pupils reaching the Advanced Benchmark –those demonstrating the most advanced skills – was more than twice the median value across all participating countries. At the other extreme, fewer pupils in Ireland (6%) than internationally (10%) failed to reach the Low Benchmark. The performance of pupils below the Low Benchmark was too poor to be assessed by the test.

In some key comparison countries, the percentage of pupils reaching the Advanced and High Benchmarks were much greater than in Ireland. Among the very top-performing countries, about two-fifths of pupils in Singapore (43%), Korea (39%), and Hong Kong (37%) reached the Advanced Benchmark. About four-fifths of pupils in each case (78-80%) reached at least
the High Benchmark, double the corresponding proportion in Ireland, while practically all pupils reached at least the Low Benchmark.

Among other Anglophone nations, relatively more pupils in Northern Ireland (24% and 59%), England (18% and 49%), and the US (13% and 47%) reached the Advanced and High Benchmarks, compared with Ireland. However, Irish performance compares well to that in New Zealand (4% at Advanced, with just 85% reaching at least the Low Benchmark) and Australia (10% of pupils at the Advanced level, similar to Ireland, but with lower percentages reaching each subsequent Benchmark).

Table 2: Percentages of pupils, in Ireland and internationally, reaching each of the four Benchmarks for mathematics

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Advanced</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ireland</td>
<td>9</td>
<td>41</td>
<td>77</td>
<td>94</td>
</tr>
<tr>
<td>TIMSS (international median)</td>
<td>4</td>
<td>28</td>
<td>69</td>
<td>90</td>
</tr>
</tbody>
</table>

Benchmark percentages are cumulative. For example, the 41% of Irish pupils at the High Benchmark includes the 9% who also reached the Advanced Benchmark.

Content and cognitive domains

In addition to overall performance, pupils’ mathematics achievement can be examined in terms of item content and cognitive processes. The TIMSS assessment framework (Mullis, Martin, Ruddock, O'Sullivan & Preuschoff, 2009) categorises each item on the mathematics test under one of three content domains – Number, Geometric Shapes and Measures, and Data Display – and one of three cognitive domains – Knowing, Applying, and Reasoning. Subscales were generated for each of these domains, allowing comparison of the relative performance of Irish pupils in each area.

Table 3 displays the scores achieved, overall and on each subscale, by Irish pupils. It also shows the relative performance of boys and girls. Although boys achieved slightly higher mean scores than girls on most domains, these differences were not significant.

Table 3: Mean scale scores, overall and by gender, for content and cognitive domains

<table>
<thead>
<tr>
<th></th>
<th>Ireland</th>
<th>Boys</th>
<th>Girls</th>
<th>Difference (Girls-Boys)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall</td>
<td>527</td>
<td>529</td>
<td>526</td>
<td>-3</td>
</tr>
<tr>
<td><strong>Content domains</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number</td>
<td>533 *</td>
<td>535</td>
<td>530</td>
<td>-5</td>
</tr>
<tr>
<td>Geometric Shapes &amp; Measures</td>
<td>520 **</td>
<td>521</td>
<td>519</td>
<td>-2</td>
</tr>
<tr>
<td>Data Display</td>
<td>523 **</td>
<td>522</td>
<td>524</td>
<td>+2</td>
</tr>
<tr>
<td><strong>Cognitive domains</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Knowing</td>
<td>539 *</td>
<td>540</td>
<td>539</td>
<td>-1</td>
</tr>
<tr>
<td>Applying</td>
<td>529</td>
<td>530</td>
<td>528</td>
<td>-2</td>
</tr>
<tr>
<td>Reasoning</td>
<td>510 **</td>
<td>512</td>
<td>507</td>
<td>-5</td>
</tr>
</tbody>
</table>

* Significantly higher than the overall national mean.
** Significantly lower than the overall national mean.
Irish pupils demonstrated a relative strength on items categorised under the Knowing cognitive domain. That is, the score achieved for Knowing (539) was significantly higher than the overall national mean score for mathematics (527). Knowing items are those where the student is expected to produce an answer to a question by recalling, recognising, computing, retrieving, measuring, or classifying. By contrast, Reasoning was found to be a relative weakness among Irish pupils, with a subscale score (510) significantly below the overall national mean. Reasoning items are characterised by higher-level processes such as analysing, generalising or specialising, integrating, justifying, and solving non-routine problems.

It should be noted that these refer to relative national performance on each cognitive domain, compared to overall performance. Although Reasoning was found to be a relative weakness for Irish pupils, their score on the Reasoning subscale was still higher than that of many other countries. A relatively poor national performance on Reasoning was also evident in Singapore, Hong Kong, Northern Ireland, England, and the US.

For the content domains, items examining Numbers were found to be a relative strength for Irish pupils. Performance on Geometric Shapes and Measures, and on Data Display, was found to be relatively poorer. Subscale scores for both categories were significantly below overall national mathematics performance.

**Trends: 1995 to 2011**

Fourth class pupils’ performance on TIMSS 2011 can be compared to the achievement of Irish pupils in TIMSS 1995 (Mullis, Martin, Beaton, Gonzalez, Kelly & Smith, 1997). As Ireland did not participate in any of the three intervening cycles, more recent data are not available, although the scaling and linking procedures used by the IEA to facilitate trend comparisons across each cycle are reliable enough to allow some general trend comparisons to be made.

Table 4 shows that there has been very little change in Irish pupils’ mathematics achievement in the period since 1995. Both the overall mean score and the percentages of pupils reaching the upper Benchmarks are similar in both cycles. However, it is worth noting that significantly fewer pupils in 2011 (6%) than in 1995 (9%) failed to reach the Low Benchmark. This can be taken to mean that, although there has been no increase in the proportion of pupils with the most advanced skills and understanding, more Fourth class pupils now have at least a basic understanding of mathematics.
Table 4: Irish mean scores and percentages reaching each Benchmark in TIMSS 1995 and 2011

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Advanced</th>
<th>High</th>
<th>Intermediate</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ireland, 1995</td>
<td>523</td>
<td>10</td>
<td>40</td>
<td>73</td>
<td>91</td>
</tr>
<tr>
<td>Ireland, 2011</td>
<td>527</td>
<td>9</td>
<td>41</td>
<td>77</td>
<td>94</td>
</tr>
</tbody>
</table>

Bold denotes a statistically significant difference between 1995 and 2011.

PUPILS’ ATTITUDES

In contrast, perhaps, to their reasonably high average achievement on the mathematics assessment, Irish pupils’ attitudes towards learning mathematics are less positive than in many other countries. Pupils were asked about the extent to which they agreed or disagreed with five statements, such as “I enjoy learning mathematics” and “Mathematics is boring”. Their responses to these statements were combined to create an aggregate scale, with pupils categorised as, on average, liking, somewhat liking, or not liking learning mathematics.

Fewer Irish pupils (41%) were described as liking mathematics than at the TIMSS international average (48%). Conversely, more pupils in Ireland did not like learning mathematics – 23%, compared to 16% internationally.

Gender differences on this measure were generally small, with 42% of girls and 40% of boys in Ireland liking mathematics (compared to 47% and 48%, internationally). However, fewer girls (21%) than boys (25%) did not like mathematics in Ireland. In both cases, but particularly so for boys, these percentages were higher than the international averages (17% girls, 16% boys).

By comparison, science, which pupils were also asked about as part of TIMSS, was relatively popular – three-fifths of Irish pupils liked science and only one-tenth did not like the subject – showing more positive attitudes to science than to mathematics among the same pupils.

Even when other variables (e.g., pupils’ gender and socioeconomic background) were considered, liking mathematics was associated with a stronger performance on the TIMSS assessment (Cosgrove & Creaven, 2013). Pupils who agreed that they like learning mathematics scored, on average, and controlling for other factors, 21 scale points higher than pupils who disagreed.

TEACHERS’ CONFIDENCE

Teachers were asked about their confidence regarding several aspects of teaching mathematics to their Fourth class pupils (Table 5). For some of the activities – providing challenging tasks for capable pupils, and adapting teaching to engage pupils’ interests – the percentages of pupils in Ireland whose teachers were confident with the specified activities were broadly similar to the international averages. More pupils in Ireland had a teacher who felt able to answer pupils’ questions about mathematics. However, somewhat fewer Irish
pupils had a teacher who felt able to help them appreciate the value of mathematics, or to show them a range of problem-solving strategies.

Table 5: Percentages of pupils whose teachers reported being very confident or not at all confident with various aspects of mathematics teaching

<table>
<thead>
<tr>
<th>Aspect</th>
<th>Ireland Very</th>
<th>Ireland Not at all</th>
<th>TIMSS Very</th>
<th>TIMSS Not at all</th>
</tr>
</thead>
<tbody>
<tr>
<td>Answering pupils’ questions about maths</td>
<td>92</td>
<td>0</td>
<td>84</td>
<td>&lt;1</td>
</tr>
<tr>
<td>Providing challenging tasks for capable pupils</td>
<td>63</td>
<td>2</td>
<td>59</td>
<td>2</td>
</tr>
<tr>
<td>Adapting teaching to engage pupil interests</td>
<td>63</td>
<td>1</td>
<td>65</td>
<td>1</td>
</tr>
<tr>
<td>Helping pupils appreciate the value of learning maths</td>
<td>61</td>
<td>2</td>
<td>69</td>
<td>1</td>
</tr>
<tr>
<td>Showing pupils a variety of problem-solving strategies</td>
<td>70</td>
<td>1</td>
<td>75</td>
<td>1</td>
</tr>
</tbody>
</table>

Rows do not sum to 100 as the third category, “somewhat confident”, is not shown.

Teachers’ responses to these individual items were combined to create a composite index. On this composite measure, about three-quarters of Irish pupils (74%) were described as being taught by a teacher who was very confident teaching mathematics, with the remaining quarter of pupils taught by a somewhat confident teacher. These proportions were very similar to the TIMSS international averages, and to the proportions in some other countries, including Australia, England, Northern Ireland, and Singapore. Almost all pupils (97%) in the Russian Federation had teachers who were very confident, although the equivalent percentage in high-performing Hong Kong and Korea was much lower, at just 48%.

By way of comparison, a much lower percentage of pupils, both in Ireland and internationally, were taught by teachers who said they were very confident teaching science – just 41% of Fourth class pupils in Ireland, and 59% of Fourth grade pupils across all TIMSS countries. Thus, while similar percentages of pupils in Ireland and internationally had teachers who were confident teaching mathematics, the percentage of pupils in Ireland whose teachers were also confident teaching science was much lower, about two-thirds of the TIMSS international average. It seems reasonable to conclude from this that, although a quarter of Irish pupils are taught by teachers who are only somewhat confident with mathematics, Fourth class teachers in general are much more comfortable teaching mathematics to their pupils than teaching science.

PROFESSIONAL DEVELOPMENT

In relation to continuing professional development (CPD), teachers were asked whether they had “participated in professional development” in each of six specified areas related to the teaching of mathematics in the two years prior to taking part in the survey (in March/April 2011). They were not asked to specify the amount of time spent on CPD for mathematics.

As shown in Table 6, Irish pupils were less likely than pupils internationally to be taught by a teacher with recent CPD in any of the specified areas. The percentage of pupils whose teachers had taken part in ICT-related CPD was reasonably similar to the TIMSS average. However, development related to mathematics pedagogy, mathematics content, or the
assessment of mathematics was much less common in Ireland. Participation in CPD aimed at supporting individual pupils’ needs was also less likely in Ireland.

Table 6: Percentages of pupils whose teachers had participated in CPD on specified topics in the two years prior to TIMSS 2011

<table>
<thead>
<tr>
<th>Topic</th>
<th>Ireland</th>
<th>TIMSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Content</td>
<td>32</td>
<td>44</td>
</tr>
<tr>
<td>Pedagogy / instruction</td>
<td>32</td>
<td>46</td>
</tr>
<tr>
<td>Curriculum</td>
<td>34</td>
<td>41</td>
</tr>
<tr>
<td>Integrating ICT into maths teaching</td>
<td>31</td>
<td>33</td>
</tr>
<tr>
<td>Assessment</td>
<td>25</td>
<td>37</td>
</tr>
<tr>
<td>Addressing individual pupils’ needs</td>
<td>33</td>
<td>43</td>
</tr>
</tbody>
</table>

TIME SPENT ON MATHEMATICS

About three-quarters of the countries that took part in TIMSS 2011, including Ireland, had national policies allocating a certain proportion of instructional time to mathematics at the Fourth grade. Generally speaking, more classroom time was allocated to reading than to mathematics, and to mathematics than to science. In Ireland, at the time of the survey, 18% of instructional time was officially allocated to reading, 13% to mathematics, and just 4% to science (Lewis & Archer, 2013). The 13% of time officially given to mathematics here was similar to the allocation in some high-performing countries such as Hong Kong (12-15%), Korea (14%), and the Russian Federation (16%), but less than in Singapore (22%). In some other nations, including Finland, England, Northern Ireland and the US, an equivalent allocation was not specified, or varied by state within the country.

In practice, Fourth class teachers spent more time on teaching mathematics in the classroom than is officially allotted. According to teachers’ reports, about 150 hours over the school year were devoted to mathematics teaching, representing about 18% of total instructional hours. As a percentage of total hours, this is similar to the time devoted to mathematics at the TIMSS international average, although the TIMSS average hours spent on mathematics teaching (162) is higher in absolute terms. The percentage time spent teaching mathematics in Ireland was also similar to that in some of our key comparison countries, such as Finland (18%) and England and the US (both 19%). Relatively less time was spent on mathematics in Hong Kong and Korea (15%) and the Russian Federation (16%), but relatively more time was devoted to the subject in Singapore (21%) and Northern Ireland (24%).

TEACHING MATHEMATICS

Fourth class teachers reported using a range of strategies to engage and teach their pupils. For lessons generally, most pupils were in classes where their teacher used questioning in every or almost every lesson to draw explanations from pupils (91%), and where pupils were praised for good effort (94%) and encouraged to improve performance (90%) with the same frequency. About half of pupils (52%) had teachers who summarised what pupils should
have learned in *every or almost every lesson*, or who related the lesson to things from the pupils’ daily lives (53%). Fewer pupils (26%) had teachers who reported bringing interesting materials to their class for *every or almost every lesson*.

Table 7 shows, with particular regard to mathematics lessons, how frequently a range of practices were utilised by Fourth class teachers. The most common practices, both in Ireland and internationally, were having pupils listen to the teacher explain how to solve a problem, and asking pupils to explain their answers. Irish pupils were more likely than Fourth grade pupils in other countries to work out problems (either by themselves or with classmates) while their teacher was occupied by other tasks, and were also more likely to work out problems as a class group under their teacher’s guidance. Relating what was learned in mathematics lessons to pupils’ daily lives was less common in Ireland than internationally. Irish pupils also took written mathematics quizzes or tests less frequently than the TIMSS average.

**Table 7: Percentage of pupils whose teachers engaged in specified teaching practices with varying frequency during mathematics lessons**

<table>
<thead>
<tr>
<th>Practice</th>
<th>Every day or almost every day</th>
<th>1 or 2 times a week</th>
<th>1 or 2 times a month</th>
<th>Never or almost never</th>
</tr>
</thead>
<tbody>
<tr>
<td>Listen to me explain how to solve problems</td>
<td>IRL 67</td>
<td>23</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>TIMSS 70</td>
<td>18</td>
<td>12</td>
<td>&lt;1</td>
</tr>
<tr>
<td>Memorise rules, procedures and facts</td>
<td>IRL 30</td>
<td>42</td>
<td>26</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>TIMSS 37</td>
<td>24</td>
<td>36</td>
<td>3</td>
</tr>
<tr>
<td>Work problems (individually or with peers) with my guidance</td>
<td>IRL 53</td>
<td>32</td>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>TIMSS 55</td>
<td>28</td>
<td>16</td>
<td>&lt;1</td>
</tr>
<tr>
<td>Work problems together with the whole class with direct guidance from me</td>
<td>IRL 53</td>
<td>32</td>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>TIMSS 45</td>
<td>27</td>
<td>27</td>
<td>1</td>
</tr>
<tr>
<td>Work problems (individually or with peers) while I am occupied by other tasks</td>
<td>IRL 24</td>
<td>27</td>
<td>34</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>TIMSS 16</td>
<td>16</td>
<td>39</td>
<td>29</td>
</tr>
<tr>
<td>Explain their answers</td>
<td>IRL 59</td>
<td>28</td>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>TIMSS 62</td>
<td>24</td>
<td>14</td>
<td>&lt;1</td>
</tr>
<tr>
<td>Relate what they are learning in mathematics to their daily lives</td>
<td>IRL 31</td>
<td>34</td>
<td>35</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>TIMSS 44</td>
<td>31</td>
<td>24</td>
<td>0</td>
</tr>
<tr>
<td>Take a written test or quiz</td>
<td>IRL 5</td>
<td>19</td>
<td>75</td>
<td>&lt;1</td>
</tr>
<tr>
<td></td>
<td>TIMSS 18</td>
<td>21</td>
<td>60</td>
<td>1</td>
</tr>
</tbody>
</table>

Following mathematics classes, the assignment of homework in Ireland can best be characterised as “frequently-given, but short”. Almost two-thirds of pupils in Ireland (62%), compared to slightly more than one-third internationally (36%), received mathematics homework *every day*. Only 5% of Fourth class pupils received mathematics homework less frequently than *three or four times per week*. By contrast, this was the case for almost one-third of pupils (32%) across all TIMSS countries.
While these figures might, at first glance, seem to suggest that Irish pupils receive more homework than their peers in other countries, the majority of Irish pupils (61%) were expected by their teachers to take no more than 15 minutes to complete their mathematics homework, and only 1% was expected to take more than half an hour. By comparison, the corresponding percentages at the TIMSS international average were 26% and 17%, respectively.

Finally, in relation to the use of ICT in the classroom, just over half (55%) of Fourth class pupils were reported by their teachers as being in classes where a computer was available for pupils to use during mathematics lessons. This was higher than the TIMSS average (42%). However, even where computers were available for use in the classroom, only small minorities of pupils used them during their mathematics lessons more frequently than once or twice a month (Table 8). Many pupils rarely or never use a computer during their mathematics class, even where it is available. When they are used, computers are most frequently utilised to allow pupils to practise mathematical skills and procedures, and to explore mathematical principles and concepts. They are less often used to look up ideas or information.

<table>
<thead>
<tr>
<th>Activities</th>
<th>IRL</th>
<th>TIMSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Every day or almost every day</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1 or 2 times a week</td>
<td>16</td>
<td>9</td>
</tr>
<tr>
<td>1 or 2 times a month</td>
<td>24</td>
<td>15</td>
</tr>
<tr>
<td>Rarely or never</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>To explore mathematics principles and concepts</td>
<td>IRL</td>
<td>TIMSS</td>
</tr>
<tr>
<td>To look up ideas and information</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1 or 2 times a week</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>1 or 2 times a month</td>
<td>26</td>
<td>21</td>
</tr>
<tr>
<td>Rarely or never</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>To practise skills and procedures</td>
<td>IRL</td>
<td>TIMSS</td>
</tr>
<tr>
<td>To explore mathematics principles and concepts</td>
<td>IRL</td>
<td>TIMSS</td>
</tr>
<tr>
<td>To look up ideas and information</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1 or 2 times a week</td>
<td>22</td>
<td>8</td>
</tr>
<tr>
<td>1 or 2 times a month</td>
<td>18</td>
<td>16</td>
</tr>
<tr>
<td>Rarely or never</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>To practise skills and procedures</td>
<td>IRL</td>
<td>TIMSS</td>
</tr>
<tr>
<td>To explore mathematics principles and concepts</td>
<td>IRL</td>
<td>TIMSS</td>
</tr>
<tr>
<td>To look up ideas and information</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>1 or 2 times a week</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>1 or 2 times a month</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>Rarely or never</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

Rows do not sum to 100 as only pupils with access to a computer are considered.

CONCLUSION

TIMSS 2011 provided internationally-comparative information on the mathematics achievement of Irish pupils at primary level for the first time since 1995, as well as their attitudes and the types of learning environments that they experience. Fourth class pupils achieved a mean score on the TIMSS mathematics assessment that was significantly above the scale centrepoint and the mean scores of 33 other countries. However, performance was significantly poorer in Ireland than in 13 other countries – most notably in the Asia-Pacific region, but also including the United States and several European neighbours such as Northern Ireland, England, Finland, and the Netherlands.
These findings complement the results of recent National Assessments of mathematics at primary level (Eivers et al., 2010; Gilleece, Shiel, Clerkin & Millar, 2012), international assessments of mathematics at post-primary level (Perkins, Cosgrove, Moran & Shiel, 2012), and – with particular regard to the contextual information relating to teaching and learning in Fourth class – the attitudes and experiences of the nine-year-old cohort of participants in the Growing Up in Ireland study (Williams et al., 2009). Work is underway in preparation for Ireland’s participation in TIMSS 2015, with a field trial scheduled for 2014. The results of the 2015 study will build on the findings reported above, and will allow more detailed analyses of change in mathematics performance, and relevant contextual characteristics, over time. A comparison of performance in 2011 and 2015 will be particularly interesting in light of the measures outlined in *Literacy and Numeracy for Learning and Life* (DES, 2011), which are aimed at raising numeracy levels among Irish children.

**REFERENCES**


This paper compares the TIMSS 2011 mathematics framework and test content for Fourth grade with the Irish mathematics curricula for Fourth class, and analyses a selection of TIMSS released items for Fourth grade where difficulty levels for Irish pupils were unusually high or low compared to international norms or were peculiar to Ireland in terms of gender differences. Findings are discussed briefly in terms of current issues in mathematics education.

Ireland last took part in TIMSS (Trends in International Mathematics and Science Study) in 1995, at both the Fourth grade and Eighth grade levels in mathematics. Since then a revised Primary School Mathematics Curriculum (PSMC) (DES/NCCA, 1999) was introduced. Relative to its predecessor, the PSMC places more emphasis on constructivist theories of learning and teaching, on problem-solving, and on communication and discussion. It advocates the use of digital technology in teaching and learning. There were also some minor changes in content including the introduction of estimation in computation and measurement and simple probability, while the use of calculators was encouraged from Fourth class onwards.

Until 2011, Ireland had not participated in any large international assessment of primary school mathematics since the TIMSS 1995 Fourth grade survey (although Ireland participated in five cycles of the OECD Programme for International Student Assessment (PISA) at post-primary level, beginning in 2000). However, National Assessments of Mathematics Achievement (NAMA) were carried out at the Fourth class level in 1999 and 2004 (Shiel & Kelly, 2001; Shiel, Surgenor, Close, & Millar, 2006) and in Second and Sixth classes in 2009 (Eivers et al., 2010). Overall performance in Fourth class in 1999 and in 2004 was not significantly different, indicating no change in overall achievement from just before the revised PSMC was introduced to immediately after. There were significant improvements on two mathematics content areas (Data, and Shape and Space) and on one skill process (Reasoning). In both assessments, relative weaknesses were identified in the content areas of Measures and aspects of Number, and in the process skills of Applying and Problem-solving. In the National Assessments 2009 (NA 2009) of Second and Sixth class, performance on the process skill of Applying and Problem-solving and on the content area of Measures was poor, relative to other process skills and content areas, especially at Sixth class.

Thus, TIMSS 2011 provided a timely opportunity to look at mathematics learning and achievement in Irish primary schools from an international comparative perspective. In broad terms, national mathematics achievement in TIMSS 2011 is similar to that reported in TIMSS 1995 (Eivers & Clerkin, 2012). Ireland’s mean of 527 was significantly above the study centre-point of 500, ranking 17th of 50 participating countries. Thirteen countries achieved
mean scores that were significantly higher than Ireland’s. Boys and girls in Ireland obtained similar mean scores on the overall assessment.

**TIMSS 2011 MATHEMATICS FRAMEWORK AND THE IRISH PSMC**

The TIMSS 2011 mathematics assessment framework describes the mathematical knowledge and skills assessed in the 2011 survey, including the percentages of items assigned to test those skills (Mullis et al., 2009). The framework has two main dimensions: a *content* dimension, describing the three mathematical content domains on the test – Number, Geometric Shapes and Measures, and Data Display; and a *cognitive* dimension, outlining the three domains of cognitive process skills– Knowing, Applying, and Reasoning.

**The content dimension**

The Number domain is assessed by approximately 50% of the items, and includes the topic areas of: *whole numbers*, *fractions and decimals*, *number sentences*, and *number patterns and relationships*. Geometric Shapes and Measures is assessed by 35% of the items, and includes *points, lines and angles*, and *two- and three-dimensional shapes*. The Data Display domain is assessed by 15% of the items, and includes: *reading and interpreting*, and *organising and representing*.

**The cognitive dimension**

The TIMSS framework assesses three cognitive domains – Knowing, Applying, and Reasoning. Knowing (assessed by roughly 40% of test items) refers to the basic facts, concepts and procedures that pupils need to be able to recall to carry out routine mathematical tasks such as computation and measuring and identification skills, which are also often prerequisites for dealing with more complex tasks such as problem-solving and reasoning. Applying, assessed by 40% of the items, is concerned with the use or application of basic facts, concepts, and procedures in representing and solving routine well-practiced problems set in familiar mathematical or practical contexts. Reasoning (assessed by about 20% of the test items) is concerned with pupils’ ability to analyse and think logically about mathematical objects, rules and relationships in the process of solving non-routine problems.

**Comparing the TIMSS mathematics framework and PSMC for Fourth class**

As with the TIMSS mathematics framework, the PSMC has two principal dimensions – a content dimension and a cognitive dimension. The content dimension has five strands: Number, Algebra, Shape and Space, Measures, and Data. The strands and content domains of the PSMC and the TIMSS mathematics framework can be loosely matched as in Table 1.

<table>
<thead>
<tr>
<th>PSMC</th>
<th>TIMSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>Number</td>
</tr>
<tr>
<td>Algebra</td>
<td></td>
</tr>
<tr>
<td>Shape and Space</td>
<td>Geometric Shapes and Measures</td>
</tr>
<tr>
<td>Measures</td>
<td></td>
</tr>
<tr>
<td>Data</td>
<td>Data Display</td>
</tr>
</tbody>
</table>
About 50% of the TIMSS 2011 items were multiple-choice, and about 50% were constructed-response. Pupils were not allowed access to calculators on the Fourth grade test.

To provide a rough comparison between TIMSS and the PSMC on the content dimension, Table 2 shows the percentage of specific teaching objectives listed in the PSMC for Fourth class for each content strand/domain, compared with the percentage of TIMSS items for each content domain. It can be seen from the table that, apart from Data, which is more heavily weighted in TIMSS, the PSMC content strands and the TIMSS content domains have fairly similar weightings.

<table>
<thead>
<tr>
<th>Domain/Strand</th>
<th>% PSMC objectives</th>
<th>% PSMC objectives (Re-categorised)*</th>
<th>% items in TIMSS 2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number, Algebra</td>
<td>45</td>
<td>54</td>
<td>50</td>
</tr>
<tr>
<td>Shape and Space**, Measures***</td>
<td>47</td>
<td>38</td>
<td>35</td>
</tr>
<tr>
<td>Data</td>
<td>8</td>
<td>8</td>
<td>15</td>
</tr>
</tbody>
</table>

*Since TIMSS includes ‘units of measure’ in the Number domain, this column includes PSMC objectives relating to ‘units of measure’ in the Number & Algebra strand.
**TIMSS combines two PSMC strands (Number and Algebra) in a domain called ‘Number and Algebra’.
***TIMSS combines the two domains (Shape and Space, Measures) in a domain called ‘Geometric Shapes and Measures’.

The cognitive dimension of the PSMC has six general process skill categories: Understanding and Recalling; Implementing; Reasoning; Integrating and Connecting; Communicating and Expressing; Applying and Problem-solving. One skill, Communicating and Expressing, was not a formal subject of assessment in TIMSS although the extended constructed-response items could provide some informal information on this domain. Table 3 shows how they align with the TIMSS cognitive domains.

<table>
<thead>
<tr>
<th>PSMC</th>
<th>TIMSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Understanding and Recalling</td>
<td>Knowing</td>
</tr>
<tr>
<td>Implementing</td>
<td></td>
</tr>
<tr>
<td>Reasoning</td>
<td>Reasoning</td>
</tr>
<tr>
<td>Integrating and Connecting</td>
<td></td>
</tr>
<tr>
<td>Applying and Problem-solving</td>
<td>Applying</td>
</tr>
</tbody>
</table>

In order to provide a rough comparison between TIMSS and the PSMC on the cognitive dimensions the item percentages for each domain from NAMA 2004 are used as a proxy, since the weightings reflected in these percentages were inferred from curriculum documents and textbooks for Fourth class at the time. Table 4 compares the percentage of items in the NAMA 2004 Fourth class test for each cognitive process skill of the PSMC with the percentage of TIMSS items for each cognitive domain. Apart from the combined category of Understanding and Recalling/Implementing in NAMA 2004, and the corresponding Knowing category in TIMSS 2011, the PSMC and the TIMSS domains are somewhat differently weighted. However, defining categories and classifying items on the cognitive dimension is more complex than is the case with the content domains and needs to be viewed in this light.
For example, non-routine problems are included in the Reasoning category in the TIMSS framework, but in the Applying and Problem-solving category in the PSMC.

Table 4: Percentages of items for the cognitive process domains of NAMA 2004 and TIMSS mathematics

<table>
<thead>
<tr>
<th>Cognitive process skill</th>
<th>NAMA 2004</th>
<th>TIMSS 2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>% items (combined)</td>
<td>% items</td>
<td>% items</td>
</tr>
<tr>
<td>Understanding and Recalling Implementing</td>
<td>12%</td>
<td>40%</td>
</tr>
<tr>
<td>Reasoning</td>
<td>21%</td>
<td>20%</td>
</tr>
<tr>
<td>Integrating and Connecting</td>
<td>7%</td>
<td></td>
</tr>
<tr>
<td>Applying and Problem-solving</td>
<td>32%</td>
<td>40%</td>
</tr>
</tbody>
</table>

Test-curriculum matching analysis

To provide further evidence of the degree of correspondence between the TIMSS 2011 framework and the PSMC, a Test-Curriculum Matching Analysis (TCMA) was carried out in which curriculum experts matched the 175 TIMSS test items with the specific objectives in the PSMC for Fourth class. Only 13 of the 175 TIMSS items (7%) were judged not to be covered in the PSMC for Fourth class. These non-matching items were all in the TIMSS Geometric Shapes and Measures domain and tested the topics of coordinates, rotational symmetry, volume of cuboids, and millimetre measures. Only nine of the 47 countries that carried out a TCMA had a higher percentage of TIMSS items that matched their national mathematics curriculum for Fourth grade.

International Benchmarks of mathematics performance

TIMSS reports pupils’ achievement using a scale with a mean of 500 (the centrepoint, anchored from the 1995 assessment) and a standard deviation of 100. In addition, four key points on this scale, 400, 475, 550, and 625, were identified for the purposes of setting and describing International Benchmarks of mathematics performance – Low, Intermediate, High, and Advanced, respectively. In order to describe what pupils can do at each of these four Benchmarks, the items used in TIMSS were located on the mathematics scale based on their difficulty. Once the items were placed and grouped on the scale they were used to derive descriptions of the knowledge and skills that pupils who scored at each International Benchmark should be able to demonstrate. (See the TIMSS methods and procedures website – http://timssandpirls.bc.edu/methods/index.html – for more detail).

ANALYSIS OF A SELECTION OF RELEASED ITEMS

After the initial achievement results were reported, 73 items from a total pool of 175 items used in the TIMSS 2011 mathematics assessment were released into the public domain (see http://www.erc.ie/documents/timss_2011_maths_items.pdf for all released items, sample responses, and information on percent correct answers for Ireland and for TIMSS overall). The released items are representative of the distribution of all TIMSS items in terms of content and cognitive domains, as specified by the mathematics framework described earlier.
Item-by-item percent correct information for each participating country can also be accessed at: [http://timssandpirls.bc.edu/timss2011/international-released-items.html](http://timssandpirls.bc.edu/timss2011/international-released-items.html). Item-level analysis of the TIMSS data provides useful information relating to the teaching and learning of mathematics in Fourth class, as well as factors contributing to the difficulty of the items.

### Table 5: ‘Unusual’ TIMSS released mathematics items, by item ID, gap between Irish and international mean for each item, and International Benchmark level

<table>
<thead>
<tr>
<th>Content domain (N released items)</th>
<th>Topic area</th>
<th>Unusually high IRL - INT ≥ 15%</th>
<th>Unusually low IRL – INT ≤ 0%</th>
<th>Unusual IRL gender gap ≥ 10% (Girls – Boys)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Item ID* (gap) IBM</td>
<td>Item ID (gap) IBM</td>
<td>Item ID (gap) IBM</td>
</tr>
<tr>
<td>Number</td>
<td>Whole numbers</td>
<td>M05_03 (+31) Adv</td>
<td>M01_03 (0) Adv</td>
<td>M01_01A (-10) Inter</td>
</tr>
<tr>
<td>(40)</td>
<td></td>
<td>M07_02 (+19) Adv</td>
<td>M01_08 (-4) High</td>
<td>M01_01B (-11) Adv</td>
</tr>
<tr>
<td></td>
<td>Fractions and decimals</td>
<td>M02_01 (+15) Inter</td>
<td>M02_04 (-2) Adv</td>
<td>M01_02 (-11) Adv</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M02_02 (+24) Adv</td>
<td>M06_02 (-6) High</td>
<td>M02_03 (-21) Adv</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M03_03 (+28) Inter</td>
<td></td>
<td>M02_05 (+10) High</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M03_05 (+19) High</td>
<td></td>
<td>M03_01 (-10) Inter</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M03_06 (+30) Adv</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>M06_05 (-13) Adv</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>M07_01 (+16) Adv</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Number sentences</td>
<td>M07_05 (+16) Adv</td>
<td>M05_06 (-1) Adv</td>
<td>M06_01 (+13) Inter</td>
</tr>
<tr>
<td></td>
<td>Patterns and relationships</td>
<td>M07_03 (+19) Adv</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Geometric Shapes and Measures (24)</td>
<td>M02_07B (+15) Inter</td>
<td>M07_07 (-12) Adv</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>M02_07A (-6) High</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Points, lines, and angles</td>
<td>M03_08 (+20) Inter</td>
<td>M03_12 (-5) Adv</td>
<td>M06_09 (-10) Adv</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M05_11 (+16) High</td>
<td>M01_07 (0) Adv</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Two- and three-dimensional shapes</td>
<td>M06_08 (+27) High</td>
<td>M06_10 (-6) Adv</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Data Display (9)</td>
<td>Reading &amp; interpreting</td>
<td>M05_07 (-14) High</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Organizing &amp; representing</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*The code for each item indicates the location of the item within a block of items (e.g. M05_01 is item 1 in block 5). The codes for the 73 released mathematics items, along with Irish and international scores, can be found at: [http://timssandpirls.bc.edu/timss2011/international-released-items.html](http://timssandpirls.bc.edu/timss2011/international-released-items.html). Examples of correct answers in cases where pupils had to write an answer, along with percent correct scores, are also available at [www.erc.ie/pirlstimss](http://www.erc.ie/pirlstimss).

Table 5 lists a selection of 35 items from the 73 released items that can be considered to be ‘unusual’ in terms of Irish performance. Since the Irish mean scale score for TIMSS 2011 (527) is significantly above the international centrepoint scale score of 500, one would expect the Irish pupils’ percent correct score on most mathematics items to be above the international mean, by up to 5%-10%. Therefore, for this section, items for which Irish pupils’ percent
scores were substantially above the international mean (i.e., difference ≥ +15%) and those that were at or below it (i.e., difference ≤ 0%) were considered to be “out of the ordinary”. Items with substantial gender differences, particularly where they are not in line with international gender differences in performance, are also included.

Inspection of these 35 “unusual” items in Table 5 shows that, in terms of the content dimension, most of the items on which Irish Fourth class pupils did unusually well were in the topic area of Fractions and Decimals (7 items) and most of the items on which they did relatively poorly were, surprisingly, on the topic area of Whole Numbers (5 items). Most items with unusual gender differences (an at least 10% gender gap in Ireland) were also on Whole Numbers (6 items). All of the “unusually high” items were in the Knowing and Applying domains and half of the “unusually low” items were in the Reasoning domain. In terms of International Benchmarks, 28 of the 35 items are at the High and Advanced International Benchmarks with just 7 of them at the Intermediate level and none at the Low level. Some of these “out of the ordinary” items are discussed further below, with a particular emphasis on items on which Irish pupils underperformed.

**Number**

**Fractions and Decimals**

Irish Fourth class pupils performed unusually well on items relating to the topic of Fractions and Decimals, relative to the TIMSS international average. This is illustrated by item M03_06 which, surprisingly for a seemingly easy item, is at the Advanced Benchmark level:

<table>
<thead>
<tr>
<th>Tom ate ½ a cake and Jane ate ¼ of the cake. How much of the cake did they eat altogether?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ans. Correct: Ireland: 53% TIMSS: 23%. Advanced Benchmark</td>
</tr>
</tbody>
</table>

Irish pupils also performed very well on items involving Decimals, such as Item M03_05 below:

<table>
<thead>
<tr>
<th>Write a number between 5 and 6.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ans. Any decimal or fraction between 5 and 6.</td>
</tr>
<tr>
<td>Correct: Ireland: 66% TIMSS: 48% High Benchmark</td>
</tr>
</tbody>
</table>

Two-thirds of Irish pupils obtained the correct answer on this item, compared to an international mean of just 48%, a difference of 18%. The inclusion of a specific teaching objective relating to ordering of decimals on the number line in the PSMC and the resulting substantial coverage of it in classroom teaching and in textbooks may help explain the much higher score of Irish pupils.

Another item, M02_01 on Decimals, on which Irish pupils did particularly well was on the application of decimals to a routine problem involving calculations with units of measurement:

<table>
<thead>
<tr>
<th>Duncan first travelled 4.8km in a car and then he travelled 1.5km in a bus; How far did Duncan travel?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ans. 6.3 km. Correct: Ireland: 75% TIMSS: 60%. Intermediate Benchmark</td>
</tr>
</tbody>
</table>
This item would be classified under Measures in the PSMC since it involves addition of units of length (km) but comes under the topic area of Fractions and Decimals in the Number domain of the TIMSS framework as it also involves decimals.

**Whole Number**

On the other hand, Irish performance on many items in the topic area of Whole Numbers was unusually low or showed unusually large gender differences. One of the poorest items in terms of Irish performance was Item M06_03:

<table>
<thead>
<tr>
<th>Circle each number which is a factor of 12</th>
<th>1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ans.</strong> Full credit: All six correct - 1, 2, 3, 4, 6, 12; Partial credit: Four or five correct.</td>
<td></td>
</tr>
</tbody>
</table>

**Full credit:** Ireland: 14% | TIMSS: 27% | **Partial credit:** Ireland: 27% | TIMSS: 44%

Advanced Benchmark.

Only 14% of Irish pupils answered this item correctly, compared to an international mean of 27%. Based on Irish performance in general, one would expect the Irish score on this item to exceed 30%. However, the concept of a factor is not formally introduced in the PSMC until Fifth class (in the strand unit Number Theory). Although pupils may be familiar with divisibility from work on multiplication and division with whole numbers in Third and Fourth classes, few would seem to be able to transfer this knowledge to generating lists of factors for numbers and solving problems involving factors.

Another item on Whole Number in which Irish pupils did less well than the international mean is M02_04:

*Mary left Apton and rode at the same speed for 2 hours. She reached this sign. Mary continues to ride at the same speed to Brandon. How many hours will it take her to ride from the sign to Brandon*

<table>
<thead>
<tr>
<th>A) 1 ½ hours</th>
<th>(B) 2 hours</th>
<th>(C) 3 hours*</th>
<th>(D) 3 ½ hours</th>
</tr>
</thead>
</table>

**Correct:** Ireland: 35% | TIMSS: 38% | Advanced Benchmark

This item involves a concept, speed, which is not introduced until Sixth class in the PSMC (in the strand unit Time). Speed is also a more complex concept as it represents a ratio of two more basic variables – distance and time (e.g., kilometres per hour). These facts may explain the relatively poor Irish performance (only 35% answered correctly) on this item. The international mean is also low, at 38%, so the absence of concept of speed in the Fourth grade mathematics curriculum may also have been a problem for some other countries.

Irish girls (45%) did substantially better than Irish boys (34%) on Item M02_05 \((23 \times 19)\) whereas the international mean gender difference was only 3%. This task normally involves knowing the steps in a long multiplication procedure, although more able pupils might use reasoning such as \(23 \times 19 = (23 \times 20) - (23 \times 1) = 460 - 23 = 437\). Given the considerable emphasis on multi-digit multiplication procedures in Irish Fourth class textbooks, a better performance might be expected. Performance on this item varied greatly from country to country. While 90% of pupils in Chinese Taipei answered correctly, less than 10% of pupils
in a number of countries did so, including Finland (5%), New Zealand (8%) and Poland (6%). This may reflect curriculum coverage or a de-emphasis on teaching formal algorithms.

**Ratio and Proportion**

A particular subsection of the topic area Whole Numbers on which Irish pupils did relatively poorly was that of simple proportions. This includes concepts such as ratio, scale, rate and the procedures of multiplication, division, and unitary method. Four of these items, M01_01A – M01A_01D, relate to one stimulus called *Trading Cards*. In the stimulus, two ratios are provided as pictorial representations: i.e.

| 1 animal card = 2 cartoon cards; and 2 animal cards = 3 sports cards |

The pupil has to use these ratios to answer the four questions.

The first question (*Becky had 5 animal cards to trade for cartoon cards. How many cartoon cards would she get?* Ans.: 10 cartoon cards) involved constructing the relationship below and carrying out the appropriate multiplication (normally a one-step problem).

| 2 cartoon cards for 1 animal card = ? cartoon cards for 5 animal cards |

This was generally easy for Irish pupils (73% correct) relative to the international mean (62%). However, substantially fewer Irish girls than boys obtained the correct answer (68% of girls, vs 78% of boys). The difference in favour of boys at the international level was just 6%.

The second question (*Jim had 8 animal cards to trade for sports cards. How many sports cards would he get?* Ans.: 12 sports cards) involved constructing the relationship below and carrying out the appropriate operations (normally a two-step problem).

| 3 sport cards for 2 animal cards = ? sport cards for 8 animal cards |

This proved to be considerably more difficult, with only 35% of Irish pupils getting the correct answer. Again, Irish boys scored considerably better than Irish girls (40%, compared with 29%) whereas the advantage for boys at the international level was just 5%.

The next question (*Steve had 15 sports cards to trade for animal cards. How many animal cards would he get?* Ans.: 10 animal cards) involved constructing a slightly more complex proportion:

| 2 animal cards for 3 sport cards = ? animal cards for 15 sport cards |

Overall, 27% of Irish pupils obtained the correct answer compared with 25% internationally. Irish boys’ performance exceeded that of Irish girls by about 9% (the gender difference at the international level was 4% in favour of boys).

Finally, the fourth question (*Brad had 8 cartoon cards to trade for sports cards. How many sports cards would he get?* Ans.: 6 sports cards) was the most complex, and involved construction of a transitive relationship among the proportions:
IF 3 sport cards = 2 animal cards AND 1 animal card = 2 cartoon cards;
THEN ? sport cards = 8 cartoon cards.

This is a multi-step problem involving a higher level of proportional reasoning and understanding than the previous three questions and one that might benefit more from formal classroom experience. This is reflected in the performance figures with just 17% of Irish pupils answering correctly (the same as the international mean) and again boys’ percent correct (22%) exceeding girls’ (11%) by over 10%, twice the magnitude of the difference at the international level.

Apart from the first one, these questions are classified as Reasoning on the cognitive dimension of the TIMSS framework and are all at the Advanced Benchmark. The performance of Irish pupils on these items is not in line with their performance on the TIMSS 2011 mathematics assessment in general. As with some earlier items showing this trend, a partial explanation may be found in the PSMC and in textbooks. The topic of Ratio and Proportion is not formally introduced in the PSMC until Sixth class. Consequently, the topic is not dealt with in the textbooks and resource materials for Fourth class. However, proportionality is a broadly-based topic affecting a number of other topics including multiplication and division, fractions, decimals, percentages, scale, and conversion of measures, all of which are covered in the PSMC for Third and Fourth classes. There should be some transfer of learning, particularly for the more able pupils, from these topics to proportionality tasks as per the four questions above. There has been considerable research on proportional reasoning (e.g., Hart, 1984; Vergnaud, 1983), indicating that its development takes place over a number of years from age eight or nine to 14 or 15 years. In this regard, it is worth noting that 11 of the 85 tasks in PISA 2003, which tested 15-year-olds, directly involved proportional reasoning with average percent correct scores ranging from 8% to 80% (OECD, 2009).

Geometric Shapes and Measures

Table 5 lists ten released items in the Geometric Shapes and Measures domain which were considered to be unusual in terms of typical Irish Fourth class performance.

Points, Lines and Angles

Three of the unusual items are in the topic area of Points, Lines and Angles and two of these, items M02_07A and M02_07B; concern the topic of Coordinates in the context of the grid map shown below. Coordinates are not on the Fourth class PSMC but familiarity with grids, particularly in the context of games, may have affected performance. The performance of Irish pupils on the two items shown next is striking in this regard as, relative to international levels, they performed unusually well (79% in Ireland; 63% internationally) on item M02_07B, but unusually poorly (43% in Ireland; 49% internationally) on item M02_07A.
M02_07. Complete the table to show where the places are. The first one has been done for you.

Correct: Ireland: 43% TIMSS: 50% High Benchmark

M02_07B. Troy lives in a house in square C4 Put an X in the square to show where Troy lives.

Correct: Ireland: 78% TIMSS: 64% Intermediate Benchmark

In item M02_07A above, pupils are required to identify the coordinates of two specified places on the grid, whereas item M02_07B requires them to identify a place on the grid given its coordinates. As Irish pupils are unlikely to have been taught Coordinates since it does not appear in their curriculum, they would have less experience of the first kind of task but would more likely experience the second kind in some game context (e.g., “go to C3” or “prize is at B5”, etc.) or in Geography lessons, where pupils construct and interpret maps. This lack of formal teaching on the topic is supported by data from the Teacher Questionnaire. Three-quarters (78%) of Irish pupils were taught by teachers who said they had not yet taught Coordinates, compared to 45% of pupils, internationally.

M02_07B: If the string in the diagram is pulled straight, which of these is closest to its length?

(A) 5   (B) 7* (C) 8   (D) 9

Correct: Ireland: 16% TIMSS: 29% Advanced Benchmark

Item M07_07B above, which involves estimating the length of a piece of string, proved to be one of the most difficult items on the test for Irish pupils. Only 16% answered correctly, 13%
below the international mean. This is unexpected given the strong emphasis on estimation of length in the PSMC for Fourth class. Research suggests that developing estimation skills in measurement among primary school pupils requires considerable learning experiences of a practical nature (Lehrer, 2003). Standard textbooks may be of limited value in this regard. Moreover, National Assessments conducted in 2004 and 2009 indicate that pupils use concrete materials, such as measuring instruments, on a very infrequent basis as they move through the primary school system.

Two- and Three-Dimensional Shapes

Recognition of 2-D shapes is easily taught and practiced and is given considerable attention in the PSMC and textbooks so the good performance on most items in this topic area is not surprising. However, one of the topics in this area – Rotation – provides another interesting case where mixed performance by Irish pupils on two items can shed light on the effects of informal learning outside the classroom versus formal classroom learning.

**M05_11:** A pattern rule says “Rotate the shape \(\frac{1}{4}\) turn clockwise each time”. What will the pattern look like? (4 choices involving different combinations of 4 rotations presented)

**Correct:** Ireland: 79%  TIMSS: 64%. High Benchmark

Item M05_11 above involves rotation in a circle through a specified angle which is not ostensibly on the PSMC, yet Irish pupils scored particularly well on the item with 79% choosing the correct response versus an international mean of 64%. This was probably facilitated by reference in the stem of the item to “\(\frac{1}{4}\) turn clockwise”, which would be familiar to most pupils, rather than specifying \(90^\circ\) as the rotation, which is not on the curriculum for Fourth class.

This view is supported by the results for a similar item, M01_07, below, which specifies a \(180^\circ\) rotation of a flag shape. The performance of Irish pupils on this item (42%) was below what would be expected based on the Irish mean performance. Unlike the previous item, the required transformation in this item is specified in degrees and there is no familiar analogy such as the clock to help pupils.

**M01_07** Which of the following shows the position of the shape above after a half turn or \(180^\circ\) rotation? (4 choices involving different rotations (90, 180*, 270, 360) presented)

**Correct:** Ireland: 42%  TIMSS: 43%. High Benchmark

Although not on the present PSMC, rotation as a geometric transformation was on its predecessor, the 1971 curriculum, but was removed as part of the review of that curriculum. In this regard, 66% of Irish pupils were taught by teachers who reported that the topic of reflections and rotations had not yet been taught or had just been introduced. Reflection is on the PSMC for Fourth class and as a result some teachers may have responded positively to the
question on whether or not reflections and rotations had been taught. In fact, performance of Irish pupils on a question which asked them to draw the line of symmetry on the picture of a kite (item ID code: M06_08, not shown here) was 74%, compared with an international mean of 47%.

**CONCLUSIONS**

The TIMSS framework and test for Fourth grade were compared with the Irish primary school mathematics curriculum (PSMC) for Fourth class in terms of content and cognitive process domains [1]. This analysis showed that the PSMC for Fourth class closely matched the content and cognitive processes tested by TIMSS 2011. The 13 items (of 175) identified as being on the TIMSS test but not on the PSMC for Fourth class related to the following topics – coordinates, rotational symmetry, volume of cuboids, millimetres, speed, factors and multiples, and ratio and proportion.

Analyses of a selection of released items provide some interesting findings relating to the Irish mathematics curriculum. Among items in the Number domain, Irish pupils scored particularly well on those in the topic area of Fractions and Decimals. In the area of Whole Numbers, consideration might be given to beginning formal work on factors and multiples, and on ratio and proportion, in Third and Fourth classes rather than waiting until Fifth and Sixth classes. TIMSS performance across countries on items relating to these areas suggests pupil readiness for learning these more complex concepts. The TIMSS results also suggest that gender appropriateness of contexts and situations used in teaching these topics should be addressed.

In the Geometric Shapes and Measures domain, the mixed performance of Irish pupils on Coordinates and the high relevance of the topic to everyday life suggest that this topic (and the related topic of describing movement between locations on plans and maps etc.), should be introduced earlier in the mathematics curriculum. The mixed performance on this topic, and perhaps the “Trading Cards” items, reflects the influence of out of school experience on the learning of mathematics. There is a need to capitalise more on such experience in classroom teaching.

Another topic in this domain where mixed performance by Irish pupils was observed is that of symmetry and transformational geometry. Axial symmetry in the form of reflection is on the PSMC, but rotational symmetry is not. Many countries include both topics in their curriculum – as indicated in responses to the TIMSS Teacher Questionnaire. The PSMC is very specific in setting out what pupils in each grade level should learn. This level of detail and lack of practical contexts in the teaching of mathematics, though it may be beneficial for some aspects of curriculum and teaching, does not encourage teachers to use problem-based teaching in which mathematical concepts may be integrated and developed in applied or practical settings. There is a need for a repository of “good” tasks aligned with high quality professional development to support teachers in moving away from over-reliance on textbook activities. Though Irish performance in mathematics on TIMSS 2011 at Fourth class can be considered satisfactory in general, there are some specific weaknesses which have been highlighted in this paper. These deficiencies were also the subject of recommendations in the
recent DES policy document setting out a national strategy for literacy and numeracy (DES, 2011). Addressing these weaknesses appropriately may not only help Irish pupils to demonstrate improvement in these areas in TIMSS 2015, but – more importantly – lead to a broader and deeper understanding of mathematics by Irish primary pupils.

NOTES
1. A more detailed version of this paper can be found in Chapter 8 in Eivers and Clerkin (2013).

REFERENCES


BRIDGES FROM PRIMARY TO POST-PRIMARY MATHEMATICS:
THE PISA ENIGMA

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To date, Ireland has participated in 5 cycles of the OECD Programme for International Student Assessment (PISA), an assessment of 15-year olds, in which mathematical literacy is key component. This paper defines mathematics in PISA, provides an overview of performance in Ireland from 2000 to 2009, identifies areas of relative strength (Uncertainty) and weakness (Space & Shape), and seeks to explain underperformance in PISA 2009, where Ireland’s mean score was significantly below the OECD average for the first time. It examines commonalities and differences between performance in mathematics at primary level, as measured in national assessments and in TIMSS 2011, and performance at post-primary level, where PISA is the only available standardized measure. The paper concludes by looking ahead to how the outcomes of PISA 2012 will be presented when they are released in December 2013.

INSIGHTS FROM PISA MATHEMATICS

In 2012, students in Ireland participated in PISA for the fifth time. The results of PISA 2012 mathematics will not be released by the OECD until December 2013, when a detailed report will be provided on the performance of students in Ireland compared to their counterparts in 67 other countries/economies. The current paper focuses on PISA mathematics outcomes between 2003 (when mathematics was a major assessment domain in PISA for the first time) and 2009.

THE PISA MATHEMATICS FRAMEWORK

PISA defines mathematical literacy as:

An individual’s capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements, and to use and engage with mathematics in ways that meet the needs of that individual’s life as a constructive, concerned and reflective citizen (OECD, 2009, p. 84).

The definition and major features of the PISA mathematics assessment remained unchanged between 2003 and 2009.

PISA mathematics seeks to assess performance on real-world problems that go beyond the kinds of situations typically encountered in school. Central to the PISA mathematics framework is the idea of mathematisation, which involves starting with a problem in a real-world context, identifying the mathematics relevant to solving the problem, reorganising the problem according to the mathematical concepts identified, gradually trimming away the reality so that the problem can be solved, and making sense of the mathematical solution in terms of the real solution. The PISA mathematics framework comprises three dimensions: (i) situations and contexts; (ii) content; and (iii) competencies. Each PISA mathematics item can be classified according to each of these dimensions.
Situations and contexts

According to PISA, an important aspect of mathematical literacy is the ability to use and do mathematics in a variety of situations such that the type of mathematics employed often depends on the situation in which the problem is presented. The situation is the part of the student’s world in which the problem is located. Four types of mathematical problem situations are identified in PISA: the personal, educational/occupational, public, and scientific. Context reflects the specific setting within a situation.

Mathematics content areas

PISA mathematics assesses four mathematics content areas (or ‘big ideas’):

- Shape & Space – recognising and understanding geometric patterns and identifying such patterns in abstract and real-world representations
- Change & Relationships – recognising relationships between variables and thinking in terms of and about relationships in a variety of forms including symbolic, algebraic, graphical, tabular and geometric
- Quantity – understanding relative size, recognising numerical patterns, and using numbers to represent quantities and quantifiable attributes of real-world objects
- Uncertainty – solving problems relating to data and chance.

Mathematics competencies and processes

PISA identifies eight types of cognitive processes involved in mathematisation: reasoning; argumentation; communication; modelling; problem-posing and solving; representation; using symbolic, formal and technical language and operations; and using aids and tools. A mathematics task may involve one or more of these processes at different levels of complexity. PISA represents these processes in three broad competency clusters:

- Reproduction – reproduction of practised knowledge (e.g., knowledge of facts and common problem representations, recollection of familiar mathematical objects and properties, performing routine procedures, application of standard algorithms, manipulation of formulate and carrying out computations).
- Connections – application of problem solving to non-routine solutions i.e., ‘integration and connection of material from various over-arching ideas or from different mathematical curriculum strands, or the linking of different representations of a problem’ (OECD, 2009, p. 110)
- Reflection – advanced reasoning and the ability to abstract and generalize in new contexts.

Close (2006) noted that, while there was a good spread of items across the three PISA competency clusters in PISA 2003, most of the items on the 2003 Junior Certificate mathematics papers fell into the Reproduction cluster, suggesting that the focus of the JC mathematics exam was more on lower-level than on higher-level mathematical processes.

Performance on PISA mathematics is reported in terms of overall (national) performance (i.e., across all students and items), and, in 2003 only, in terms of different mathematical content
areas. Performance is also reported with respect to proficiency levels on the overall mathematics scale, and, in 2003 only, on the content subscales. To date, performance has not been reported with respect to mathematical situations or competencies.

In each PISA cycle, about one-quarter of items are allocated to each content area, while approximately one-half of items assess connections, one quarter assess reproduction, and one-quarter assess reflection. In 2009, when there were 35 mathematics items, 46% were multiple-choice (students select one correct answer) or complex multiple-choice (students respond to a series of yes/no statements) in format, 32% required a short written response, and 23% required a longer written response.

**PISA VS. TIMSS**

Students in Second year in Ireland have not participated in TIMSS mathematics since 1995. However, a broad comparison across TIMSS and PISA can provide insights into the nature of PISA mathematics and what is seeks to assess. The following represent some points of comparison between the two studies:

- TIMSS assesses a grade-level cohort (Grade 8, or Second year in Ireland) while PISA assesses an age-cohort (15-year olds, who, in many countries, are distributed over several grade levels). In Ireland, students in Second, Third, Transition and Fifth years participate in PISA, with the majority (over 60%) in Third year.
- Compared with TIMSS, PISA allocates fewer items to the Reproduction competency cluster, and more to the Connections cluster. Hence, PISA seems to place more emphasis on higher-level mathematical processes, compared with lower-level ones (Hutchison & Schagen, 2006).
- Compared with TIMSS, PISA allocates fewer items to Number & Algebra, more to Measurement, and considerably more to Data (Hutchison & Schagen, 2006).
- A majority of TIMSS items are multiple choice, while a majority of PISA items are constructed response.
- The reading load in PISA is greater than in TIMSS. According to Ruddock et al. (2006), ‘It is the quantity of reading that marks PISA out, not the complexity of the language, which is similarly unfamiliar in both international studies’ (p. 123). Ruddock et al. claim that a consequence of the high reading load in PISA is the relatively low demand in the mathematics required, which reflects the low level of mathematics that students can apply in new contexts, as opposed to very familiar ones.
- While both PISA and TIMSS collect background information on schools and students through questionnaires, TIMSS also collects information from teachers. Hence, TIMSS is concerned with mathematics instruction and with the content of mathematics, while the focus in PISA is on ‘determining the extent to which students can use what they have learned’ (OECD, 2009, p. 122).
- TIMSS includes a category of test items not found in PISA – namely ‘content rich’ mathematics, which can include items with a strong mathematical language, and context-free items.
In comparing performance on TIMSS Grade 8 and PISA among countries involved in both assessments, Wu (2009) noted that Western countries tend to perform better on PISA, and Asian countries on TIMSS. In a subsequent paper, Wu (2010) attributed differences in performance across the studies to the content balance of the two tests, and to sampling definitions (age-based vs. grade-based). For example, Wu showed PISA students in Western countries tended to have higher number of years of schooling than students in Asian and Eastern European Countries, and this was associated with stronger performance on PISA mathematics and weaker performance on TIMSS mathematics, while TIMSS’ grade-level sample suited Asian countries better.

OVERALL PERFORMANCE ON PISA MATHEMATICS IN IRELAND

Table 1 documents the performance of students in Ireland on PISA mathematics compared with the OECD average across three PISA cycles (2003 to 2009). In 2003 and 2006, Ireland’s overall mean scores were not significantly different from the corresponding OECD average scores. Then, in 2009, the performance of students in Ireland dropped to a level that was significantly below the OECD average. This is also reflected in Ireland’s ranking relative to other countries taking PISA mathematics. In 2003, for example, Ireland ranked 17th among 29 participating OECD countries, and 20th among OECD countries and partners. In 2009, Ireland ranked 26th of 34 participating OECD countries, and 32nd of 65 among OECD and partner countries.

The OECD average of 500 was set in 2003 (when mathematics was a major assessment domain for the first time). By 2009, this had dropped to 495.7, primarily due to the addition of new OECD member countries (Chile, Israel, Slovenia).

Table 1: Mean scores on overall PISA mathematics of students in Ireland and on average across OECD countries (2003-2009)

<table>
<thead>
<tr>
<th></th>
<th>2003 Mean (SE)</th>
<th>2006 Mean (SE)</th>
<th>2009 Mean (SE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ireland</td>
<td>502.8 (2.45)</td>
<td>501.5 (2.8)</td>
<td>487.1 (2.54)</td>
</tr>
<tr>
<td>OECD Average</td>
<td>500.0 (0.63)</td>
<td>497.7 (0.5)</td>
<td>495.7 (0.89)</td>
</tr>
</tbody>
</table>

Table 2 shows the percentages of students in Ireland and on average across OECD countries performing at each proficiency level on PISA mathematics (2003 to 2009). We can see that, even in 2003 and 2006, when overall performance in Ireland was not significantly different from the OECD average, there were proportionately fewer higher-achieving students in Ireland – that is, students performing at Levels 5 or 6. For example, 10.2% of students in Ireland performed at Levels 5-6 in 2006, compared with compared to an OECD average of 13.3. On the other hand, fewer students in Ireland (16.4%) than on average across OECD countries (21.3%) performed at or below Level 1 in the same year.

In 2009, just 6.7% of students in Ireland, compared with an OECD average of 12.7%, achieved Level 5 or 6. Twenty-two percent of students in Ireland and on average across OECD countries performed at or below Level 1.
Table 2: Percentages of pupils at each proficiency level on the PISA overall mathematics – Ireland and OECD average (2003-2009)

<table>
<thead>
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<tbody>
<tr>
<td>Level 6</td>
<td>2.2</td>
<td>1.6</td>
<td>0.9</td>
<td>4.0</td>
<td>3.3</td>
<td>3.1</td>
</tr>
<tr>
<td>Level 5</td>
<td>9.1</td>
<td>8.6</td>
<td>5.8</td>
<td>10.6</td>
<td>10.0</td>
<td>9.6</td>
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<tr>
<td>Level 4</td>
<td>20.2</td>
<td>20.6</td>
<td>19.4</td>
<td>19.1</td>
<td>19.1</td>
<td>18.9</td>
</tr>
<tr>
<td>Level 3</td>
<td>28.0</td>
<td>28.6</td>
<td>28.6</td>
<td>23.7</td>
<td>24.3</td>
<td>24.3</td>
</tr>
<tr>
<td>Level 2</td>
<td>23.6</td>
<td>24.1</td>
<td>24.5</td>
<td>21.1</td>
<td>21.9</td>
<td>22.0</td>
</tr>
<tr>
<td>Level 1</td>
<td>12.1</td>
<td>12.3</td>
<td>13.6</td>
<td>13.2</td>
<td>13.6</td>
<td>14.0</td>
</tr>
<tr>
<td>&lt;Level 1</td>
<td>4.7</td>
<td>4.1</td>
<td>7.3</td>
<td>8.2</td>
<td>7.7</td>
<td>8.0</td>
</tr>
</tbody>
</table>

The performance of students in Ireland on PISA mathematics stands in marked contrast to their performance on reading literacy and science. In 2000, 2003 and 2006, students in Ireland achievement mean scores in reading literacy that were significantly higher than the corresponding OECD average scores. In 2009, the mean score of students in Ireland was not significantly different from the OECD average. Students in Ireland achieved a significantly higher mean score than the OECD average in all PISA assessments between 2000 and 2009. Hence, performance on PISA mathematics is weaker than on reading literacy and science. Ireland is perhaps unique among PISA countries in performing differently across domains. For example, the top 15 countries in PISA 2009 science were above the OECD average on PISA 2009 mathematics.

While Ireland’s overall performance on PISA 2009 was disappointing, there are some positives to be taken from it. The standard deviation in Ireland was 85.6, compared with an OECD average of 91.7. The relatively narrow range of achievement in Ireland can be interpreted as indicating equity in mathematics outcomes, though of course, a higher mean score, combined with a small standard deviation, would have been preferable. Similarly, the finding that male and female students in Ireland did not differ significantly in their performance on PISA mathematics in 2009 can be taken as evidence of equity in learning outcomes. Again, however, it would have preferable if both males and females achieved higher average scores.

WHY DID PERFORMANCE ON PISA MATHEMATICS DECLINE IN PISA 2009?

As noted above, the mean score on overall mathematics in Ireland declined by one-sixth of a standard deviation between 2003 and 2009, with most of the decline occurring between 2006 and 2009. This was the second largest drop between 2003 and 2009. The largest occurred in the Czech Republic, where the mean score fell by 24 points (one-quarter of a standard deviation). Other countries with significant declines between 2003 and 2009 include Sweden (-15), France (-14), Belgium (-14) and the Netherlands (-12), while Mexico (+33), Brazil (+30), Portugal (+21) (all low performers in earlier PISA cycles) and Germany (+10) had significant increases.
The decline in the mathematical performance among students in Ireland coincided with a much greater decline in performance on reading literacy (one-third of a standard deviation) and no change in science.

In Ireland, the percent correct scores on the 32 items that were common to the 2003 and 2009 assessments (the items used to link performance in 2009 to the overall mathematics performance scale established in 2003) were 49.0% in 2003 and 46.7% in 2009. Hence, the overall decline in mathematics performance in Ireland is also evident in the reduction in percent correct scores. The percent correct scores for both 2003 and 2009 also point to the difficulty of PISA mathematics items for students in Ireland.

A number of hypotheses have been put forward to explain the decline in mathematics performance in Ireland (see Perkins, Cosgrove, Moran & Shiel, 2012 for more details). None of these on its own is likely to explain the full extent of the decline. They include:

Changes in demographics

The percentage of first- and second-generation migrant students\(^1\) in the PISA sample increased from 3.5\% in 2003 to 8.3\% in 2009. In 2009, the mean mathematics score of second-generation migrant students was significantly lower (by 25 points) than the mean score of native students. Interestingly, the mean score of first-generation migrant students was higher than that of native students, but not to a significant degree. There was also an increase in Ireland between 2003 and 2009 in the proportion of students speaking a language other than the language of the test, from 0.8\% to 5.6\%. In 2009, the gap between those who reported speaking the language of the test at home and those who spoke another language at home was 35 points, which was statistically significant. It was not possible to compute reliable mean scores for second-generation migrant students or students who spoke a language that was different to the language of the test in 2003, as too few students were categorised in these ways.

Changes in distribution of pupils across grade levels

The proportion of 15-year olds in Ireland enrolled in Transition year increased from 16.7\% in 2003 to 24.0\% in 2009, and students in Transition year showed the largest decline in mathematics between 2003 and 2009 (33 points, or one-third of a standard deviation), perhaps reflecting variation in opportunity to learn mathematics in Transition year programmes.\(^2\) In contrast, the decline in Third year was just 12 points.

Changes in PISA test administration procedures

There was a change to the procedure for test administration in 2009 in Ireland, whereby the PISA test was administered by teachers (76\% of schools) or by external personnel (24\%). This is in line with practice in other countries, and was implemented in an attempt to increase

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\(^1\) The OECD defines migrant students as students who were born in a country with both parents born outside the country (second-generation migrants) and those students who were born outside the country with both parents also born outside the country (first-generation migrants).

\(^2\) See Moran, Perkins, Cosgrove and Shiel (2013) for a description of the approaches to teaching mathematics in Transition year in schools in PISA 2012.
interest in PISA in schools. In earlier cycles of PISA, the test was always administered by an external administrator. In 2009, the mean average mathematics score of students in schools in which the test was administered by teachers was some 6 scale points lower than that achieved by students in schools in which the test was administered by an external administrator. However, the difference can be accounted for by the lower average socio-economic composition of schools in which teachers administered the PISA test.

**Sampling factors**

An analysis of school-level performance in PISA 2009 identified a number of low-performing schools – schools with exceptionally low achievement in reading literacy, and it was found that mathematics achievement was also exceptionally low in these schools (a school-level average of 390 points, compared with a school-level mean reading score of 371). The presence of these schools in the 2009 sample can be attributed to random sampling fluctuation, and it is unlikely that as many will appear in the PISA 2012 sample.

**Disengagement from PISA testing**

There is some evidence that students in Ireland had lower levels of engagement with the PISA 2009 assessment than with earlier PISA assessments. According to Perkins et al. (2012), when students took block M1 in mathematics (one of three clusters of mathematics items) at the beginning of a test booklet in PISA 2009, there was an increase of 2 percentage points in the proportion of items in this block that were skipped or were not reached by students in Ireland. However, when block M1 appeared in the final quarter of a test booklet, there was an increase of 7.2% in the proportion of skipped or not reached items. In contrast, on average across OECD countries, there were no changes in the proportions of skipped or not reached items between 2006 and 2009. It may be that ‘survey fatigue’ or a perception that PISA was not ‘high stakes’ contributed to students in Ireland deciding to skip test items, especially those towards the end of the test booklets.

A number of measures designed to improve mathematics performance among students in post-primary schools have been put forward. For example, in the *National Strategy to Improve Literacy and Numeracy* (DES, 2011), there are proposals for the introduction of standardized testing for all students in mathematics (and reading) in the second year of post-primary schooling and a target has been set for the proportion of students taking Leaving Certificate Mathematics at Higher Level (30% by 2020). However, even prior to the publication of the outcomes of PISA 2009, there was a sense that student performance and interest in mathematics needed to improve.

**PERFORMANCE ON PISA MATHEMATICS CONTENT AREAS**

We need to go back to PISA 2003 for data on the performance of students in Ireland on PISA mathematics content areas. Table 3 provides mean scores for students in Ireland, and on average across OECD countries. In Ireland, students performed significantly above the OECD average Change & Relationships and Uncertainty, not significantly different from the OECD average on Quantity, and significantly below the OECD average on Space & Shape.
Table 3: Mean scores on the PISA 2003 mathematics content scales by gender – Ireland and OECD average

<table>
<thead>
<tr>
<th>Content Area</th>
<th>All Ireland</th>
<th>Male Ireland</th>
<th>Female Ireland</th>
<th>All OECD</th>
<th>Male OECD</th>
<th>Female OECD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change &amp; Relationships</td>
<td>506.0</td>
<td>512.2</td>
<td>499.6</td>
<td>498.8</td>
<td>504.3</td>
<td>493.3</td>
</tr>
<tr>
<td>Quantity</td>
<td>501.7</td>
<td>497.2</td>
<td>506.1</td>
<td>500.7</td>
<td>503.8</td>
<td>497.6</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>517.2</td>
<td>524.9</td>
<td>509.4</td>
<td>502.0</td>
<td>508.3</td>
<td>495.6</td>
</tr>
<tr>
<td>Space &amp; Shape</td>
<td>476.2</td>
<td>488.9</td>
<td>463.4</td>
<td>496.3</td>
<td>504.6</td>
<td>487.9</td>
</tr>
</tbody>
</table>

Students in Ireland performed particularly poorly on Space & Shape, with a score that was some 20.1 score points below the corresponding OECD average. Further, the gap on Space & Shape in favour of male students in Ireland was 25.5 points, compared with an OECD average difference of 16.7 points in favour of males. Oldham (2002) has pointed out that PISA Space and Shape does not cover synthetic geometry, which was the main focus of geometry in the JC mathematics syllabus at the time (along with transformational geometry). Similarly, Close (2006) found that none of PISA Space & Shape items could be mapped onto the Junior Certificate mathematics content areas of geometry or trigonometry. It might, of course, be argued that Irish students need more experience with some aspect of PISA Space & Shape, including the interpretation of 3-D shapes in different dimensions. Readers are referred to Shiel, Perkins, Close and Oldham (2007) for annotated examples of PISA Space & Shape items, as well as items in other PISA content areas.

The strong performance of students in Ireland on Uncertainty was unexpected since probability was not emphasised to any great extent on the pre-Project Maths Junior Certificate syllabus. However, students may have acquired some knowledge of Uncertainty through their study of the primary school curriculum (NCCA, 1999), which includes probability in the Data strand.

**PRIMARY LEVEL MATHEMATICS AND PISA**

We cannot establish direct links between performance on PISA mathematics and the performance of pupils in Ireland at primary level. However, some broad comparisons can be drawn between the performance of 15-year olds on PISA, and the performance of pupils in primary schools on the National Assessments of Mathematics Achievement, which was administered to pupils in Second and Sixth classes in 2009 (see Eivers et al., 2010) and on the Trends in International Mathematics and Science Study (TIMSS), which was administered to pupils in Fourth class in Ireland and in 50 other countries in 2011 (see Eivers & Clerkin, 2012; Close, 2013).

A clear limitation of national assessments is that they do not provide information on the relative position of pupils in Ireland and their counterparts in other countries. Test items are selected with reference to the curriculum for primary schools, and the performance of students in Ireland. Performance is scaled with reference to students in Ireland only. However, we can look at performance across mathematics content areas and processes, vis-à-vis PISA. In the
2009 National Assessment of Mathematics Achievement, pupils did quite well on Shape & Space in Second class (72.7% correct), but less well in Sixth (58.8%) (Clerkin & Gilleece, 2011). Similarly, pupils did well on Measures in Second class (49.0%), but less well in Sixth (38.2%). Performance was more stable across the Second and Sixth classes on Number & Algebra (58.7% vs. 57.6%), while there was an increase in performance on Data items between Second and Sixth (56.1% vs. 63.5%).

The disappointing performance on Measures at both Second and Sixth classes is of concern, as many of the measures items are also classified as Apply & Problem Solve. Pupils in Second class achieved a mean score of 48.6% on Apply & Problem Solve, while pupils in Sixth class achieved a mean score of 44.2%. On the other hand, performance on lower-level processes was stronger. For example, pupils in Second class achieved 58.2% on items classified as Implement, while pupils in Sixth achieved 58.6%.

TIMSS 2011 allows us to benchmark the performance of pupils in Ireland against that of pupils in other countries, though care should be exercised in interpreting TIMSS data, since, unlike PISA, many developing countries contribute to the international average scores. Students in Ireland performed relatively better on Number, and relatively less well on Geometric Shapes & Measures and on Data, compared with their performance on the test as a whole. Pupils in Ireland had a mean score on Number that was 79 points below that of pupils in the highest-scoring country on that domain (Singapore). The gap between pupils in Ireland and pupils in the highest scoring country on Geometric Shapes & Measures (Korea) was 87 points, while the gap between pupils in Ireland and in the highest-scoring country on Data Display (also Korea) was 80 points.

As discussed by Close elsewhere in this volume, TIMSS 2013 also revealed weaknesses among pupils in Ireland on items classified as Reasoning (510 points), compared with those classified as Knowing (539) and Applying (529). These outcomes are not inconsistent with PISA, where, as noted earlier, performance on Space & Shape items requiring reasoning was poor. The results of national assessments at primary level and of TIMSS 2011 support the view that overall performance on mathematics could be better, and that there are significant weaknesses with respect to Geometry/Measurement and to higher-level skills such as reasoning and problem solving.

CONCLUSION AND A LOOK AHEAD TO THE PISA 2012 OUTCOMES

The poor performance of students in Ireland on PISA 2012 will have surprised many educators. However, performance on PISA mathematics in general was not significantly different from the OECD average in previous PISA cycles, and performance on Space & Shape was especially problematic. Although, clearly, the function of Project Maths is not to raise scores on future PISA assessments, it would seem import to undertake an in-depth comparison between Project Math and PISA, in terms of underlying frameworks and examinations/test items, with a view to ascertaining in more detail the specific aspects of mathematics on which students in Ireland struggle. It would also be useful to perform an audit on Project Maths using the PISA process categories of Reproduction, Connections and Reflection, with a view to examining the balance between the three.
Significant gaps in the mathematical knowledge of pupils in primary schools need to be addressed. These include weaknesses in areas such as Geometry, Measures, and, more broadly, Reasoning and Problem Solving. A stronger emphasis on these aspects at primary level could well contribute to improvement in mathematics performance at post-primary level.

The timing of PISA 2012, in which mathematics was a major assessment domain, was not ideal from an Irish point of view. This is because students in Third year (just over 60% of the sample) would not have experienced any instruction under Project Maths, as they comprised the last cohort to study under the old Junior Certificate mathematics syllabi. Nevertheless, there will be much that is of interest to mathematics educators in the PISA 2012 mathematics outcomes, to be released by the OECD in December 2013. In particular, like PISA 2003, PISA 2012 will summarise performance on mathematical content areas, and will also provide data on the performance of students on three new sub processes (described as subcomponents of matheamatisation): *formulating* situations mathematically; *employing* mathematical concepts, facts, procedures, and reasoning; and *interpreting*, applying and evaluating mathematical outcomes (OECD, 2013). Results of a computer-based assessment of mathematics, which was administered in a subset of PISA 2012 countries, including Ireland, will also be provided. Early in 2014, the Educational Research Centre will publish a report comparing the performance of students in the Initial Project Maths schools with that of students in other PISA 2012 schools. This will complement earlier reports on the implementation of Project Maths (Cosgrove et al., 2012; Jeffes et al., 2012). PISA 2012 will also provide an opportunity to reflect on the outcomes of PISA 2009 in more detail and determine if they represented a once-off drop, or a more permanent decline in achievement.

**REFERENCES**


USING LESSON STUDY TO HELP PRE-SERVICE TEACHERS BRIDGE THE THEORY-PRACTICE GAP

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The challenge in bridging theory and practice within the teaching of mathematics is not a new problem for researchers. It has been widely reported that a major challenge for teacher educators is helping pre-service teachers put into practice what they have learned in their teacher education programme (Allen, Butler-Made, & Smith, 2010; Cheng, Cheng & Tang, 2010; Korthagen, 2010). Indeed, Allen et al (2010) identified “being able to strike a balance between theory and practice” (p.647) as one of the greatest challenges for all pre-service teacher education programmes since the professionalization of teaching. Although vigorous attempts have been made to address this issue it remains a central problem in teacher education and has become ever more urgent (Wittmann, 2001). This paper examines this theory-practice problem by reporting on a study that researched how a curriculum specialisation in mathematics education, modelled on the principles of Japanese Lesson Study, could assist pre-service primary teachers to bridge the theory-practice gap. Through examining the pre-service teachers’ lesson planning, their weekly journals, their lessons, their reflections and their interview data, the findings revealed that using the Lesson Study model is an effective approach to help pre-service teachers bridge the theory-practice gap. They showed vast improvements in their lesson planning and implementation including: demonstrating a greater understanding of important components of the lesson and displaying a more knowledge-based anticipation of students’ response

INTRODUCTION

The objective of teacher education programmes is to provide pre-service teachers with a set of skills which enable them to cope with the complex situations they find themselves faced with in their everyday teaching (Cheng et al 2010). However, it seems the challenge of teacher education is to help these pre-service teachers put what they have learned in the teacher education programme into practice (Allen et al, 2010; Cheng et al 2010). Studies of teacher education have repeatedly revealed a disparity between the theory learned by pre-service teachers in their teacher education programmes and the subsequent classroom practice of these teachers (Allen et al 2010; Cheng et al 2010; Korthagen, 2010). Indeed one of the main criticisms of teacher education programmes is their failure to enable students to bridge this theory-practice gap (Allen, 2009).

Given that as early as 1904, Dewey reported on the gap between theory and practice and presented proposals to bridge this gap, it is remarkable that it remains such a central issue in teacher education today. However, closer examination of the contributory factors to the theory-practice gap reveals the sheer complexity of the issue. Korthagen (2007) highlights “the complex psychological and sociological phenomena influencing educational processes” as posing particular difficulty in finding suitable solutions to the problems causing the theory-practice gap (p. 306). Robinson (1998) also recognised the complexity of the issue.
emphasising that “narrowing the research-practice gap is not just a matter of disseminating research more effectively or of using more powerful influence strategies” (p. 17). This viewpoint acknowledged that the cause of the theory-practice gap was not simply the disconnect between university researchers and classroom practitioners but that the root of the problem lay much deeper. Several factors have been identified as contributing to the theory-practice problem. These include: the complexity of teaching (Allen et al 2010; Korthagen 2010; Leikin and Levav-Waynberg, 2007; Hiebert, Stigler, Jacobs, Bogard Givvin, Garnier, Smith, Hollingsworth, Manaster, Wearne & Gallimore, 2005), pre-service teachers’ preconceptions (Cheng et al, 2010; Hiebert, Morris, Berk & Jansen, 2007; Joram & Gabriele, 1998; Korthagen, 2010; Lortie, 1975), socialisation towards patterns existing in schools (Cohn, 1981; Frick, Carl & Beets, 2010; Veenman, 1984; Zeichner & Tabachnick, 1981), the relationship between researchers and practitioners (Klein, 1992; Korthagen, 2007; Leikin & Levav-Waynberg, 2007; Wittmann, 1984) and the inadequacy of the theory (Cheng et al, 2010; Korthagen, 2010; Hiebert, Gallimore & Stigler, 2002; Hiebert et al, 2007).

Several solutions have been suggested throughout the years. These include ideas such as school-university partnerships and alternative forms of knowledge provision e.g. practice-based knowledge and alternative teaching strategies such as teacher modelling. However one approach which has been gaining increasing support is using the Japanese model of lesson study to reform teaching (Cohan & Honigsfeld, 2007; Hiebert et al, 2002; Lewis, Perry & Murata, 2006; Sims & Walsh, 2009). Lesson Study has been cited as a suitable solution as it directly connects “to the work of teachers and their students”, is “participant driven and grounded in enquiry, reflection and experimentation”, is collaborative in nature and involves “the sharing of knowledge” (Darling-Hammond & McLaughlin 1995, p.2). Cohan and Honigsfeld (2007) found that incorporating lesson study into teacher preparation was very beneficial. “Every candidate benefited from a high level of collegial support, started developing a positive professional self-concept, and exhibited dispositions that teacher educators hope to expect from all future teachers” (p. 87).

**BRIEF DESCRIPTION OF LESSON STUDY**

Lesson Study was introduced in Japan as a form of professional development. Lesson Study is a translation of the two Japanese words: jugyo and kenkyu, which mean lesson and study, respectively (Fernandez & Yoshida, 2004). As suggested by the term, lesson study is a process, Japanese teachers regularly engage in, to examine their teaching practice through the careful planning and observation of lessons (Cohan & Honigsfeld, 2007). Stigler and Hiebert (1999) describe lesson study as an opportunity for teachers to examine their practice “with new eyes”. Figure 1 graphically represents the lesson study cycle. As can be seen from Figure 1, lesson study consists of three critical phases: the planning phase, the implementation phase and the post-lesson phase. The focus of the lesson study cycle is the research lesson. Although these research lessons are taught in the teachers’ actual classrooms, they differ from everyday lessons in that they are comprised of a number of special features. Lewis and Tsuchida (1998) identify the following special features of research lessons:
They are carefully planned, sometimes over several months, typically in collaboration with at least one colleague.

They are focused. They focus either on a specific goal, such as helping students be active problem-solvers, or developing a successful approach to a specific topic, for example subtraction with regrouping.

They are observed by other teachers. Sometimes the observers are limited to the other teachers involved in the lesson study process whereas sometimes they can be open to observers from the whole of Japan.

They are recorded. This can be done in a number of ways: videotaped, audiotaped, narratives or copies of students’ work.

They are discussed. Subsequent to the teaching of the lesson, the strengths and weaknesses of the lesson are discussed. Particular emphasis is placed on the effectiveness of the lesson on achieving the learning goals.

**Figure 1: Lesson Study Cycle (Lewis, Perry, Friedkin, Roth, Baker & McGrew, 2012)**

The Japanese approach of lesson study, which treats theory and practice as inseparable entities, has been gaining increasing support in the reform of teaching (Cohan & Honigsfeld, 2007; Hiebert et al, 2002; Lewis, Perry & Murata, 2006; Sims & Walsh, 2009) and more recently researchers have examined how the lesson study approach can be adopted in pre-service teacher education (Hart, Alston & Murata, 2011; Burroughs & Luebeck, 2010; Sims & Walsh, 2009; Cohan & Honigsfeld, 2007).

**LESSON STUDY IN IRISH INITIAL TEACHER EDUCATION**

As teacher educators have started to recognise the potential of lesson study to improve pre-service teacher practice, lesson study has also been adapted and developed in several Irish
Colleges of education. Corcoran (2007) acknowledges that lesson study provides an opportunity for prospective teachers to develop a meaningful understanding of the primary mathematics curriculum by studying children during mathematics lessons, by optimising the use of the available supporting documents and organising classrooms to maximise the development of mathematics process skills (p.286). Studies of lesson study undertaken in colleges of education in Ireland have shown that lesson study has the potential to greatly influence pre-service teacher education. Corcoran (2007) found that through lesson study pre-service teachers can improve their mathematical subject knowledge. Similarly Leavy (2010) reported that lesson study allowed her pre-service students to deepen their understanding of statistics. Corcoran and Pepperell (2009) found that lesson study enabled pre-service teachers to develop both their mathematical and pedagogical proficiencies. The pre-service teachers in their study showed significant knowledge development particularly with respect to their foundation knowledge, as classified by Rowland (2003). Leavy and Sloane (2008) reported that the experiences in observing the impact of teaching design lessons on student learning served as the springboard for the development of understandings than could not have been facilitated within college-based clinical contexts (p.168). These studies also highlighted several aspects of lesson study that were pivotal in pre-service teacher learning. These were: attending to what and how students learn mathematics (Corcoran & Pepperell, 2009), collaborative planning (Leavy & Sloane, 2008), observing and reflecting on the practice of teaching (Leavy & Sloane, 2008), and learning about the effects of diverse methods of teaching on students’ learning (Corcoran, 2007).

METHOD
The participants in this study were a group of final year pre-service primary teachers in Mary Immaculate College who had selected the teaching of mathematics as their specialist area of study in the final semester. At this stage in their degree programme the pre-service teachers had completed all the compulsory mathematics education courses and all their required teaching practice placements of their degree. The group consisted of 25 students, 11 of which were male and 14 of which were female. 24 of the students had chosen this mathematics specialisation course as their first choice. The one remaining student had chosen it as their third choice. Three members of the mathematics education faculty were in charge of the course and they acted as mentors to the pre-service teachers during the lesson study cycle.

OVERVIEW OF THE CURRICULUM SPECIALISATION COURSE
The Lesson Study research was carried out over a 12-week spring semester in the context of a curriculum specialisation in mathematics education course offered to third year pre-service teachers in Mary Immaculate College. As part of this course the pre-service teachers were required to take part in a Lesson Study cycle. The course reflected the main components of the Japanese Lesson Study process as outlined in Figure 1. Each of the participants was involved in every aspect of the process; the planning, teaching, analysing and revising of the mathematics lesson. The initial weeks of the semester involved introducing the pre-service teachers to the Lesson Study process and preparing for Lesson Study. The participants were divided into five groups of five and then each group was assigned a different topic of either
algebra or probability. The algebra research lesson focused on growing patterns. The probability lessons were sequential and focused on describing likelihoods, comparing and explaining likelihoods, ordering likelihoods and sampling. The groups then researched the relevant theory surrounding their topic. The members of the group met twice weekly to collaboratively work on the lesson preparation. Three members of the mathematics education faculty were responsible for instructing and supervising the lesson study process. The groups met with at least one faculty member three or four times during this planning phase where they received feedback on their lesson planning.

The next phase of the lesson study process was the implementation stage. This involved one pre-service teacher from each group teaching their respective lesson to a Fifth class in a primary school while the other group members observed the lesson. Their observations involved evaluating student thinking and learning in relation to the concepts being taught, engagement with the content of the lesson and behaviour of the students during the lesson.

Following the teaching of the lesson all group members met with at least one faculty member for a post-lesson collaborative reflection. The group members and faculty members shared their observations on the first lesson and suggestions were proposed on how the lesson should be modified. The group members then met twice and modified the lesson accordingly, in preparation for the re-teaching of the lesson. The next phase involved the re-teaching of the lesson to different 5th class students in a different primary school. In some of the groups, a different teacher taught the second lesson whereas in some groups the same teacher re-taught the lesson. Once again the rest of the group members observed the teaching of the lesson.

After the final teaching of the lesson the groups met to reflect on revised lesson. These meetings included: one ninety-minute class, where groups were given the opportunity to discuss their lessons with the faculty members and watch a video recording of their second lesson, and three or four meetings in their small groups. The aim of these meetings was to consolidate what they had learned from the lesson study process and to prepare for the presentations where they had to report back to their peers on their lesson study experience. Finally the groups had to report on their lesson study experience. This included a group presentation to their peers, submitting an individual reflective journal and submitting the logs they kept from their meetings.

**DATA COLLECTION**

In the study only qualitative data were collected. However, a variety of data collection techniques were used. The primary qualitative methodology was participant observation which included in-class observation of teacher practice and observations of Lesson Study group meetings (which were also recorded and transcribed). Other data collection methods included pre-service teacher questionnaires, examples of pre-service teachers work, pre-service teacher interviews (recorded and transcribed), pre-service teacher presentations and pre-service teacher reflective journals. Modelled on the procedures used by Sloane and Leavy (2008) and by Leavy, Murphy and Farnan (2009), data collection methods adopted were closely synchronised with the stages of the lesson study process. Table 1 illustrates the relationship between the data collection procedure and the Lesson Study cycle.
### STEPS OF THE LESSON STUDY CYCLE | DATA COLLECTION STRUCTURE AND METHOD
---|---
**STEP 1: Collaboratively Planning the Research Lesson** | Audio taped meetings with researcher
| Written logs of group discussions
| Record of resources used to research and design lesson
**STEP 2: Seeing the Research Lesson in Action** | Observation of lesson by researcher
| Observation notes of lesson study group members
**STEP 3: Discussing the Research Lesson** | Audio taped group meetings of researcher, faculty member and lesson study participants following the lesson
**STEP 4: Revising the Lesson** | Written logs of group discussion
| Record of changes made to revised lesson and justification of those changes
**STEP 5: Teaching the New Version of the Lesson** | Videotaped lesson
| Observation of lesson by researcher
| Observation notes of lesson study group members
**STEP 6: Sharing Reflections about the New Version of the Lesson** | Written logs of group discussion
| Record of changes made to revised lesson and justification of those changes
| Videotaped group presentation of their work
| Group interview with researcher

Table 1: Synchronisation of the data collection methods with the lesson study process

### DATA ANALYSIS

Nvivo 10 was used to analyse the qualitative data. Throughout the Lesson Study process dominant themes were identified from the data collected and these themes were then further classified into categories. These categories were then validated across the various data sources. These in turn provided rich insights into the growth of these pre-service teachers during the Lesson Study process.

The Knowledge Quartet (Rowland, Huckstep & Thwaites, 2003) was used as a framework for the identification of content knowledge observed in teaching. The knowledge quartet (Figure 2) is a framework which allows mathematics educators “identify ways in which the trainees' mathematics content knowledge 'played out' in their teaching” (p. 97). It focuses on both subject matter knowledge and pedagogical content knowledge. In this way it allowed the researcher to analyse what theory they had learned from their degree programme they were putting into practice. The knowledge quartet was used as a tool by the researcher in their observation of the classes the pre-service teachers taught. While the researcher observed the lessons they identified aspects of the teacher behaviour which signified knowledge from a particular dimension of the knowledge quartet. For example, one group of pre-service teachers used pie charts to represent the law of large numbers in their lesson. This representation was particularly effective and demonstrated the pre-service teachers’ transformation knowledge.
FINDINGS

Examination of the changes the pre-service teachers made to their lesson plans, observations from the two lessons each group taught and observations of the group meetings provided valuable insights into the pre-service teachers’ learning throughout the lesson study process. The changes that they made to their lessons were based on the feedback they received from the faculty members teaching the course, from their reading of research relevant to their topics and from the observations that were made during the first teaching of the lesson. The findings from this data show that the lesson study process helped the pre-service teachers become more aware of several theoretical aspects of teaching by carefully analysing the lessons they planned and taught. The following paragraphs look in more detail at the particular aspects of teaching which were most greatly affected.

Using a context in mathematics teaching

The importance of using a context in the teaching of mathematics is strongly encouraged in the mathematics primary school curriculum. The guidelines state that “for children to really understand mathematics they must see it in context” (Department of Education and Science, 1999). The need to use a context to explore their mathematical concepts was something which
all the groups recognised at the beginning of the lesson study process. However the contexts originally chosen by three of the groups were not age appropriate or meaningful for a Fifth class group - for example, one group had chosen the fairy tale, Goldilocks and the Three Bears, as the context for their lesson. This issue was flagged by the faculty members teaching the course in the initial group meeting and the groups subsequently changed their contexts. These changes were then proven to be very successful in the subsequent teaching of the lessons, in particular for the probability group teaching describing likelihoods. Rather than the fairy tale context they had initially chosen they elected to show a video clip of the TV programme, Top Gear, and discuss the probability of two racers winning a race. In their meeting after the first teaching of the lesson, all of the pre-service teachers remarked how important their context had been in immediately sparking the students’ attention and engaging them in the lesson.

**Anticipating students’ responses**

In the planning phase of lesson study particular attention is paid to anticipating student responses in order to enable teachers to be better prepared to deal with issues that might arise during the course of the lesson (Fernandez & Yoshida, 2004). Although many of the Japanese teachers involved in lesson study are able to “draw on their past experiences” and “observations of their current students” (Fernandez & Yoshida 2004, p. 7) in order to anticipate student responses, the pre-service teachers relied on the experience of the faculty members teaching the course and research relevant to their topics to help them to anticipate student responses. Anticipating student responses allowed the pre-service teachers deal effectively with misunderstandings which occurred during the lesson. One example of this was in the algebra lesson where a student wrongly identified the algebraic pattern the class were working on. The teacher whose group had prepared for this error was able to guide the student to the correct algebraic pattern using suitable manipulatives. The teacher later commented that knowing this error may arise meant she had a solution ready when it did actually arise. This helped maintain the flow of the class and gave the teacher confidence.

**Role of a mentor in the Lesson Study process in initial teacher education**

However in the same algebra lesson mentioned above some responses which had not been anticipated arose and the teacher found some of these difficult to deal with. This occurred when the students were working in pairs to identify an expression to represent a new algebraic pattern. The students were calling out the expressions they had come up with in their pairs and the teacher was finding it difficult to recognise “on the spot” if their solutions were correct or incorrect. The members of faculty teaching the course suggested several strategies the teacher might use to deal with student responses. One of these was getting the students to test their answers themselves. In the following class she successfully implemented this strategy by getting the student to solve their expression and check their answer using cubes. Similarly, in other groups who were having difficulty dealing with student responses particularly if they were incorrect the members of faculty teaching the course suggested that they get the pupils to justify their answers and then discuss them as a class group. This proved to be an effective strategy for the pre-service teachers. In the second teaching of the lesson
incorrect responses were explored and corrected, whereas in the first lessons many incorrect responses were ignored.

Another extremely important aspect of the initial phase of lesson study is the identification of precise learning goals. The first draft of three of the groups’ lesson plans either included no learning objectives or vague learning objectives. As a result some of the activities they included in the lesson served no purpose in achieving the objectives they had for the lesson. In other cases in meant that the activities planned for the lesson were not sequential, students were expected to complete harder tasks before they had built up the required understanding. Also because it was not clear what the students were supposed to be learning, it was impossible to determine if it was accomplished at the end of the lesson (Hunt, Wiseman & Touzel, 2009). Again the groups, received feedback from the faculty members teaching the course, to this effect and the pre-service teachers developed much clearer precise objectives and changed the activities accordingly. This led to the pre-service teachers assessing the success of their lessons in the feedback meetings by referring to how well they felt the learning goals were achieved in the lesson. Hiebert et al (2007) “propose that focusing on students’ learning and explaining such learning (or its absence) in terms of instructional episodes provides a targeted but comprehensive and systematic path to becoming an effective teacher over time” (p. 48).

Finally the conclusions of the pre-service teachers’ lessons often failed to reflect the purpose of a conclusion. Many of them were very lower order or unrelated to the tasks which the students had previously done. Hunt et al (2009) say that “the conclusion of a lesson is often neglected by some teachers because they tend to concentrate their attention on the body of the lesson” (p. 70). At the early feedback meetings the members of faculty teaching the course stressed the importance of the conclusion in consolidating, reinforcing and reviewing what had been covered in the lesson. They also highlighted the diagnostic assistance it offers teachers in preparing to teach future lessons. Following this feedback the pre-service teachers adapted their conclusions and they remarked in the feedback meetings after the first teach how the concluding activities such as thumbs up, thumbs down and the use of whiteboards had given them instant feedback on the success of their lessons.

DISCUSSION

Implementing the lesson study model as part of the primary pre-service course proved to be an effective approach to help pre-service teachers bridge the theory-practice gap. The pre-service teachers developed several valuable skills; they learned the importance of context in engaging their students with the mathematics they are teaching, they learned the importance of understanding students’ thinking about the concepts and anticipating their responses; they learned valuable instructional techniques and they learned to analyse their lessons in view of learning goals.

Similar to the findings of previous studies, which looked at implementing lesson study approaches in pre-service teacher training, the researcher found that the pre-service teachers learned to think more deeply about learning goals (Sims & Walsh, 2009). They began to analyse the success of their own lessons in terms of these learning goals. Hiebert et al. (2007)
suggest “that assessing whether students achieve clear learning goals and specifying how and why instruction did or did not affect this achievement lies at the heart of learning to teach from studying teaching” (p.48).

However the important role of the mentor in this process cannot be overlooked. The pre-service teachers had already been taught the importance of objectives and conclusions numerous times during their initial teacher training. The mentors, in this case the members of faculty teaching the course, brought aspects of teaching which the pre-service teachers were overlooking, such as objectives and conclusions to their attention. Then during the course of the lesson study process they began to see how these aspects of teaching they were overlooking affected the overall success of their lesson.

“Learning from teaching is a critical component of successful teacher education” (Sims and Walsh 2009, p.732). The Lesson study cycle these teachers engaged in offered them an opportunity for this to happen. Implementing a lesson study approach, with the necessary support, in initial teacher education has the potential to help pre-service teachers bridge the theory-practice gap.

REFERENCES


This paper reports on a study related to young children’s understanding of 2-D shapes. Task-based interviews were conducted with twenty children aged 4 – 8 years from a large urban girls’ school in north County Dublin. The foci of the study were: shape drawing and recognition, the assignation of van Hiele levels, the recording of differences between class levels and investigation of whether calls for an earlier van Hiele level were justified. The findings indicate that an overuse of inadequate prototypical images may lead to necessary attributes of shapes being ignored. There was not the significant improvement in the performance on shape identification tasks that one may have expected between the class levels. However, the relatively poor performance of the youngest children on tasks involving non-examples of shapes implies some progress in geometric understanding as children move through junior primary classes. Children used both visual and property-based responses which suggested that children may move between van Hiele levels depending on the task involved. The study shows the need to expose children to a much greater variety of experience with 2-D shapes than seems to be the case at present.

INTRODUCTION

Spatial abilities are becoming ever more important in an increasingly technology driven society. Lowrie and Smith (2003) note that since children are exposed to a variety of visual stimuli in their daily lives, they need to develop visual skills that allow them to make sense of the world around them. In the Irish context, the strand of Shape and Space in the mathematics curriculum (Government of Ireland, 1999) facilitates development of these geometric skills. The focus of this paper is on the recognition and definition of four common 2-D shapes - the circle, square, triangle and rectangle – by children aged 4 – 8 years. Pierre and Dina van Hiele became prominent in the area of research into geometry in the 1960s when they devised levels of geometric awareness. Zevenbergen, Dole and Wright (2005) explain that the van Hieles regarded effective teaching, rather than age or maturation, as necessary for progression in geometric thinking. One of the authors of this paper, Edel Collins (EC) conducted research in a primary school to see what changes were occurring in pupils’ knowledge of abovementioned 2-D shapes between Junior Infants and Second class. A series of shape selection and drawing tasks were used to assess children’s understanding and knowledge. A snapshot of current levels of geometric awareness highlighting strengths and inaccuracies in pupil knowledge, is provided. In the concluding section, we will give some consideration to the implications of these findings for the teaching of geometric concepts.

VAN HIELE LEVELS OF GEOMETRIC THINKING

Pierre and Dina van Hiele posit five levels of geometric thinking. Drawing on the work of Hoffer (e.g., 1981), Burger and Shaugnessy (1986, p.31) describe the levels as follows:
Level 0 (Visualization). The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.

Level 1 (Analysis). The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established.

Level 2 (Abstraction). The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.

Level 3 (Deduction). The student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, definitions, and theorems.

Level 4 (Rigor). The student can compare systems based on different axioms and can study various geometries in the absence of concrete models.

The levels were considered by the van Hieles to be sequential, discrete and hierarchical (van Hiele, 1959/1984). However, this static view of the levels has been disputed. For example, on the basis of interviews with students from early primary through to college level, Burger and Shaugnessy (1986) suggest that the levels are dynamic and that students are likely to oscillate between levels particularly at points of transition. While the movement back and forth between levels emerged for those in their study between levels 1 and 2, their suspicion that it was also true for those between levels 2 and 3 has been confirmed by others (Gutiérrez & Jaime, 1998; Gutiérrez, Jaime, & Fortuny, 1991). Indeed Gutiérrez et al. (1991) propose that students use different levels of reasoning depending on the problem to be solved and that there are degrees of acquisition within each level. Furthermore, Clements et al. (1999) who conducted clinical interviews with young children aged 3 – 6 years, maintain that a prer.recognitive level exist before the visual level where children cannot distinguish (2-D) shapes such as circles, rectangles and triangles from non-exemplars of classes of these shapes. According to them, these children “are in transition to, instead of at, the visual level” (p.205, italics in original). They also advocate a renaming of the visualization level as “syncretic”, since a level does not consist of ‘pure’ forms of knowledge – for example, visualization includes both visual/imagistic knowledge and declarative knowledge (‘knowing what’).

The Inspectorate Evaluation of primary schools carried out in 86 Irish primary schools in 2005 found that pupils in a quarter of classrooms, while able to recognise shapes, had little understanding of their properties. It was also found that planning included an overemphasis on number work and in a third of classrooms there was a dependence on textbook and paper exercises and didactic exposition by the teacher (DES, 2005). In the 2009 national assessment of mathematics, pupils’ performance in the domain of Shape and Space decreased between Second and Sixth class (Eivers et al., 2010). The recently published TIMSS results show that reasoning in the domain of Geometric shapes and Measures is an area of relative weakness for 4th class pupils in Ireland (Eivers & Clerkin, 2012). In particular, it was found that while these pupils did well on recognition of 2-D shapes, they performed relatively poorly on items
involving reasoning and problem-solving with geometric shapes. We were interested in finding out if children had sufficient experience of the visualization level to allow progress to the next level (analysis) and if this might be at the root of difficulties that children experience in the area.

**METHODOLOGY**

Task-based interviews adapted from those used by Clements et al. (1999) were administered to twenty children from a large urban vertical girls’ school in north County Dublin. There were four pupils from each class stream: Junior Infants (JI), Senior Infants (SI), First class and Second class. Children were chosen by their class teachers on the basis of their willingness and confidence in communicating. Children were interviewed individually and interviews were videoed. A range of mathematical abilities was represented.

Children were first asked to draw their favourite shape and discuss their knowledge of the shape including examples from the environment and what was important when drawing this shape. Following on from this, variations of drawing tasks were introduced. For example, children were asked to draw “another” triangle. Burger and Shaughnessy (1986) used this task to investigate the properties that students varied to make “different” figures and explore whether the students thought the number of possible triangles was finite or infinite. If the child could conceive an infinite number of triangles he/she was placed on a higher van Hiele level than a child who could only draw one type of triangle. In the case of rectangles, children were requested to draw a “bigger rectangle”, a task used by Elia et al. (2003) to determine a van Hiele level. Children were also required to complete shape identification tasks in turn for the circle, the square, the triangle and the rectangle. In each task, the child was shown a set of drawings of examples and non-examples of the particular shape and was asked to colour, say, all squares.

Goldin (2000) describes the structured, task-based interview for the study of mathematical behaviour as involving minimally an interviewee (the problem solver) and an interviewer (the clinician) interacting in relation to one or more tasks (questions, problems or activities) introduced in a pre-planned way. Thus the interviewees’ interactions are not merely with the interviewers but also with the task environment. Clinical interviews were pioneered by Swiss psychologist Jean Piaget. Ginsburg (1997) explains that in the clinical interview the examiner must treat each individual child differently and constantly interpret the child’s responses in order to follow up incisively, as the child will respond more to their subjective interpretation of the task than the objective reality. Because children were likely to interpret the tasks differently, a section in the interview schedule was created for possible contingencies such as the child not being familiar with one of the shapes being investigated or failing to follow instructions on the task.

While the drawing task facilitated quick insight into whether children know what each shape looks like, the development of motor skills might affect drawing of the shapes and thus any assessment was interpreted in terms of to how the child had responded to the identifying and defining tasks. For example, if a child did not draw straight lines on the rectangle, square or triangle, EC checked the child’s accuracy on the shape identification tasks before making any
judgement on whether she was aware that ‘straight lines’ was an attribute of these shapes. Following the interviews children’s descriptions of shapes were classified as “visual-based” or “property-based”. Reliance upon a mental image indicated a visual-based rather than a property-based understanding of the shape while identification of correct attributes suggested that responses be categorised as property-based. Below we report on some of the findings that emerged from these interviews.

**FINDINGS**

**Circle**

The circle was the shape that children recognised most easily. Reasons given included “I don’t like the points on the other shapes, I like round” (Dorothy) to “It’s like an ‘O’” (Caitlin), “It’s like a face” (Blair) and “An orange is made out of a circle” (Abigail) [1]. The circle was also compared to the ‘stop’ sign, the sun, traffic lights, drains and the top of pencils. All pupils were capable of drawing the circle with older pupils’ drawing being more accurate. Inconsistencies in children’s knowledge were highlighted when asked to draw a bigger version of the shape, for example, one child drew an oval in response to the request to draw another bigger circle.

After drawing the circle, EC asked pupils to tell her something about the circle. Most responses for the circle were property-based. For example 10 children used the word “round” and 10 children said the circle had “no corners”. In the identification task each child was given a page on which there were examples and non-examples of the circle (See Figure 1 below). It was explained to the children that on the page some of the shapes were circles and some of the shapes were not and participants were asked to colour only the circles. Whilst children were colouring EC asked them about their reasons for selecting certain shapes and excluding others. The circle task highlighted the relationship between misconceptions and mathematical language. For example, pupils justified their inclusion of the ellipse on the basis that it looked “round”.

![Figure 1: Circle identification task adapted from Razel and Eylon (1991) in Clements, Swaminathan, Hannibal and Sarama (1999)](image-url)
As can be seen from Table 1, there was very little difference in children’s capacity to identify correctly the circle in the shape identification task - in fact, JI pupils performed at the same level as the pupils in Second class. More interesting perhaps is the picture that emerges around the identification of “non-examples” (e.g., shape 10) that were marked as circles. In general pupils seem to have improved in this as they moved from JI to First class (indicating that the younger pupils were at a pre-recognitive level). The disimprovement in Second class is noteworthy. Possible reasons for this will be discussed later.

<table>
<thead>
<tr>
<th></th>
<th>Total</th>
<th>Junior Infants</th>
<th>Senior Infants</th>
<th>First class</th>
<th>Second class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of circles marked by participants.</td>
<td>93%</td>
<td>98%</td>
<td>93%</td>
<td>82%</td>
<td>98%</td>
</tr>
<tr>
<td>Percentage of non-examples marked.</td>
<td>9%</td>
<td>15%</td>
<td>5%</td>
<td>4%</td>
<td>11%</td>
</tr>
</tbody>
</table>

Table 1: Percentage correct responses in circle identification task

Square

While pupils’ freehand drawings of the square included similar length sides their descriptions often excluded mention of the lines being equal length. As the sample from an interview transcript below shows, it required prompting in the square identification task to elicit that the lines of a square were equal.

EC: Ok so we’ll move on now and we’ll do the squares ... Straightaway you picked that one, why did you pick that one?
Danielle: Because it has four sides; one at the bottom, one at the top and one at the side and one at the other side
EC: Ok ... (Danielle continues to colour)
Danielle: Think that’s all
EC: Ok now that’s got four sides doesn’t it? (pointing to rectangle)
Danielle: (murmurs assent)
EC: Is it a square?
Danielle: No, yeah, because it looks like a rectangle
EC: Ok but why is a rectangle different to a square?
Danielle: Because it is longer.
EC: Ok and which parts are longer?
Danielle: Em, the bottom line has smaller sides
EC: Ok does this (pointing to rectangle which has the horizontal lines longest) have smaller sides?
Danielle: No.
EC: What can you say about the sides on this? (Pointing to a square she had previously coloured)
Danielle: They’re all equal, all the same.
Pupils did not perform well on the squares task as many of the older pupils excluded the square when it was in a different orientation (for example, the shape numbered 13 in Figure 2) and all twenty children coloured in one shape which had vertical lines slightly longer than the horizontal lines.

**Figure 2: Square identification task (First class pupil)**

Frobisher, Frobisher, Orten and Orten (2007) describe the pre-recognition level as that at which children treat quadrilaterals and other polygons as the same kind of shape. In the course of the interviews the square was frequently compared to the rectangle. Frobisher et al. (2007) explain that at the visualisation level children are concerned with size and orientation rather than properties. Second-class pupils were more likely to use the vocabulary of lines and corners. Although property-responses were frequent, children sometimes became fixated on one incomplete attribute, e.g., 11 pupils mentioned “four corners or points” and six mentioned “four lines or sides”. One of the pupils, Abigail, determined that because a square has four sides she would colour in all of the four-sided shapes in the identification task.

<table>
<thead>
<tr>
<th></th>
<th>Total</th>
<th>Junior Infants</th>
<th>Senior Infants</th>
<th>First class</th>
<th>Second class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of squares marked by participants.</td>
<td>61%</td>
<td>66%</td>
<td>66%</td>
<td>54%</td>
<td>57%</td>
</tr>
<tr>
<td>Percentage of non-examples that were marked.</td>
<td>27%</td>
<td>27%</td>
<td>37%</td>
<td>27%</td>
<td>17%</td>
</tr>
</tbody>
</table>

**Table 2: Percentage correct responses in square identification task**

As can be seen from this table, children’s capacity to identify squares correctly seemed to disimprove as they moved from JI to Second class. However, given that children in 2nd class were less likely to mark a ‘non-example’ as a square, it can be conjectured that they were more inclined to identify shapes on a property rather than a visual basis.

**Triangle**

When children were asked to draw a triangle all twenty children competently drew a prototypical equilateral triangle with the point on top and the base of the triangle horizontal to the bottom of the page. When asked to draw a “different” triangle, many pupils were unable to do so and often made an irrelevant change such as orientation - for example one Second Class child said that the point could be at the bottom and that was the only way that a triangle could differ from what she had drawn.

**Figure 3: Drawing of triangles (Second class pupil)**
Changing the size was another popular method. One JI pupil drew five versions of the equilateral triangle. She explained their differences in terms of size with one being a “normal” one, another was a “thin” one, next was a “middle” one, a “big” one and finally a “small” one. Frobisher et al. (2007) claims that because of the prototypical image of the triangle as an equilateral triangle with one side parallel to the horizontal side of the page, children fail to recognise less “regular” triangles and even regular ones that are orientated differently. Schifter (1999) in one small-scale study of 27 children aged nine years presented children with a set of ten different triangles and all 27 children agreed that only one shape (the prototypical equilateral triangle) was a triangle. The results of this research are similar.

<table>
<thead>
<tr>
<th></th>
<th>Total</th>
<th>Junior Infants</th>
<th>Senior Infants</th>
<th>First Class</th>
<th>Second Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of the</td>
<td>78%</td>
<td>95%</td>
<td>60%</td>
<td>75%</td>
<td>80%</td>
</tr>
<tr>
<td>triangles that were</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>marked by participants.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Percentage of the</td>
<td>46%</td>
<td>70%</td>
<td>52%</td>
<td>28%</td>
<td>32%</td>
</tr>
<tr>
<td>non-examples that</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>were marked.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Percentage correct responses in triangle identification task

Rectangle

The rectangle was drawn most often going from left to right starting with the top line. There was a tendency to draw the rectangle with the longest lines horizontal. One Second class pupil described the rectangle as “It just has longer, two longer lines, one on the top and one on the bottom and two shorter lines down the sides”. Similarly another Second class pupil said the rectangle had “a big line at the top and a big line at the bottom and there is a small line on the left of it and a small line on the right”. Although the rectangle is introduced in JI, one SI child was unable to draw a rectangle and one First class pupil could not remember how to draw the rectangle until after she had drawn and discussed the square.

When pupils were asked to draw a bigger rectangle two strategies were adopted: they increased one dimension or they increased both dimensions. Elia, Gagatsis and Kyriakides (2003) suggest that the application of the first of these strategies may correspond to the pre-recognitive level. The eight participants in this study who adopted this strategy were not all at a pre-recognitive level in other tasks.

Figure 4: Increasing one dimension (First class pupil)
On the other hand, Elia et al. (2003) propose that children who apply the second strategy seem to have the characteristics of the “syncretic” level (Clements et al., 1999), since they conserve the components and the properties of the figure they transform. Seven children adopted this second strategy; two from JI, two from SI, one from First class and two from Second class.

Figure 5: Increasing two dimensions (First class pupil)

The prototypical rectangle dominated in the rectangle identification tasks. Pupils were able to explain that two sides were long and two sides were short unlike the square where length of sides was not often mentioned. The perpendicularity requirement was often ignored in the rectangle identification task. According to Frobisher et al. (2007), young children are less successful in recognising rectangles than the other three common shapes and their mental image of a rectangle appears to be a shape that has four sides and two long parallel sides which may lead children to select ‘long parallelograms’ or right angles trapeziums as rectangles. This was evident in this research.

<table>
<thead>
<tr>
<th>Percentage of the rectangles that were marked by participants.</th>
<th>Total</th>
<th>Junior Infants</th>
<th>Senior Infants</th>
<th>First class</th>
<th>Second class</th>
</tr>
</thead>
<tbody>
<tr>
<td>50%</td>
<td>50%</td>
<td>50%</td>
<td>50%</td>
<td>50%</td>
<td>50%</td>
</tr>
<tr>
<td>Percentage of the non-examples that were marked.</td>
<td>38%</td>
<td>45%</td>
<td>40%</td>
<td>25%</td>
<td>42%</td>
</tr>
</tbody>
</table>

Table 4: Percentage correct responses in rectangle identification task

DISCUSSION

In general it was found that children come to school with a good understanding of 2-D shapes. In the main, children performed best on identification of the circle and least well on the identification of the rectangle. This may be because orientation is not a confounding factor in the identification of the circle. Younger children seemed to perform better on the identification of shapes. However this may well be because they were more inclined to include rather than exclude figures. Furthermore, First class children showed less inclination to mark non-examples than JI or SI children. This indicates the existence of a pre-recognitive level as proposed by Clements et al. (1999). An interesting pattern that emerged was that the children from Second class were more likely to mark non-examples of the circle, triangle and
rectangle than the First-class group. This may be due to insufficient experience with 2-D shapes. It might also be that they were in transition between levels. Spitler (2003) in her study of one prekindergarten pupil’s path of discovery to understanding a triangle noted that two parallel schemes, a “visual triangle” and a “three-sided figure” co-existed, the threads of which were not interwoven, but rather, loosely connected, during the prekindergarten year. These loose connections began to strengthen as the child broadened her image of “triangle” to sometimes include non-prototypical triangles, and as she became explicit about certain attributes of triangles. Similarly in this study, while participants could often say that a triangle had three sides, they drew the prototypical equilateral triangle and did not include shapes on the identification task that had three sides. The preschooler in Spitler’s study showed instability in her inaccurate but stable concept of a triangle as the school year progressed. Spitler explains this instability was brought on by the cognitive conflict she encountered between her fixed, prototypical idea of a triangle that she originally had and interference with this prototype, as non-prototypical triangles were introduced. During the course of the interviews this ‘cognitive conflict’ was occasionally evident. One JI pupil asked during the rectangle identification task “Should I colour that one?” and followed this with “It has 4 sides ... I don’t know”. A SI child determined that a square had four corners and this justified her inclusion of the trapezium in the square identification task. In general, there was evidence that the van Hiele levels were more dynamic than fixed and that the tasks given to children influenced their reasoning (for example, see interview with Danielle above).

The influence of prototypical shapes is something that surfaced frequently during the interviews. Vurpillot (1976) as cited in Spitler (2003) reported on Piaget’s findings concerning young children”s preference for the “good form” — closed and symmetrical (p. 6). In the triangle identification task there were two shapes that were not closed and two Junior Infant pupils drew in the rest of the line themselves assuming that it was simply a mistake. The symmetrical triangle was also coloured in by all participants while other triangles were not successfully identified. Matthews (2005, p.30) correctly warns that “an image of a standard, three equal-sided figure is being promoted as ‘the’ triangle and the understanding and geometric conceptualisation of students is being limited to this overused and cliché image.”

Children’s incomplete understandings were evident in the way they used mathematical language. One Second class pupil said that a triangle had one straight line and traced the horizontal base. She described the other two lines as “diagonal”. She did not appear to realise that a line could be diagonal and straight. While pupils recited a familiar definition of a triangle having three corners they failed to recognise shapes with three corners in the identification task. Many pupils coloured in all of the shapes that had three corners and did not consider how the lines were formed.

The teaching of shape necessitates much discussion to refine understanding and discover inaccuracies in knowledge. Students are facilitated if multiple learning styles; visual auditory and kinaesthetic are supported. The Inspectorate Evaluation of Curriculum Implementation (DES, 2005, p. 25) recommend that “teaching should be cross-curricular, with integration and linkage encouraged, to make Mathematics more meaningful for pupils and to avoid overload
for teachers”. Geometry can easily be integrated into the visual arts curriculum. The Irish National Teachers Organisation/INTO produced a report titled “Maths in Primary School” (Carr, 2006). In this report it was noted that access to concrete materials is considered necessary for pupils at all class levels, from infants through to Sixth class and that language also plays an important role. The Inspectorate Evaluation of Fifty Schools (DES, 2002) also noted that the majority of recommendations regarding Mathematics refer to the need for an increased use of materials and resources in teaching and learning, and an emphasis on the language of mathematics and on oral work in mathematics. This is something supported by this research. The scalene triangle was excluded by twelve pupils although all twenty pupils could say that a triangle had three corners or three sides. Clements and Sarama (2000) advocate that when teachers are drawing attention to the properties of shapes they vary the types of shapes shown to pupils in order to help them to strengthen their concepts of shapes. Delaney (2010) includes “the mathematical work of selecting examples” as one of the mathematical tasks of teaching as it “requires mathematical knowledge to select shapes that can be tested by the definition of the shape but which pupils encounter less frequently than ‘typical’ examples of the shapes” (p. 15).

CONCLUDING REMARKS

The two immediate limitations of the study are that it is concerned only with female participants and occurred in just one setting. While the findings of this small scale study cannot be generalised to represent the geometrical knowledge of all children from JI to Second class, it points to the need to give greater attention to the strand of Shape and Space in primary mathematics and to build on the rich understanding of shape that children have from an early age.

Fuys, Geddes and Tischler (1988) reasoned that children enter elementary school at the first level of geometric thought, the visual level, and then do not progress; rather, they remain at this visual level throughout their elementary years due to lack of exposure to descriptive/analytic experiences. While the 2nd class children in this study used more sophisticated language and more property-based descriptions of shapes than the younger pupils, there was not the increase in knowledge that one might expect. For example, two Second class pupils did not describe the square as having sides of the same length. Furthermore, some of the younger children appeared to have rich understandings of 2-D shapes and it seems likely that their thinking is not being sufficiently challenged as they progress through school.

Teachers need to ensure that accurate concepts are held before new shapes are introduced throughout primary school and alternatives to the tasks typical textbooks must be explored. In particular, greater attention must be given to children’s experience with and reasoning about 2-D shapes in the junior classes so that they are prepared for more challenging experiences in senior primary school.

NOTES

1. Pseudonyms are used throughout.
REFERENCES


Matthews, S. (2005). The truth about triangles, they’re all the same or are they? Australian Primary Mathematics Classroom, 10(4), 30-32.


This article seeks to explore the links between an Undergraduate Data Analysis for Business module and Project Maths (PM), an Irish secondary school curricular development initiative involving the phased development and introduction of new syllabuses in Mathematics. In particular we focus on the level of preparedness Project Maths offers students of an applied mathematical subject, namely Data Analysis for Decision Makers (DADM), at third level. We examine whether these students think that their experience of Strand 1 (Probability and Statistics) in the Leaving Certificate PM course affected their performance on their DADM module.

INTRODUCTION

Maths has been identified as a key driver for the knowledge economy, scientific advancement and as a fundamental requirement for active citizenship (Department of the Taoiseach, 2008). Project Maths (NCCA, 2013) is an initiative by the Irish education authorities which involves the introduction of revised mathematics syllabuses at second level. It involves changes to what students learn, how they learn it and how they are assessed. It aims to develop the mathematical knowledge, skills and understanding needed for continuing education, life and work. It also purports to develop a flexible, disciplined way of thinking in students to enable them to solve problems in mathematical and real world contexts. Similar initiatives are on-going across the United States, (Weiss, Pasley, Smith, Banilower & Heck, 2003), and Europe (Eurydice, 2011) in countries such as Belgium, the Czech Republic, Slovenia and Spain, encouraging pupils to participate in their own learning through discussions, project work, practical exercises and other ways that help them reflect upon and explain their mathematics learning. The latter report explores the common challenges and national policies for teaching mathematics in Europe. It found that the use of problem-focused learning is the main objective of many countries. Some countries emphasise contexts which are familiar to students so as to provide a meaningful frame of reference for their learning (Spain, Poland and Italy). In other countries (Estonia), students are encouraged to participate in outdoor learning, relating their mathematical knowledge to creativity, architecture and visual arts. Similarly, active learning and critical thinking is advocated in many jurisdictions (for example, Belgium, the Czech Republic, Slovenia and Spain), encouraging “pupils to participate in their own learning through discussions, project work, practical exercises and other ways that help them reflect upon and explain their mathematics learning” (Eurydice, 2011, p.56). Indeed some of the central aims of the PM syllabus (Project Maths Development Team, 2013) are: the application of mathematics to real-life settings; forming links between maths topics; using mathematical language and verbal reasoning to convey ideas; and planning and conducting collaborative investigative activities. Strand 1 of the revised syllabus in Ireland concerns statistics and probability.
In 2011, the Quinn School of Business, UCD, Ireland restructured its undergraduate business degree programmes. During this process quantitative skills were identified as central to the holistic education of business students. Data Analysis for Decision Makers (DADM) was designed during this review as a core module for all (circa 600 per annum) undergraduate School of Business students. It is delivered in Semester 1 of first year.

In this paper we aim to analyse undergraduate business student perceptions of Data Analysis themes and examine whether earlier exposure through Project Maths at second level, to similar themes assists students in meeting the DADM learning outcomes.

To aid our understanding of the research we ask the following question in this article:

Did exposure to probability and statistics concepts, via Project Maths, assist students meet their DADM learning outcomes?

BACKGROUND MATERIAL

An interim report on Project Maths (PM), based on research commissioned by the NCCA and conducted by the National Foundation for Educational Research in England (Jeffes, Jones, Cunningham, Dawson, Cooper, Straw, Sturman & O’Kane, 2012) looked at the pilot schools where Project Maths has been taught since 2008. This report also examined the rest of the country’s schools which have taught PM since 2010. It findings suggest:

• Students at Leaving Certificate level appear to be performing well and are broadly confident in their abilities in many aspects, particularly Statistics and Probability.

• Students find tasks that require higher order skills such as reasoning and transferring knowledge to new contexts harder than more mechanically demanding tasks.

• Students are regularly engaging with the application of mathematics to real-life situations; making connections and links between mathematics topics; using mathematical language and verbal reasoning to convey ideas; and planning and conducting investigations.

• Whilst, in general, students following the revised syllabus performed better than their comparison group peers, this difference is only statistically significant in relation to a particular item which explores students’ abilities in Strand 1, Statistics and Probability (assessing students’ understanding of the outcomes of simple random processes).

• Many Leaving Certificate students were planning to pursue further study and/or careers in mathematics, favouring professions such as accountancy and business management.

While we acknowledge that it will be 2016 before the first full cohort of First to Sixth Year (the secondary school cycle in Ireland) students will have taken Project Maths, we hope this paper will serve to promote a constructive discussion on the readiness of our second level mathematics students for a more contextualised data analytics course at third level. Apart from Jeffes et al. (2012), no empirical research on the impacts of Project Maths on student learning and achievement has yet been published. In this discussion we offer some comparative data on second and third level student achievement scores in the quantitative disciplines.
A report by the NCCA (2012) of mathematics teachers in the 24 initial schools providing Project Maths surveys students’ experiences with PM. In this article we give an airing to some of the student’s attitudes towards PM and its relevance to the third level module DADM.

UNDERGRADUATE DATA ANALYSIS

The state end of secondary school exam in Ireland is the Leaving Certificate. A minimum of a B3 (70%-74%) at ordinary level or a D3 (40%-44%) at higher level in the Irish Leaving Certificate Maths is stipulated as an entry requirement for undergraduate business programmes at UCD. The majority of entrants to business programmes in UCD come via this access route, (89% in 2012). Students who sat the Leaving Certificate in 2012 were the first cohort to have come through the Project Maths Strand (Strand 1) on Probability and Statistics. Equivalent entry requirements are stipulated for other access routes. Business students come into the business programmes with a wide variety of mathematical abilities. Successful completion of the DADM module means that students are able to:

1. Prepare spreadsheet models to store, manipulate and analyse quantitative data using common probability distributions and statistical functions.
2. Calculate, analyse and present useful statistical measurements from large-scale data sets.
3. Create and interpret inferential statistical statements about population parameters.

DADM Delivery Approach

The module is delivered through a two hour lecture in large groups (circa 150) with a follow-on tutorial in smaller groups (circa 45). We use the principles of Active Learning (AL) in DADM contact time. AL can be defined as a process of keeping students mentally active in their learning through activities that involve them in gathering information, thinking and problem solving. The interested reader may consult Michael (2006) for evidence that this approach is effective.

A business related problem with a small exercise is used during lectures as the basis for developing theoretical concepts. Students are then asked to engage in class exercises, usually with their peers, to answer similar problems and interpret their solutions. This approach is aligned well with the student’s experience of Project Maths at second level where discussion and collaborative learning are encouraged.

DADM also employs eLearning and ICT resources extensively which students can access online at any time. The learning management system used in UCD is Blackboard, through which the eLearning content is made available. The eLearning authoring software Articulate Storyline (Articulate, 2013) is used to develop the eLearning content. Captivate (Adobe, 2013), is also used to record “How to” demonstrations in Excel. Students were also invited to complete Excel spreadsheet training given by an external IT training provider in weeks 3 and 4 of the semester.

Students are instructed to complete additional online short exercises in advance of tutorials. Answers to the online exercises (and feedback on any calculation steps) are given when a
student submits an attempt. The short exercises focus on methodology and demonstrate the steps in formulating and solving problems.

Students can attempt these exercises as many times as they like. We encourage students to use these as practice exercises. There is no continuous assessment (CA) credit awarded for attempting these online exercises. At the tutorial, tutors review student progress with the online exercises, responding to student queries and clarifying concepts.

Online review and more extensive case study exercises are available for students to work through in their own time at the end of each section of the module. These exercises, called Theory into Practice (TIP) focus on sample business applications and demonstrate the applicability of data analysis within a business context. Sample analysis and interpretation are given when a student submits an attempt. Open source data sets are used where possible.

Again sample analysis and interpretations are given as feedback when a student submits an attempt.

Our intention is to facilitate different student learning styles and abilities so as to enable students to engage fully with the learning process. We try to engage students in ways of learning which students prefer while maintaining academic rigour with respect to the content. While there is much debate currently as to whether CA should be introduced on the Leaving Certificate programme the approach taken on the DADM module is to allocate 40% of credit for CA and this leads to a very different learning environment to the secondary school classroom.

PROJECT MATHS – STRAND 1: STATISTICS AND PROBABILITY

Concern that the teaching and learning of mathematics in secondary schools in Ireland was failing to make the connections between mathematics and its place in real life, led to a curriculum review and the roll out of the Project Maths initiative. An in-depth study of the Irish mathematical classroom (Lyons, Lynch, Close, Sheerin & Boland, 2003) showed that most of the learning taking place in the maths classroom was of a procedural and mechanical style and that the teaching and learning approaches promoted this learning. Indeed Ireland’s relatively poor showing in international comparison tests such as PISA in 2003 (Ireland ranked 17th of 29 OECD countries and 20th of the 40 participating countries) and TIMSS in 2011 (Ireland placed 17th again here whereas Ireland was placed significantly above the international mean and was ranked 6th of 17 countries when it last took part in TIMSS in 1995) also prompted these major syllabus changes. While the rationale behind Project Maths has been widely welcomed by the key educational stakeholders, its execution has drawn mixed reaction and often heated debate. Oldham & Close (2009) argue that more in-service teaching must be provided for the teachers to become more comfortable with a style of teaching that may not be what they themselves are used to or even contrary to their own philosophy of mathematics education. A major professional development initiative (The Professional Diploma in Mathematics for Teaching) designed in UL and NUIG, to up skill the large cohort (48%) of out-of-field secondary maths teachers is now underway across 10 Irish third level institutions including UCD.
The PM syllabus and its assessment have also attracted attention. For example, Engineers Ireland (2010) and some third level educators (Grannell, Barry, Cronin, Holland & Hurley, 2011) have expressed concern at the omission of key maths topics such as integration, matrices and vectors and the over influence of the PISA type mathematical philosophy. Conversely the topics of statistics and probability have been given twice as much coverage within the syllabus compared to previous syllabuses. Lubienski (2011, p. 31) has noted the length and difficulty of the statistics strand, particularly for senior cycle students and indications (Jeffes et al., 2012) show that these topics are being learned well in line with the PM objectives upon international comparison.

There is considerable overlap between the learning outcomes of Project Maths Strand 1 (Probability and Statistics) and the DADM module. Learning outcomes for Strand 1 specify that students should be able to:

- discuss basic probability rules and concepts;
- calculate probabilities associated with the outcomes of random processes;
- use statistical reasoning with the aim of becoming a statistically aware consumer;
- discuss and design plans for data gathering;
- describe data by means of descriptive statistical and graphical techniques;
- draw inferences from data; and
- synthesise and solve related problems.

Traditionally, there has been a trend to ‘teach to the test’ in leaving certificate subjects such as maths. There is an argument that the introduction of coursework would alleviate that situation and facilitate a deeper knowledge and promote exploration of subject content. Problem Based Learning (PBL) and Enquiry Based Learning (EBL) are effective teaching strategies but may be more appropriate for adult or distance learners and third level environments. Lubienski (2011) notes the high emphasis on the Leaving Certificate examination, which, in her view, constrains instruction and places the second level instructor in the role of “exam coach”. Initial feedback suggests that teachers feel under pressure to revert to old style ‘drill and practice’ teaching and abandon student-centred, inquiry-based methodologies because the Leaving Certificate result is so important in gaining a place at college (Cosgrove, Perkins, Shiel, Fish & McGuinness, 2012). An investigation by the authors of how Continuous Assessment may support learning at second level will be the topic of a related discussion to the one presented here.

**Assessment strategies**

To enable the students to meet the specified learning outcomes and achieve the goal of deeper learning we endeavour to use a variety of assessment methods. Grading criteria are employed to quantify to what extent individual students have met the learning outcomes. The DADM assessment tasks are summarised in Table 1.
<table>
<thead>
<tr>
<th>Item</th>
<th>Weight (%)</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excel work</td>
<td>3</td>
<td>Spreadsheet exercises</td>
</tr>
<tr>
<td>Excel test</td>
<td>12</td>
<td>Data analysis</td>
</tr>
<tr>
<td>2x MCQs</td>
<td>15</td>
<td>Theory &amp; practice</td>
</tr>
<tr>
<td>Team project</td>
<td>10</td>
<td>Open ended</td>
</tr>
<tr>
<td>Terminal exam</td>
<td>60</td>
<td>2 hr. written exam</td>
</tr>
</tbody>
</table>

**Table 1: DADM assessment**

**RESEARCH METHODOLOGY**

We obtained ethical clearance to conduct our action research project. Both quantitative and qualitative data were gathered in an attempt to answer the research question stated in Section 1. We compared the students Leaving Certificate results from 2012 with their performance on the DADM module 2012.

We also implemented an optional online survey to ascertain student’s attitudes toward the DADM module and their experiences with Project Maths. We acknowledge the limitations of web-based surveys and also accept that the use of an optional online survey is limited in use for a scientific study and we refer the reader to Couper (2000), Dillman & Bowker (2001), and Umbach (2004) for a discussion on the problems arising from online surveys. Considering the timeframe constraints of the academic semester this was deemed by the authors to be the only approach available to elicit student opinion and was judged to be useful in giving some insight into the research question under discussion, notwithstanding the issue of a self-selecting response mechanism.

The survey contained a mix of five-point Likert items and open ended questions. The Likert items were scaled from 1 (*strongly disagree*) to 5 (*strongly agree*). Using this scale, the more students agree with a statement, the higher the Likert average. A justification of the appropriateness of Likert scales to gauge attitudes is given in Waples, Weyhrauch, Connell & Culbertson (2010). A copy of the survey and anonymised data are available at the author’s home page http://www.ucd.ie/cba/members/paulacarroll/

The survey was delivered via Blackboard. A pilot pen and paper survey was run in week 8 of semester, following which minor changes were made to improve readability. The full survey was made available to students from week 11. Of the 546 students enrolled for the module, a small number withdrew from their programme of study by the end of semester 1 for various personal and academic reasons. This left 540 students of whom 127 responded to our invitation to participate in the survey giving a response rate of almost 24%. We acknowledge that in social scientific terms this is a low response rate and note also that not all respondents answered all the survey questions. Students are also invited to complete an end of semester evaluation for each module they complete through UCD’s centralised evaluation system. For
the DADM module 135 students completed this evaluation. Our survey response rate is in line with similar evaluation surveys conducted by the university.

ANALYSIS AND FINDINGS

Question 10 and 11 respectively, of the survey stated: On completing (Having completed) DADM (Project Maths) I will have (I have) a good understanding of Probability and Statistics.

Table 2 shows a summary of the responses to questions 10 and 11. It shows the percentage of responses in each Likert category for each course. The last two rows show the Likert mean and standard deviation excluding students who did not answer that question. We see for example that 41% (10.2%+30.7%) of respondents did not agree to having a good understanding of Probability and Statistics on completing DADM. While 12.6% of respondents said they strongly disagreed that they had a good understanding of the topics having completed PM.

<table>
<thead>
<tr>
<th></th>
<th>DADM</th>
<th>Project Maths</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unanswered %</td>
<td>12.6</td>
<td>11.8</td>
</tr>
<tr>
<td>Strongly Disagree %</td>
<td>10.2</td>
<td>12.6</td>
</tr>
<tr>
<td>Disagree %</td>
<td>30.7</td>
<td>16.5</td>
</tr>
<tr>
<td>Neither Agree nor Disagree %</td>
<td>18.1</td>
<td>13.3</td>
</tr>
<tr>
<td>Agree %</td>
<td>25.2</td>
<td>32.2</td>
</tr>
<tr>
<td>Strongly Agree %</td>
<td>3.1</td>
<td>1.6</td>
</tr>
<tr>
<td>NA %</td>
<td>0</td>
<td>11.8</td>
</tr>
<tr>
<td>Likert mean*</td>
<td>2.77</td>
<td>2.91</td>
</tr>
<tr>
<td>Likert Stdev*</td>
<td>1.1</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Table 2: Student attitudes to DADM/PM levels of understanding (*excluding non-response)

As stated earlier the learning outcomes for PM (Strand 1) and DADM are similar but the results above would suggest that the DADM respondents are struggling with the ‘applications and context to business’ of the probability and statistics learning material. This is a worrying development as an analysis of the Project Maths text books (Morris, Cooke & Behan, 2012; Keating, Mulvany, Murphy & O’Loughlin, 2012), and the State Examinations Commission (2013) show a heavy emphasis on business type applications and financial contexts/scenarios in relation to questions on statistics, data and probability.

We analysed the DADM results of students who completed the module and whose access method was the Leaving Certificate in 2012 (this accounted for 429 of the entire DADM class, 179 females and 250 males). The left side of Figure 1 shows a histogram of the leaving cert points in maths (excluding the bonus points for higher level maths). A bimodal
distribution is apparent. The right side of Figure 1 shows the DADM equivalent grade points for the same cohort of students. The distribution is less bimodal, leaning toward a right skewed unimodal distribution.

The Pearson correlation coefficient measuring the relationship between the LC 2012 and DADM results is 0.498. At \( p = 0.05 \), using a t-test, we conclude that there is evidence of correlation although it could be considered to be relatively weak.

There is evidence to suggest that the more capable students (those who achieved an honour in the higher level leaving certificate) may have become complacent in thinking that Project Maths was very similar to DADM and as such didn't do the necessary work to attain the same high results they achieved at second level. While the students who took the ordinary-level course (or those who had gotten a D at higher level or those from GCSE entry route) worked harder. In fact those who studied maths at the ordinary level (126 students) scored an average of 47.73% overall on the DADM module while the higher level people (303 students) scored an average of just 57.39% with standard deviations of 12.09% and 12.06% respectively.

![Figure 1: Leaving Cert 2012 Maths points (left) and DADM grades (right)](image)

One student commented that “the methods through which we learned probability and statistics for project maths and this module are different”. In terms of the final assessment, the Concepts and Skills section of the PM Leaving Certificate higher level paper (paper two) mirrors the short questions section of the DADM terminal exam paper, both in terms of content and style. The Contexts and Applications section of the PM paper is again closely aligned to the long questions (applied) on the DADM paper. The average mark scored on the long questions (more applied and inferential, PM-type questions) was just 39% while the short (more methodical and formulaic) questions scored an average of 56%.

A comparison of the word count for the longer sections of both papers shows no significant statistical difference. The sample papers for 2011, 2012 and 2013 combined with the actual 2012 PM paper had an average word count of 486 (there were 598 words on the 2012 paper taken by the respondents in question). The word count for the 2011 and 2012 DADM papers averaged at 422 with the 2012 paper containing 429 words. Considering that the Leaving Cert paper is 2.5 hours in duration compared with 2 hours for the DADM exam, these word counts are notably similar. The word count for the Maths for Business module (the other numerical module taken by these students) exam papers (also 2 hours in length) for the last 3 years was
728 words. Note that this is the word count for the entire exam as there is no differentiation between short and long questions here. The IMTA (Irish Maths Teachers Association) said the wordy nature of some questions [on the Leaving Cert paper] has posed problems for students with literacy difficulties. In relation to Project Maths one student said: “The maths became too English based and it was difficult to figure out at times what the actual sum was”; another said:

I find statistics and probability difficult anyway, and studying Project Maths didn’t help me in any way shape or form. I found it difficult enough to understand and interpret the amount of writing and English that came with the questions before I was even able to begin understanding and attempting the maths side of it. I don’t find the way that Project Maths was taught to be effective, especially when the teachers didn’t even seem to fully understand it.

Students of the DADM module expressed a preference for concrete tasks like calculating a measure in Excel and found higher level synthesis tasks more challenging. Students also found the language and terminology difficult to grasp. One student commented:

Some of the slides on Blackboard were helpful but alot them (sic) used very complicated language which meant that I was at a loss to what many of them were actually explaining.

Preliminary analysis of the Maths for Business module, shows an increase in results compared to previous years, which is not apparent in DADM. This is likely due to the increased number of students with higher-level maths due to the bonus points initiative. The fact that there is an emphasis in DADM on the higher level of learning/understanding required to synthesise problems may also be reflected in this analysis. Two opposing views stated by students were: “The Project Maths at pass level did not prepare me for the level of maths in Data analysis [DADM].” and “DADM is a very realistic topic which can be applied in real life problems and maybe should be focused on more in project maths.”

To determine second-level students’ views on the broader application of mathematics, they were asked to comment on the extent to which they perceived it to be useful in the following ways:

• to help in daily life
• to aid learning in other school subjects
• to enable them to get into the university of their choice
• to enable them to get the job of their choice.

80-100% of these students said mathematics as a career was relevant to professions such as

• Accountant
• Engineer
• Owning a business
• Scientist
• Working with technology
Students of DADM were asked to reflect on their experiences with DADM and with Project Maths, and to indicate if they thought the skills and understanding developed were relevant to consumers, business leaders, scientists or engineers and government or public bodies policy makers. Their responses are summarised in Table 3.

<table>
<thead>
<tr>
<th>Category</th>
<th>Considered relevant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unanswered</td>
<td>14.17%</td>
</tr>
<tr>
<td>Consumer</td>
<td>22.05%</td>
</tr>
<tr>
<td>Business leader</td>
<td>66.14%</td>
</tr>
<tr>
<td>Scientist or engineer</td>
<td>44.09%</td>
</tr>
<tr>
<td>Government policy maker</td>
<td>33.86%</td>
</tr>
</tbody>
</table>

**Table 3: Student perception of the relevance of DADM skills**

The business leader category was selected by 66.14% of survey respondents as having relevance for the DADM module. Interestingly only 22.5% thought the DADM skills relevant to a consumer.

**CONCLUSIONS**

Over 84% of students registered to DADM achieved a sufficient level of understanding as indicated by the pass ratio. There was a relatively weak correlation between exam scores on the 2012 Leaving Certificate mathematics and the semester 1 2012/13 DADM module. The performance of the higher-level Leaving Certificate student was not as high as expected when compared to their ordinary-level counterparts. Analysis of the more procedural questions versus the context-based question on the DADM exam resonates with the finding by Jeffes et al. (2012) which concluded

> In general, items requiring higher order skills (such as reasoning and an ability to transfer knowledge to new contexts) are found more difficult than those which are more mechanical in demand (p.8).

The readiness of PM students to study a more contextualised data analytics course at third level is unclear and perhaps it is too early in the development of the PM initiative to draw any significant conclusions. It is interesting in this MOOC saturated era that 55% of students think that lectures do not help them learn how to implement data analysis techniques and interpret the results. If the university experience is about promoting self-directed learning we should not be surprised that first year students find this a discomforting experience for which they do not thank us. As reported in Jeffes et al. (2012), the proposed teaching and learning approaches promoted by PM are being felt by the students but we cannot yet tell if the students’ new mathematical experiences are as a result of the content and delivery of PM in terms of the teaching they receive or if their own learning style is in fact changing. As Project Maths is as much about changing teaching and learning practices as it is about changing
content, third-level instructors also must take on board the two-way process that is PM and adapt to the changes taking place in second-level mathematics teaching and learning. We suggest that this paper be viewed, not as a critique of whether PM is working or not, but as a wider commentary on the educational aspirations of improved literacy and numeracy for our students and how we at third level can contribute positively to empowering this.

ACKNOWLEDGMENTS

We thank the DADM students for participating in this research project. We also thank the Quinn School of Business eLearning and programme office teams for their support.

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http://www.ncca.ie/en/Curriculum_and_Assessment/Post-Primary_Education/Project_Maths/


http://www.projectmaths.ie/overview


WHAT WAS I THINKING? ONE MODULE (METRIC SPACES), TWO MODES (OF TEACHING) AND THE IMPACT FACTOR: A STUDENT’S PERSPECTIVE

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National University of Ireland, Galway

This paper seeks to explore the mathematical learning experience of a specific advanced level module (Metric Spaces) facilitated by two different lecturing/teaching approaches in two institutions. It is a response, from a student’s perspective, to recent and ongoing research enquiry into how students can learn such a subject area in a deep and enduring manner. In considering the two approaches, we draw upon Vygotsky’s socio-cultural theory of learning and Pirie-Kieren’s dynamical theory for the growth of mathematical understanding. This paper is primarily a personal statement of a student, as witness to and as participant in the teaching and learning of metric spaces in the two different environments. It offers insight into the impact, on the ground, of teaching and classroom practices employed; it probes the crossing of boundaries in mathematical growth and development at advanced undergraduate level.

INTRODUCTION

Typically students in the penultimate or final year of their mathematics-major degree programmes embark upon the new and abstract analytic subject called Metric Spaces. The subject is characterised by theory and proof, and its abstraction allegedly offers considerable scope for extension to further important mathematical ideas. One student is offered two versions of the module: the whistle-stop tour, during which much is seen and catalogued from a distance, and the deep archaeological dig, where exploration of and with the key building blocks is the central theme and, inevitably, where less is seen.

That student is me. I signed up to both, during the course of a three year period, in two institutions. I am a recent graduate (2012) from a four year mathematics degree programme, during the second year of which I took the first version of the module Metric Spaces – the whistle-stop tour. I had previously completed a commerce degree and a Masters degree in accounting and had worked as a taxation consultant for four years, before returning to fulfil a long held ambition to pursue mathematics for its own sake. My experience of learning mathematics and my interest in mathematics education compelled me to embark upon a structured PhD in mathematics education at university level. I am concerned about what constitutes achievement in learning mathematics in senior undergraduate students.

Recent research by the second-named author (A) has concerned the quality and depth of mathematical learning experienced by students in Metric Spaces under her direction as lecturer (McCluskey, 2011a, 2011b). Thus a unique opportunity to extend this work presented itself. Accordingly, I joined A’s Metric Spaces class as a regular student participant in the first semester of 2012/2013 with the task of analysing the classroom experience, and to respond as student witness. This second version of the Metric Spaces module I termed the deep archaeological dig.
This paper has emerged due to the serendipity of time and circumstance. While there may be a perceived limitation, in that the same student took both modules in two different learning environments, nonetheless, the paper offers a relatively unique ‘student witness’ perspective.

**THEORETICAL FRAMEWORK**

There is an extensive body of literature in university mathematics education (Alcock & Weber, 2008; Asiala et al., 1996; Edwards & Ward, 2004; Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Moore, 1994; Seldon & Seldon, 1995; Tall, 1991; Weber, 2001) that in essence seeks the Holy Grail, namely an understanding of how to foster deep, meaningful and enduring learning of mathematics. Its extent bears testament to the importance, and to the challenge, of such a pursuit.

It is useful to negotiate one’s own development in mathematical learning and understanding with the support of established theories of learning. In this work, I refer to Vygotsky’s socio-cultural theory of learning and to the Pirie-Kieren theory for the growth of mathematical understanding. These theories have lent structure to the ensuing analysis of my learning and understanding in the two modules on Metric Spaces.

**Vygotsky’s socio-cultural theory of learning**

Lev Vygotsky proposed a socio-cultural theory of learning (Vygotsky, 1978), arguing that social interaction, language and the social environment are essential in a child’s development from birth onwards. Two main principles of Vygotsky’s work are the More Knowledgeable Other (MKO) and the Zone of Proximal Development (ZPD) (Vygotsky, 1978). The MKO refers to someone who is more knowledgeable than the learner with regard to a particular idea or concept, normally a teacher, but the MKO could also be the learner’s peers or even a computer. Vygotsky introduced the idea of the ZPD as the difference between what a learner can perform by themselves and what they can enact when given guidance and help from an MKO expert. He regarded the ZPD as the zone within which learning occurred. Vygotsky also emphasised the importance of private or inner speech, in aiding the development of the learner. He believed that language developed from the social environment, later becoming internalised as thought and inner speech.

**Pirie-Kieren Theory for the dynamical growth of mathematical understanding**

The Pirie-Kieren Theory is a theoretical framework for the dynamical growth of mathematical understanding and it has been well-established in the literature (Martin, 2008; Martin & Pirie, 2003; Pirie & Kieren, 1994; Pirie & Martin, 2000; Towers, Martin, & Pirie, 2000), building on a constructivist approach to mathematical understanding. Meel (2003) acknowledges that the model has generally focused on the development of understanding in younger children. However, a number of studies within university mathematics education have utilised the model to document understanding of mathematical concepts including, for example, quadratic functions (Borgen & Manu, 2002), Taylor series (Walter & Gibbons, 2010) and taxi-cab geometry (Martin, 2008), thus broadening the applicability of the model to university mathematical understanding.
The model contains eight potential layers for describing understanding, labelled as Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Inventising. A diagrammatic representation of the theory can be seen in Figure 1 below. This representation shows the layers as eight nested disks, with each layer being contained in the subsequent layers and also containing all previous layers. Growth in understanding is understood to occur as learners work by moving back and forth between these different layers of understanding (Martin, 2008), emphasising that understanding need be neither linear nor one-directional (Towers et al., 2000). This idea of expressing mathematical understanding as non-linear has been considered by others, including Asiala et al. (1996).

Figure 1: The Pirie-Kieren model for the growth of mathematical understanding (Pirie & Kieren, 1994, p. 167)

*Primitive Knowing* is the inner-most layer of the model and this includes all the previously constructed knowledge that the learner has, apart from any knowledge about the topic currently being considered. Any knowledge the learner has about the particular topic already, will be found in one of the other outer layers (Martin, 2008). Wrapped around the *Primitive Knowing* centre layer are the layers of understanding of the concept, each becoming increasingly abstract in form. The next level is *Image Making* and this is the level at which learners engage with tasks in order to get an initial idea about the topic being considered. At the *Image Having* layer, the learner carries a mental image for the concept, without having to rely on the previous tasks carried out at the *Image Making* stage. *Property Noticing* is the next level of understanding. The learner is working at this level when they are investigating those images for properties and connections and successfully articulating these to either themselves or others (Martin, 2008). Learners at the *Formalising* layer of understanding can work with the concept as a formal object, by abstracting based on their previous images. At the *Observing* layer, learners investigate the connections between formal statements about a concept and search for ways to define their ideas as theorems (Walter & Gibbons, 2010). Students at the *Structuring* layer are starting to create logical arguments in the form of proof. The outmost level is called *Inventising* and a learner is working here when they have gained a
full structured understanding of a concept, enabling them to create new questions which may grow into a totally new concept (Pirie & Kieren, 1994, p. 171).

Included in the Pirie-Kieren framework is the idea of folding back (Pirie & Kieren, 1994). Pirie and Kieren consider that when a student is confronted with some problem that they cannot solve, they will need to fold back to an inner level of understanding, in order to lead to a thicker and deeper understanding than originally held. Pirie and Kieren acknowledge this activity as vital to the growth of understanding (Pirie & Kieren, 1994, p. 173). Martin (2008) considers effective folding back as a powerful facilitator of a developing mathematical understanding.

**METRIC SPACES – TAKE ONE (MS1): THE WHISTLE-STOP TOUR**

My first encounter with Metric Spaces was during the first semester of the second year of a four-year undergraduate degree in mathematics (2009/2010). The module, at 5 ECTS, spanned 11 weeks of the semester, with an allocation of three 50-minute lecture slots per week. Generally, one of these slots per week was given over to a tutorial class, led also by lecturer. The module offered an introduction to metric space theory, and included a three week coverage of normed vector spaces. There were six assignments (two of which related to normed vector spaces) over the duration of the module, issued in weeks one, two, three, five, seven and nine. Usually the completed assignments were to be submitted exactly one week later. Model solutions to assignment numbers two through to six were provided to students when the module was completed after week 11.

**Teaching mode**

The module was taught in the traditional ‘definition-theorem-proof’ (DTP) format, widely accepted as the usual teaching method of advanced mathematical subjects at university (Weber, 2004). In practical terms this meant that the lecturer wrote on the blackboard and that I copied down the notes. The course material was presented in a strictly logical sequence, and consistent with Weber (2004), the logical nature of the content appeared to be prioritised over its intuitive nature. Class discussion was minimal and class activity consisted primarily of transcription. Summarised printed notes were uploaded to the lecturer’s personal website periodically, with details of the definitions, examples and statements of theorems that had been met in class previously. The module was assessed in the final examination in the summer session using the DTP format.

As an example, during the first four weeks of the module I transcribed notes on a total of 18 definitions and 22 theorems, and their associated proofs. As an indication of the material met in lectures, a summary of the content over this period is outlined in Table 1 below. This content and its volume were indicative of the course in general.

**My learning in MS1**

Meeting the key ideas and concepts provided me with a broad awareness of this abstract mathematical subject, very much in the manner of a whistle-stop tour. Due to the shortage of time and the number of new concepts and ideas that arose in each lecture, I needed to be disciplined in my learning of the subject. I felt that we flew through it (or over it).
<table>
<thead>
<tr>
<th>Week</th>
<th>Definitions</th>
<th>Theorems</th>
</tr>
</thead>
</table>
| 1    | • Metric space (m.s.) and some examples  
• Open ball  
• Bounded subset  
• Diameter of a subset | • Cauchy-Schwarz (C.S.) inequality  
• An associated corollary of C.S.  
• \( \mathbb{R}^n \) is a m.s.  
• The diameter of an open ball is less than or equal to twice its radius |
| 2    | • Interior point  
• Open set and some examples  
• Interior of a set  
• Convergent sequence  
• Limit point of a set  
• Closed set | • Every open ball is an open set  
• Open sets are closed under arbitrary union and finite intersection  
• Every convergent sequence in a m.s. is bounded  
• A subset of a m.s. is closed iff its complement is open |
| 3    | • Continuous mapping  
• Isometry  
• Cauchy sequence  
• De Morgan’s laws | • Closed sets are closed under arbitrary intersection and finite union  
• Sequential convergence characterisation of continuous functions  
• A map is continuous iff inverse images of open sets are open  
• Isometries between m.s. are continuous  
• Every convergent sequence in a m.s. is Cauchy |
| 4    | • Complete m.s.  
• \( l^w \) and \( c_0 \)  
• Dense subset  
• Completion of a m.s. | • \( \mathbb{R} \) is a complete m.s.  
• Every convergent sequence in a m.s. has a unique limit  
• A complete subspace of a m.s. is closed  
• A closed subspace of a complete m.s. is complete  
• Continuous functions agreeing on a dense set are identical  
• Every m.s. has a completion  
• \( \mathbb{R}^n \) is a complete m.s. |
Table 1: Summary of content of first four weeks of MS1

As a survival learning strategy, it was necessary to commit some ideas to memory. If a student had a question regarding an assignment, he or she was expected to ask the lecturer during the tutorials. I was somewhat intimidated by the analytic nature of the subject and rarely asked for guidance during these tutorials. Academic books were the primary method that assisted and supported my learning of the subject. With reference to Vygotsky’s theory of learning, the more knowledgeable other in this regard was not the lecturer, but in fact my academic books. As a class we rarely discussed the ideas from the module with each other.

From the Pirie-Kieran perspective, of necessity I began working at the Formalising level, due to the DTP approach taken within the learning environment. As a learner, I did not truly engage in the folding back activity that Pirie and Kieren refer to in their theory. I carried out very little of the Image Making, Image Having or Property Noticing of the inner layers of the model, while engaging with the course material.

As a result, I lacked a connected understanding of the concepts and ideas. I had understood the formalised theorems and proofs on the course, I had validated the proofs - but I did not own them. Due to the considerable lack of time and the large volume of content from this course and others, I was unable to immerse myself wholeheartedly in the proofs until studying for the final exam. The ideas fascinated me once I had the opportunity to engage with them, but unfortunately it was too little, too late. There was little chance to peel back the concepts and to flourish at this late stage, with 11 other semester-long subjects to get a handle on, in such a short space of time. The proofs I gave back to the lecturer in the exam were my lecturer’s, and not my own.

In this regard, Marton and Säljö (1976) identified two different levels of processing of material by students, which were subsequently termed the deep/surface approach to learning dichotomy. Students seeking a thorough level of understanding adopted a deep approach and students relying on learning just those pieces of information expected to appear on a test adopted a surface approach to learning. Mason (2012, p. 29) argues that as a university course advances and as the concepts and techniques build up, many students of mathematics, despite their original intentions, are forced to adopt a form of surface learning in order to cope better with the volume of material they are encountering.

METRIC SPACES – TAKE TWO (MS2): THE DEEP ARCHAEOLOGICAL DIG

My second encounter with Metric Spaces was during the first semester of my structured PhD (2012/2013). The module, also at 5 ECTS, spanned 12 weeks of the semester, with an allocation of two 50-minute lecture slots per week and two 50-minute tutorials per week. The tutorials were led by an experienced tutor and there was an option to attend one or both tutorial sessions each week. There were four assignments over the duration of the module, issued in weeks one, four, seven and ten. Usually the assignments were to be submitted exactly three weeks later. Sample solutions for each assignment, chosen from selected students’ work, were published online shortly after the assignment due date.
Teaching mode

While the students got settled during the beginning of most MS2 lectures, the lecturer left the following quote from the famous mathematician Paul Halmos on screen:

Don’t just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis? (Halmos, 1985, p. 99)

The lecturer was a strong proponent of the notions of interrogating, questioning and reasoning, so that the above almost became the course slogan. On the first day of class the lecturer provided the students with the usual housekeeping rules, including a course outline of the topics to be covered in the module. She informed the class that the module would be successfully completed through consistent cooperative and collaborative learning effort and participation in class discussion. From my perspective, this was new and I was intrigued. The lecturer also provided a list of learning outcomes, separate to the course outline. My previous experience of learning outcomes was that they usually turned out to be course outlines or topic lists in disguise. However, for this module, learning outcomes were presented to the students as follows:

On successful completion of Metric Spaces, students should be able to write down, explain and use definitions of key concepts encountered; demonstrate how key definitions emerge naturally from the parent example given by the real line; establish that each example from a given list forms a metric space and illustrate other properties which such examples may have; construct proofs which connect and relate metric concepts; produce examples which illustrate and distinguish definitions; write down all mathematical work with rigour and precision; create new or other lines of mathematical enquiry on the basis of mathematical ideas encountered.

The lecturer varied her exposition of the course material through presenting orally, writing on a tablet and on the whiteboard, and discussion with the class. This was incidental however to the actual business of doing the mathematics, accomplished by the lecturer and the class in collaboration.

My learning in MS2

I considered the above learning outcomes to be generalisable across any abstract mathematics course. Weber (2001) hypothesised four types of strategic knowledge that undergraduates appeared to lack when constructing proofs – knowledge of proof techniques, which theorems are most important, when particular facts and theorems are likely to be useful and when one should or should not try and prove theorems using symbol manipulation. MS2 attempted to develop the students’ strategic knowledge, enabling us to make proofs for ourselves. I learnt to construct my own proofs connecting and relating concepts, writing my work down with rigour and precision. This was evident from detailed annotations kept by me while attending the module. I maintained these notes to supplement my efforts to gain an understanding of my own personal learning of abstract mathematics. This procedure was not possible in MS1, due to the necessity of full-time transcription.
A typical classroom experience

Doing the mathematics was an important aspect of the classroom experience of MS2. As an example of what I did while learning MS2, I have chosen to outline my understanding of a particular theorem. The first theorem that was introduced in MS2 was the following:

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and let \(f : X \rightarrow Y\) be a map. Then \(f\) is continuous if for all open sets \(G \subseteq Y\), \(f^{-1}(G)\) is open in \(X\).

Prior to the introduction of this theorem, a considerable amount of time was spent by the lecturer on developing the ideas related to it. The definitions of metric space, inverse image, open ball, continuous function and open set had been developed and discussed in extensive detail, during both the lectures and tutorials and as part of the first worksheet.

The concept of continuity of a function had been introduced initially in terms of a real-valued function and then translated from this specific context into the more general case of a function on an arbitrary metric space. The lecturer emphasised the importance of pictures when trying to get to grips with these concepts. When explaining the notion of continuity of a function \(f\), a picture was introduced on the plane first, followed by a more abstract picture of a function between two arbitrary metric spaces. I identified with the use of pictures as being very important for developing an understanding of concepts met on the module. When meeting a new definition, I always began by drawing a picture in order to get an idea of what the definition was saying. In doing this I was working at the Pirie-Kieren level of Image Making.

Throughout the module, the lecturer constantly emphasised the importance of making proofs for oneself. Once the above theorem was introduced in a particular lecture, she worked through the proof that every open ball is an open set in collaboration with the class. The next lecture began by the lecturer writing the following four conjectures on the blackboard:

Suppose you have a metric space \((X, d_X)\) and that you know the definition of an open set in it. Is \(\emptyset\) open in \(X\)? Is \(X\) open in \(X\)? If \(G_1\) and \(G_2\) are each open in \(X\), is \(G_1 \cap G_2\) also open? If \(G_i\) is open for each \(i\) in some index set \(I\), is \(\bigcup_{i \in I} G_i\) also open?

The proofs were worked through together, by the lecturer and the students as a collective. The lecturer began by asking the students to provide the definition of an open set. She emphasised that this was the third definition we had met on the course, and that if we internalised these definitions, then we should be able to do any of these tasks quite easily. This was the lecturer’s established approach from the outset of MS2. With respect to the above tasks, she emphasised that there were only two things required to be known by the student – the mathematical definition and how to apply reasoning. Edwards and Ward (2004) emphasise the need for students to have experiences that focus on the use of mathematical definitions. They also argue that the special nature of mathematics definitions should be identified more directly in courses at all levels. This indeed was a tactic that the lecturer was using on the ground, hoping that students would come to understand the important role of mathematical definitions and thus internalise the ideas being met through them. The emphasis that was put on the role of mathematical definitions was an important factor in the development of my understanding of the subject.
Only now did the lecturer attempt to begin a collaborative proof of the theorem. The first part of the proof (left to right) was set out initially by her in the form of a discussion with the students, as follows:

1. We began by drawing a picture encapsulating what it meant for $\mathbf{f}$ to be continuous (via $\varepsilon - \delta$).

2. The lecturer then posed the question: “Do we know what open set means?”

3. Next she asked: “Do we know what inverse image means?”

Drawing the picture is an example of my Image Making; where I engaged in an activity to get some idea of what a continuous function between metric spaces looks like. This is the point (Image Making) at which the mapping seen in Figure 2 starts. Through this activity, I had an image for continuity and could explain it to myself as follows: to be a continuous function means that for every open ball with a positive radius in $\mathbb{R}$, there is some open ball of positive radius in $\mathbb{R}$ (Image Having in the Pirie-Kieren model).

The proof was started by introducing an open set $G \subseteq Y$. To identify with this notion of an open set, I reverted back to making a picture (Image Making) of two arbitrary shapes for metric spaces $X$ and $Y$, with a function map $\mathbf{f}$ between them. I drew an open set $G$ in $Y$, together with its inverse image $f^{-1}(G)$ in $X$.

Taking an arbitrary point $x$ in $f^{-1}(G)$, I identified that it was necessary to find some open ball centred on $x$ inside $f^{-1}(G)$, in order to show that $f^{-1}(G)$ was indeed an open set. This was an example of me working at the Property Noticing stage of the Pirie-Kieren model.

I noticed that the assumption of continuity of $\mathbf{f}$ meant that it was necessary to start over in $Y$ (I was again working at the Property Noticing level). The lecturer then posed the question: “What allows the introduction of an open ball around $x$ in $G$?”

I noticed from my picture that $G$’s property of openness meant that it was possible to introduce an open ball around $x$. Translating this into mathematics I wrote the following in my notes: So, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq G$.

I was formalising my idea into mathematical language in order to create a proof, an example of my progression from the Formalising layer to the Structuring layer of the Pirie-Kieren model.

Returning to Property Noticing, I made the connection between the assumption of continuity and the fact that there was an open ball in $X$ for this open ball $B_\varepsilon(f(x))$ in $Y$. The written mathematics structured my argument: By continuity, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x)$ exists in $X$.

I went back to working at the Image Making level by noting that “the function $\mathbf{f}$ grabs everything in this $\mathcal{G}$-ball and sends it to the $\mathcal{T}$-ball”.

Structuring this mathematically, I wrote: $f(B_\varepsilon(x)) \subseteq B_\varepsilon(f(x))$.

I then wrote this question to myself: “Where is the $\mathcal{T}$-ball? Inside $G$”, therefore completing the above with: $f(B_\varepsilon(x)) \subseteq B_\varepsilon(f(x)) \subseteq G$. Thus, this ball $B_\varepsilon(x) \subseteq f^{-1}(G)$. 

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I finally wrote down that “each point in the $d$-ball gets mapped into the set $g$”, therefore completing that part of the proof.

The above example shows me using the process folding back in order to gain a better understanding of the theorem I was working with. The extent to which I folded back is illustrated in Figure 2. Martin (2008) and others consider effective folding back as a powerful facilitator of a developing mathematical understanding.

My understanding of the theorem was supported by my social learning environment and the lecturer’s role as scaffolder to my learning. I used the processes of inner speech and image making as mechanisms to develop my learning, aspects that are harmonious with Vygotsky’s socio-cultural theory of learning (Vygotsky, 1978).

CONCLUSION

In the end, I believe that my experience in MS1 is captured succinctly by Dreyfus (1991, p. 28) in his reference concerning the typical undergraduate mathematics experience:

“...they [students] have been taught the products of the activity of scores of mathematicians in their final form, but they have not gained insight into the processes that have led mathematicians to create these products.”

In MS1, I had received an introduction to Metric Spaces in a very organised and comprehensive manner. I had been able to process and validate most of the proofs given and, during exam preparation, I had even begun to glimpse and consequently appreciate their meaning. But I had no sense of doing mathematics. The doing had already been done - by others – and my job, at best, was to understand the proofs given and to return them to the lecturer at exam time. I could not make connections on my own. I did not learn how to think like a mathematician as there was neither time nor space in class for thinking.

In MS2, the introduction given was similarly well-organised and comprehensive. The lecturer’s style was in many ways to assume the form of a student. Proofs were (re)discovered in class as a normal activity. The lecturer thought her way through each proof, as if for the
first time, and she sought students’ contributions to an unfolding proof. Whether a ruse or not, her occasional revelation of ‘being stuck’ often produced valuable student intervention. Class time was spent thinking and doing mathematics; we were prodded and pestered for our participation. I began to think like a mathematician because I had time and opportunity to do so. I could make connections, I could ‘do proofs’, not just read them or write them out. I just needed time and insight into how mathematicians think.

In view of the above, there may well be implications for university teaching, especially concerning the development and enhancement of advanced mathematical thinking.

REFERENCES


CROSSING BOUNDARIES INTO SECONDARY MATHEMATICS CONCEPTS: THE CASE OF FUNCTIONS

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This paper reports on the use of a particular function machine which was based on the context of the ‘faulty oven’ to develop conceptual understanding of the concept of function among 4th class children. While the concept of functions is traditionally associated with secondary level mathematics, this research shows that the identification of appropriate models and contexts supports primary level children in accessing complex mathematical ideas. This approach was researched, developed, and tested in classrooms as part of a Lesson Study project with pre-service elementary teachers and mathematics teacher educators.

INTRODUCTION

Agreement exists that “The quality of teaching of mathematics … is essential to improving mathematics proficiency” (EGFSN, 2008, p. 7). Research highlights the significance of mathematics at primary level. It is reported that early successes in mathematics facilitate the development of the necessary foundation, interest and motivation to successfully study mathematics during formal education (NCCA, 2006; EGFSN, 2008). Therefore while there was a time when people doubted that kindergarten and elementary grade children were capable of learning algebra, over time it has been accepted that algebra should be addressed from the early elementary grades given its potential to develop children’s ability to analyse, represent and generalise mathematical relationships (NCTM, 2000; Suh, 2007).

While some subdomains of algebra, e.g. quadratic equations, are too advanced for young learners there are many algebraic ideas that are within their reach e.g. equality. One aspect of algebra that is suitable to teach to elementary school children is the concept of functions due to its potential to develop appropriate algebraic thinking (Reeves, 2005/06; Van de Walle, 2003). While a formal treatment of function does not occur until secondary school in the Irish education system, the concept of function can be introduced informally in the primary classroom. By definition, a function is “…a rule that uniquely defines how the first or independent variable affects the second or dependent variable” (Van de Walle, 2003, p. 436). Research has shown that this concept can be introduced in primary classrooms provided that adequate supports are provided to children when reasoning about this concept (Reeves, 2005/06; Suh, 2007). In order to make this concept of function accessible to children, the way in which the concept is introduced is crucial. In reality to teach algebra as most of us were taught would be potentially disastrous.

Mathematics education research plays a crucial role in investigating the effects of “artificial objects” on the development of mathematical understanding through the design and cyclical process of testing, reiteration and modification of objects (Wittman, 1998). When an emphasis is placed on both objects and instructional approaches, children can be supported in accessing powerful mathematical ideas. While many tools (e.g. technology) and approaches are used in primary classrooms to develop understanding of complex mathematical concepts,
we briefly outline one approach we drew on in the design of lessons that supported primary children in gaining access to the concept of function: models. *Models and manipulatives are well established in primary mathematics education particularly in relation to place value and number work. However, in order to be optimally effective, young children need support in making connections between the analogies embodied by the tools/manipulatives and the mathematical ideas, i.e., that the children use the models/manipulatives to represent the formal mathematics within the problem to be solved* (Gravemeijer, 1999, p. 159). Popular approaches when teaching functions include the use of visual/geometric growing patterns, physical function machines and appropriate children’s literature (e.g. *Two of Everything* by Lily Toy Hong (1993)). Function can be introduced in tandem with work on patterns and relations. However, physical function machines provide the potential to develop a conceptual understanding of function as it allows children to model the problem situation and analyse the change which comes about as a result of a function using simple tables (Reeves, 2005/06; Suh, 2007; Billings et al., 2007/08). Given that within the Irish context, various studies have reported unsatisfactory use of resources in the strand of Algebra (Shiel et al., 2006), the authors were anxious to explore the potential uses and impact of models/manipulatives within this mathematics strand.

This paper examines the mathematics content area of algebra; a domain traditionally considered to ‘belong’ to the domain of secondary level mathematics. It focuses in particular on the concept of function, describing the use of a specific tool to mediate learning. A physical function machine was used to provide a meaningful context and concrete vehicle to support children in developing conceptual understanding of function.

**METHOD**

**Participants**

This study was carried out with 21 final year pre-service primary teachers during the concluding semester of their teacher education program. Participants had completed their mathematics education courses (three semesters) and all teaching practice requirements (at junior, middle and senior grades) and self-selected into mathematics education as a cognate area of study. Four participants were male; the remainder were female. Two participants were international Erasmus students.

In the particular group working on the concept of function, there were five pre-service elementary teachers who worked alongside three mathematics educators to design a series of instructional activities to promote development of a deep understanding of function. Initial instructional activities developed were field-tested and refined through working with two different 4th class groups in different schools in Limerick city.

**Lesson Study**

All pre-service teachers, and three mathematics educators, engaged in *Japanese Lesson Study* which is an approach for studying teaching that utilizes detailed analyses of classroom lessons (Fernandez & Yoshida, 2004; Lewis, 2002; Lewis & Tsuchida, 1998). In this study, lesson study was used to examine the planning and the implementation of lessons in classrooms and
thus facilitated the design of tools and sequences of instruction to support the development of algebraic reasoning with primary children. This paper reports on the work of one lesson study group - the ‘Function’ group.

The research was conducted over a 12-week semester. Participants worked collaboratively in groups of 5-6 on the design and implementation of a research lesson. While the first phase involved the research and preparation of a study lesson, i.e., researching the concept of function in order to construct a detailed lesson plan, the implementation stage involved one pre-service teacher teaching the lesson in a 4th class primary classroom while the remainder of the group and the researchers observed and evaluated classroom activity and student learning. Subsequently, following discussion, the original lesson design was modified in line with their observations. The second implementation stage involved re-teaching the lesson with a second class of primary children and reflecting upon observations. The second implementation was videotaped. This cycle concluded with each lesson study group making a presentation of the outcomes of their work to their peers and lecturers at the end of the semester.

EXPLORING THE CONCEPT OF FUNCTION

This paper provides a detailed description of the teaching of the concept of function, with a particular focus on the tool used to facilitate the development of algebraic reasoning.

Setting the scene: The context of the ‘Faulty Oven’

To launch the lesson, the teacher presented the scenario, “Yesterday I was baking. I have to admit I like baking.” The children were immediately engaged with the context and eager to find out more. After asking the children to indicate whether they liked baking through a show of hands, the story continued with the teacher pointing to a picture of two cakes (see figure 1) stating, “I put these two cakes into my oven and something very strange happened. When I opened the oven door something had gone terribly wrong.” Children were then invited to predict what might have happened. Students’ responses were initially generic, e.g. “They were burnt” (Bernadette); “They were the wrong colour” (Callum); “Did they taste bad?” (Ryan). The teacher revealed that when the oven was opened there were not two but rather four cakes (see figure 2). On prompting, the children came up with mathematical explanations of what had happened:

Teacher: What do you think happened?
James: The oven added two. (+2)
Grace: The oven doubled them. (x2)
Joseph: Maybe they used self-raising flour?

Figure 1: Picture of cakes placed in oven
Additional scenarios were presented to provide children with further opportunities to predict and identify the nature of the function, e.g. “Later I made five lasagnes. Can anyone guess what happened when I opened my oven?” Children made predictions (e.g. seven lasagnes (rule: +2), ten lasagnes (rule: x2 or double)). Again children were presented with the image of the lasagnes which came out of the oven (10 lasagnes). By working through this second example, children tested predictions and made a decision regarding the ‘rule’ being used. When the class was asked what was wrong with the oven, a student Keith, responded “It’s a cloning machine.” This statement highlighted that the children had quickly grasped the concept of function. Through the use of probing questions such as “What happened to the food each time?” and further scenarios (e.g. “If I then made three apple pies…” ) students successfully identified the rule e.g. “it multiplied” (Ross); “it doubled” (Joseph).

Subsequently children were informed that the oven was a magic machine called a “function machine”. Some participating teachers chose “math rule maker” as an alternative label, believing that it provided guidance to children regarding the working of the machine (“So it makes math rules”), thus making the concept of function more accessible. The physical ‘faulty oven’ function machine (an aluminium foil covered box with coloured buttons) was revealed to the class (see figure 3) and generated great mystery and amusement amongst children. The following excerpt demonstrates how the features of the function machine (input, output, and rule buttons) were introduced.

**Teacher:** So this is the faulty oven we use to make math rules. It has two doors. We’ve got a little door here. This is where we put in our food.

*(Teacher points to the input drawer, see figure 4)*

There is another door at the back where the food comes out.

*(The teacher opens the output door, see figure 5)*
Anything we put in here is the ‘input’. *(Pointing to the input door)*

Why do you think we call it the ‘input’?

Cian: It’s what you put in.

Teacher: It’s what you put in it, excellent. So ‘put’ and ‘in’ – ‘input’… What do you think we call what comes out?

Lorna: The output?

Teacher: The output- in and out. It’s easy to remember isn’t it? Now do you see these buttons down along the side of the machine. *(The teacher points to the front of the machine, see figure 3)*

What do you think they do?

Louis: Am …that’s what makes them multiply.

Teacher: That’s what makes them multiply-ok- and would each button do the same thing?

Bernadette: They could add, multiply, divide or minus.

Teacher: Excellent, so they might all have different…?

Children: Rules.

One child subsequently volunteered his interpretation of the various parts of the machine

Robert: The button is a switch and the output is what the switch activates.

This thinking was quite sophisticated and suggests that the function machine context provided a relevant concrete vehicle to facilitate children in developing conceptual understanding of function.

**Using the ‘Faulty Oven’ function machine**

The focus then moved to children experiencing the math rule maker in action. Base ten materials (e.g. Dienes’ blocks) were used to represent the food (input and output). All inputs and outputs were prepared in advance in plastic trays (see figures 4 and 5). While children could see and examine input trays, output trays were hidden (unknown to children) in a secret compartment in the back of the function machine (see figure 5).

A child was invited to the top of the class (see figure 6). He/she counted the number of Dienes’ blocks in the pre-prepared input tray (e.g. 7 cubes), placed the tray in the input slot and pressed the appropriate (function) button, e.g. blue. The teacher, not the child, then opened the slot at the back of the function machine (figure 5), discretely hid the input tray and selected the appropriate output tray (figure 6). On handing this tray to the child, the child counted the output (e.g. 11 cubes). Children were not aware of the exchange of trays which took place in the function machine and were surprised when the output tray had a different number of cubes than the input tray. Led by the teacher, the class recorded the 3-4 inputs and outputs for each selected button in an effort to predict the rule. While the teacher used a poster to record inputs and outputs for the selected button (see figure 6), children recorded on a pre-prepared worksheet.
Figure 4: Input drawer and tray

Figure 5: Back view of Faulty Oven (output drawer and trays)

Figure 6: Using the function machine
On presenting a number of related inputs and outputs, the teacher gave the class time to work out the possible rule/function (e.g. blue rule could be +16 or x9 after the first input/output—see figure 8).

<table>
<thead>
<tr>
<th>Blue Button</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>23</td>
</tr>
</tbody>
</table>

To promote success, initial rules for the buttons focused on one-step rules (e.g. “add five” or “double”). However, the level of challenge increased incrementally culminating with the final button (yellow in this case) which consisted of a more complex 2-step rule, e.g. “multiply by two and add three.” When being introduced to two-step rules, children were informed that all rules so far were one-step rules, e.g. x2, +15, x9. They were invited to provide an example of a two-step rule. We found there was a tendency for children to provide examples such as “+3+5” (Monica) and “+25-20” (William). The teacher highlighted that such examples e.g. +3+5 could be described as a one-step rule i.e. +8. Children also had the opportunity to work out the input when given output i.e. work backwards and justify their approach, e.g. “If the output is fifty-four, what is the input?...how did you know that?”

Subsequently, children gave examples of two-step rules. The following excerpt illustrates the nature of the dialogue which took place in one classroom when working out a two-step rule.

Teacher: Our input was?
Pavol: Seven
Teacher: …and our output is?
Class: Seventeen
Teacher: Now if that was a one-step rule what could the rule be? Emma?
Emma: Add ten
Teacher: Add ten
But it’s a two-step rule so there are two steps in it. O.K.
(The teacher gestures to William to come help with the next input/output for the yellow button)

So what’s our input this time?

William: Eight

Teacher: Put it (Refers to the Dienes’ blocks) in and press the yellow two-step button.

(William places the tray into the input drawer and presses the yellow button. The teacher discreetly selects the appropriate output tray from the back of the machine and presents it to William)

What is the output?

William: Nineteen.

Teacher: Your input was eight and your output was nineteen. What could the rule be? It could be adding - what else could it be doing?

Monica: Multiplying.

Teacher: Multiplying and adding. Has anyone any other idea?

(Students given some time to think, some children start to raise their hands)

Ok I am going to give you a hint: the first step is ‘multiplied by two’. So we multiply it by two first, then we do something else to it. Have another look at it now.

(The teacher circulates and observes while the children work and raise their hands)

Lorna, what did you get?

Lorna: Times two plus three?

Independent work

Once the four rules had been identified, children then worked in groups to generate and identify their own rules/functions. Groups were provided with record sheets, these sheets reflected many of the features of the original function machine – they used the same language (input/output) and contained the same perceptual cues to identify rules/functions (coloured buttons similar to that found on the function machine). Each group member took turns to act as the function machine. This involved the ‘math rule maker’ child secretly constructing a rule (e.g. +5). She then received an input (between 0 and 10) from each group member and calculated the corresponding outputs (figure 9). Children were encouraged to use multiplication and addition only in their rules, given that rules involving division and subtraction may lead to outputs being fractions or negative numbers. Children were encouraged to create a two-step rule if they wished. Each group member recorded all the relevant information (inputs, outputs) in order to promote prediction and checking. The rules which students generated ranged from “plus five” to “times seven” to “times nine plus one”. Children experienced little difficulty making and testing predictions regarding the rule. Two-step rules proved more challenging in this regard, and required children to demonstrate perseverance when using strategies such as trial and error, guessing and checking.
As a concluding activity, children were given the opportunity to share their rule with the class. While only one example of a two-step rule or function was taught during the lesson, the independent work highlighted that teaching mathematics using effective models and representations helps develop sophisticated mathematical reasoning and understanding among children. The next excerpt reflects the reality in many classes where the complexity of some of the children’s rules was a source of amazement for the teacher as well as a source of genuine challenge for the children.

Teacher: Robert’s rule is so complicated it only works for even numbers.
Teacher: Can you give Robert an input? Remember it could be division or subtraction in this rule. It’s a two-step rule, is it?
Robert: Yeah.
Teacher: Can we have an input?
Josh: Two.
Teacher: Two. Robert, what’s your output?
(\textit{Robert writes ‘7’ as the related output on the board})
We’ll take another input.
Callum: Ten
(\textit{Robert writes ‘11’ as the related output on the board})
Teacher: O.K., David?
David: Four.
(\textit{Robert refers to his worksheet and then writes ‘8’ as the related output on the board. Students are given some time to think and work})
Robert: The first step is addition and the second step is division.
(\textit{Students given further time to think and work, additional students raise their hands})
Teacher: Another hint: the first step is ‘plus twelve’. O.K., and the second step is division. So what you will have to do is imagine adding twelve to our input and try and figure it out then.
IN CONCLUSION

The development and use of the tool described in this study were critical in supporting relatively young children in reasoning about functions. The research indicates that functions can be taught to primary school pupils and while these were children were taught by relatively in-experienced (pre-service) teachers, we believe that the design of the physical function machine which facilitated the use of concrete materials; alongside the selection of the context which the children could relate to; meant that the concept of function was accessible to these relatively young children. The function machine acted as an effective tool in promoting conceptual understanding of function among children in the participating schools, where all of the children engaged in the lessons and developed the appropriate conceptual understandings. In fact some of the children exceeded the expectations.

This research carried out in classrooms also shows that understanding functions can be fun! The use of the ‘faulty oven’ function machine led to much enjoyment. While the creation of a function machine requires effort in sourcing and assembling the component parts, once made, the function machine is an invaluable resource. Beyond algebra, the function machine can be used to promote understanding of operations and number facts. We found that due to the novel appearance and unconventional working of the function machine, children were interested and excited in exploring and creating functions, and the classrooms we visited were buzzing with questions and predictions. However as is evident from this paper, using the function machine does not mean reducing the mathematics challenge provided to children. In short, the approach has the potential to promote students’ motivation, enjoyment, involvement and most importantly understanding of the concept of function.

REFERENCES


We are witnessing the rise of MOOCs as university-based commercial experiments in global education. These MOOCs (including www.coursera.org, www.udacity.com, and edx.org) provide an historical opportunity to democratize access to educational materials, while at the same time, challenging the very notion of the traditional “brick and mortar” university. This paper will describe an unfolding set of analyses on how to improve the effectiveness of massive online open courses, using an example from online calculus learning. The target MOOC, which has been in existence for almost a decade – WEPS.com – provides a best practice example to help ensure the educational effectiveness of the emerging global phenomenon of university forays into cyberinfrastructure. In the presentation, we will summarize insights from two National Science Foundation workshops on MOOCs that involved George Mason University in the US, and the University of Helsinki, Finland.

MASSIVE ONLINE OPEN COURSEWARE

Significantly large numbers of students signed up for free AI courses at Stanford University (160,000 students in a single course, for example), which led to the founding of a company to run similar courses (https://www.udacity.com/). Over 3.7 million students are part of Coursera, a consortium that offers 376 courses from over 80 educational partners. A similar consortium with course offerings has been created, called EdX, made up of leading institutions such as Harvard and MIT. EdX is developing coursework with 12 universities from the US, Canada, the Netherlands, and Australia. The Gates Foundation recently gave community college financial support for students to take EdX courses (http://www.gatesfoundation.org/How-We-Work/Quick-Links/Grants-Database/Grants/2012/06/OPP1061475).

Collectively, these free course offerings provided at a global scale are called massive online open courses (or MOOCs). The rise of MOOCs has drawn national attention, including the attention of the White House’s President’s Committee of Advisers on Science and Technology (http://www.whitehouse.gov/administration/eop/ostp/pcast). While MOOCs have been hailed, perhaps prematurely, as the solution to global education needs (http://www.technologyreview.com/news/506351/the-most-important-education-technology-in-200-years), little is known about how to design, evaluate, and pay for them. For example, of the 44501 students who took an education MOOC at Stanford, 95% did not take the final, and only 291 (7 in every 1000) completed papers to receive a certificate (http://suse-www.stanford.edu/news/online-course-attracts-40000-participants-and-questions-gse-students).
Principles in MOOC design applicable to an online calculus project

Our goal in this paper is to review some principles for the design of MOOCs in education, following the experiences of offering online courses in calculus for almost a decade (www.myweps.com). The paper will use as its blueprint a successful massive online open course (MOOC) system in mathematics offered by the University of Helsinki, involving 17 institutions in 12 countries (Descamps, Bass, Evia, Seiler & Seppälä, 2006). This MOOC (https://myweps.com/moodle/course/category.php?id=106) has received international recognition by its inclusion in a unique relationship between the US National Science Foundation and the Academy of Finland (Kelly & Seppälä, 2012).

WEPS (developed in Moodle) already collects the following data: (a) a mathematics placement test, (b) a survey of students’ experiences and expectation for the course, (c) the outcome of peer-review workshops of challenging conceptual problems in various topics in calculus. The system provides a model solution and grading suggestions; five students grade each other’s work, and report grades to the system; the system takes an average of the five grades. Students earn a grade for the problems, and a grade for grading other’s work, (d) automatically generated mathematics quizzes with step-by-step feedback on errors, (e) when (by date), how often, and for what duration each student accessed each instructional resource (e.g., videos, quizzes, PDF files), and (f) the IP of the student’s computer, which can provide GIS identifiers for student geographic location.

WEPS resources include: (a) video files, (b) pdf files, possibly with audio for accessibility, (c) chat rooms, and discussion forums, (d) practice exercises and quizzes, and (e) tutoring sessions.

The existing MOOC has been subject to two experimental comparisons. In one study, the MOOC students had less attrition, and outperformed a traditional course at the University of Helsinki (Descamps et al., 2006). Seppäälä reports that a proof-based advanced calculus course offered online (in 2011) also outperformed two traditional courses (Seppäälä, personal communication, 2012).

The current NSF-Academy of Finland project has held two meetings with approximately 30 US and international experts in mathematics, mathematics education, cognitive science, data analytics, data mining and statistics to upgrade the capacity of the MOOC to: (a) better understand students’ knowledge of mathematics, (b) continuously assess their progress, and (c) use student data to dynamically improve the instructional resources in the learning environment. The proceedings of these meetings are being prepared for publication and will be reported on during the MEI5 conference. Topics to be covered include: (a) the larger literature on online education, (b) the need for specifying precursor data on students, (c) the opportunity to experimentally explore novel pedagogical manipulations, e.g., the use of spatial abilities studies to inform instruction, and (d) gaining access to video data logs.
Review of online learning

The research framework in which MOOCs are designed is not well described. We present a brief review of the pertinent studies on online learning by Phelan (in press), who notes that: a) no significant differences between comparable online and face-to-face instruction tend to be found; b) very few rigorous studies—especially in K-12; but c) there is promise in blended learning environments. Whether technological advances have the potential to provide flexible access to instruction and course content to a larger number of students cost-effectively has yet to be adequately researched.

Phelan also notes that the Department of Education (2010) quantitative meta-analyses included studies of web-based instruction in which objective measures of student learning were examined using experimental/quasi-experimental designs. Many studies did not fit these criteria and so were reserved for narrative synthesis. Only half of the 99 studies provided adequate data to compute effect sizes. Thus, even though some studies show modest differences in learning outcomes, more rigorous exploration to determine all the dimensions in which both the instruction and student population differ along with the learning environment. The issues of rising costs and accessibility were not considered.

Precursor data

Student progress in a MOOC may be tied to data collected on prior occasions. We will report on the work of Carlson, and Bressoud and Rasmussen on micro and macro data that may be pertinent for analysis in a calculus MOOC.

Micro data may be in the form of a calculus readiness test (Carlson, Oehrtmann & Engelke, 2010). The table summarizes what is measured by the test¹.

### Topic

<table>
<thead>
<tr>
<th>Reasoning Strands</th>
<th>Number of Test Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quantitative Reasoning</strong> involves identifying and relating measurable attributes of an object or situation in a problem context.</td>
<td>9</td>
</tr>
<tr>
<td><strong>Proportional Reasoning</strong> involves thinking about how two quantities change such that their ratio remains constant; attending to how one variable changes so that it is always a constant multiple of another variable.</td>
<td>3</td>
</tr>
<tr>
<td><strong>Covariational Reasoning</strong> involves thinking about how two quantities in a functional relationship are changing together; attending to how one variable changes while imagining successive amounts of equal changes in another variable. It involves coordinating two varying quantities that change in tandem while attending to how the quantities change in relation to each other.</td>
<td>14</td>
</tr>
<tr>
<td><strong>Process View of Function</strong> involves thinking of a function as an entity that accepts a continuum of input values to produce a continuum of output values; it views a function as a generalized process that accepts input and produces output, and appropriately coordinates multiple function processes.</td>
<td>11</td>
</tr>
<tr>
<td><strong>Notational Reasoning</strong> involves making sense of symbols used in mathematical expressions and giving meaning to the mathematical ideas communicated by conventional notation.</td>
<td>9</td>
</tr>
<tr>
<td><strong>Graphical Reasoning</strong> involves making sense of graphs that represent functions, and interpreting the meaning of attributes of a graph that convey aspects of a function’s behavior.</td>
<td>9</td>
</tr>
<tr>
<td><strong>Computational Abilities</strong> refers to facility with manipulations and procedures needed to evaluate functions, solve equations, compose functions, and invert linear and exponential functions, within the context of algebraic representations.</td>
<td>6</td>
</tr>
</tbody>
</table>

### Content Areas

| Proportions: Ratios of quantities in constant proportion | 2 |
| Algebra: Algebraic expressions, equations, inequalities | 9 |
| Functions: Concept, properties, operations | 13 |
| **Representations of Functions**: Symbolic, graphical, tabular, contextual (verbal) | 9 |
| Analytic Geometry: Circle, parabola, line | 7 |
| Trigonometry: Functions and applications | 5 |
| Models: Functions as models | 5 |

Macro data come from Bressoud, Carlson, Mesa and Rasmussen’s (2013) work from the first large-scale national survey of mainstream Calculus I instruction. Supported by the Mathematical
Association of American and a grant from the National Science Foundation, the survey was conducted across a stratified random sample of two- and four-year undergraduate colleges and universities during the fall term of 2010 (see Lutzer, Rodi, Kirkman & Maxwell, 2007 for selection criteria). Of the 521 colleges and universities that were selected, 222 participated: 64 Associate degree colleges, 59 Bachelor degree colleges, 26 Master’s degree universities, and 73 Doctoral universities. 660 instructors and over 14,000 students responded to at least one of the surveys. Preparation for the surveys included a literature review leading to a taxonomy of potential dependent and independent variables followed by constructing, pilot testing and refining the survey instruments.

The results we report from these surveys provide a picture of who takes calculus in college: their personal and academic background, their motivation and career intentions, their attitudes and beliefs about mathematics. We also report on students’ classroom experiences that contribute the most toward improving student confidence in mathematical abilities, enjoyment of mathematics, and desire to continue the study of mathematics (e.g. Tinto, 2004).

**Experimental opportunity**

Uttal and Cohen (2012) showed that spatial skills are strongly predictive of achievement, persistence, and attainment in STEM fields. This relation holds true even after accounting for other variables that also predict entry and persistence in STEM fields, such as mathematical and verbal aptitude.

We know that spatial skills are related to STEM achievement; however, we don’t know why. Ostensibly, the reason for the relation seems obvious: STEM fields are spatially demanding, and thus those individuals with higher levels of spatial skills will do better in STEM courses. However, a review of research on the relation between spatial skills and STEM learning suggests that the relation between spatial skills and levels of achievement decreases as students move from novice to more expert levels. Experts may rely more on factual semantic knowledge and less on spatial skills such as mental rotation. Consequently, many people may drop out of STEM fields because they have difficulty with the spatial demanding entry-level courses. If some method could be found to help students through this early period of learning, then perhaps more students could succeed in STEM fields. MOOC environments may offer a unique test bed for such studies.

**Gaining access to video data**

Emerging Massive Open Online Courses (MOOCs) rely heavily on video files. Since videos contain multiple frames that are not “tagged” they are of little value to educational research since they cannot be indexed and searched without major budgets for “hand coding” of individual video frames. To search engines, video files are effectively “black boxes.” We will briefly present a new technology for unlocking video files for search and metadata tagging since videos are central to many educational interventions (see the papers by Wang and colleagues, 2003-10).
The method used involves first stripping the audio track and visual frames of the video. Audio is then transcribed using a proprietary algorithm. The initial results are then cross-referenced with commercial algorithms to achieve higher accuracy. Visual frames are analyzed with image recognition techniques to identify words, objects, and people displayed. Both sets of data are then directly indexed to the original video file. Because the processed data is structured and accessible to both humans and computers, these data can be used as a proxy for the contents of the complete video file. Thus, analytical and search techniques can be applied to these structured data streams to provide unique analysis of the underlying video file.

Initial research has shown the effectiveness of this approach on small sets of videos. Research is currently in progress to a) test the model on a larger corpus of educational video data and b) to develop a new iteration of the software that will allow for its use on a much larger and potentially commercial scale. If successful, this research will open major opportunities for educational research and classrooms, particularly in the area of online and distance learning.

REFERENCES


A STUDY OF MULTI-DIGIT ADDITION IN FIRST CLASS: THE IMPORTANCE OF MENTAL STRATEGIES AND INVENTED PROCEDURES IN DEVELOPING NUMBER FLUENCY

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In primary schools, standard algorithms remain the focus of emphasis and repetitive ‘drill and practice’. Research shows that children become preoccupied in trying to remember the steps of algorithms, and often do not understand the procedure they are applying. In this study, I sought to develop children’s number fluency, by focusing on their own invented algorithms. Analysis of the data generated by the teaching experiment revealed a significant improvement in the children’s ability to use number flexibly, and also showed the considerable progress made by every child in inventing algorithms for adding multi-digit numbers. Furthermore, enabling the children to communicate and express their mathematical practices emerged as a very important theme of the study. In this paper, I will give the reader insight into the children’s engagement with the teaching experiment and demonstrate how the experiment helped the children to become ‘proceptual thinkers’.

INTRODUCTION

In a pre-technological era, arithmetic skills and accurate calculation were crucial to the functioning of society (Brown, 2001). The teaching of pen and paper algorithms was the foundation of mathematics curricula (Reys and Nohda, 1994) as an algorithm was considered to be reliable, accurate and efficient and presented the user with a written record of work (Usiskin, 1998). In a considerably, more advanced society, people are less dependent on arithmetic skills and algorithms, as calculators and computers are widely available and are often more efficient. However, the teaching of standard algorithms is still common practice in primary schools and much emphasis is placed on applying the algorithm (Anghileri, Beishuizen, and Van Putten (2002); Kamii and Dominick, 1998; Usiskin, 1998).

The Primary School Mathematics Curriculum

The Primary School Mathematics Curriculum (PSMC) advocates that children learn the operation of addition, of numbers up to 99, with and without renaming, in First class. Highlighted in the curriculum is the development of arithmetic skills as an “important part of the child’s mathematical skills” (p. 7). The PSMC places less emphasis than previous curricula on complex paper and pencil calculations and a greater emphasis on mental calculations, estimation and problem solving (ibid.). Calculators are recognised as a worthwhile and beneficial addition to school mathematics but the availability of such a resource does not lessen the need for understanding and basic skills in computation (ibid.). The PSMC does not overtly advocate or insist upon the use of standard written algorithms in the teaching of number operations, however, it is covertly recommended as one of several exemplars to achieve this objective but is not regarded as ‘the way’ to teach addition. Yet in schools, students are taught conventional or standard algorithms for operations within the number strand of mathematics and in my opinion, too much emphasis is placed on learning
the procedure of the standard algorithm as if it were a curriculum objective instead of an exemplar.

**Personal experience**

Prior to instruction on the standard algorithm, pupils in my class displayed good understanding of place value and had developed mental strategies for the addition of single-digit numbers, including, ‘locking the bigger number in the head’ and counting on, doubles, near doubles, etc. which they found successful. Exploration of multi-digit numbers followed, with children finding means to calculate solutions mentally, e.g. estimating. The introduction of the standard algorithm for addition, involving ‘carrying’, taught according to school policy, brought with it difficulties and challenges. While many children found this ‘efficient’ task quite simple, a few struggled with the algorithm. It appeared that the repetition of ‘drill and practice’ was having no effect as these children did not understand the procedure or why they were using the procedure; for example, when solving an addition task involving renaming, many children would use the phrase “carry the one”, diminishing their understanding of place value because this ‘one’ was in fact ‘one ten’ and on many occasions, children would mix up the algorithmic procedures for addition and subtraction. Consequently, the children in my class were so focused on carrying out the steps of the algorithm correctly, that they dismissed their own original ideas and ‘invented methods’ of calculation which may have lead them to answering the questions successfully.

**Opposition to the standard algorithm**

Much research has highlighted the potential expediency of using an algorithm. Barnett (1998) describes an algorithm as a step-by-step procedure that “guarantees” the correct answer to a problem, provided that the steps are “executed correctly” (p. 69). Similarly, Maurer (1998) defines an algorithm as a “precise, systematic, method for solving a class of problems” (p. 21), while, Usiskin (1998) depicts an algorithm as a recipe. However, Hedrén (1999, p. 235) argues that such “ready-made mechanical rules for computation” do not conform with the idea of social constructivism which forms the foundation of most educational curricula. He believes that letting students invent and discuss their own methods of computation with peers and teachers would better adhere to the ideas of social constructivism. It is the opinion of Hedrén that a teacher should never force a standard algorithm on students. In agreement, Kamii and Dominick (1998) argue that the teaching of algorithms is founded on the invalid assumption that mathematics is a “cultural heritage” that must be “transmitted” from the adult to the child, from one generation to the next (p. 132). The authors propose that algorithms are harmful to children in Grades 1-4 and “hinder children’s development of numerical reasoning” (p. 131).

Kamii and Dominick outline the danger of algorithms encouraging children to give up their own thinking and “unteaching” place value. The authors state that children, when working with algorithms, have a tendency to think about every column as ones, and therefore the algorithm weakens their understanding of place value. Thompson (2000) identifies two distinct characteristics of place value: **quantity value** and **column value**. **Quantity value** refers to understanding that 38 is ‘30’ and ‘8’; while using **column value** demonstrates
understanding that 38 is ‘3’ in the tens column and ‘8’ in the units column. The author notes that children’s errors in applying an algorithm often show that they do not understand the column value basis of the standard algorithm.

Kamii and Dominick suggest that while educators may consider standard algorithms to be efficient, they found that once students began to invent their own methods, this argument no longer held true. The authors maintain that teaching standard algorithms to lower attaining students only serves to send them the message that they are incapable of coming up with their own methods, so following the steps of the predetermined algorithm will get them answers. Hiebert and Wearne (1996) believe that students in conventionally instructed classrooms are more likely to calculate correctly before being able to explain their procedures. In contrast, the authors note that alternative instruction seems to facilitate higher levels of understanding and skill. Students are encouraged to develop their own procedures and significant time is given to discussing and analysing these invented procedures. Researchers argue that logical, rational mental structures are remembered better and can be accessed more easily because they have many potential links with new information and experiences (Bruner, 1960; Hiebert and Wearne, 1996). Hiebert and Wearne state that children who understand and use mental strategies are in a better position to acquire and use procedures effectively. Acquisition of procedures may result from inventing new procedures; either creating new procedures or adapting previously learned procedures for future use, or adopting procedures demonstrated by others. Thompson (1994) remarks that in the correct environment “children can develop into creative mathematicians, inventing unorthodox written algorithms which not only work but which sometimes possess a degree of unexpected elegance” (ibid, p. 343).

This body of research indicates that flexibility in teaching and learning in mathematics is crucial. Children must be encouraged to think flexibly about number, to invent their own procedures for solving problems and to explain and reason their methods as this will enhance understanding of mathematical concepts. This idea of thinking flexibly about number is encapsulated by the notion of “proceptual thinking”.

Proceptual thinking

Gray and Tall (1994) recognise the duality between process and concept in mathematics in putting forward the notion of a “procept” - an amalgam of a mathematical process and concept. The authors recognise the “ambiguity of notation” in allowing the learner to move between the process required to carry out the task and the mathematical concept (ibid. p. 4). According to Gray and Tall, a procept consists of a collection of “elementary procepts which have the same object” (p. 6). It is my understanding that 8 is a procept; it includes the concept of number, the process of counting and can be expressed as 4+4, 6+2, 11-3, 4×2 etc. involving the use of different processes. Gray and Tall coined proceptual thinking as “the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object” (p. 7). It is a combination of conceptual and procedural thinking which requires flexibility of thought on the part of the learner.

Gray (1991) conducted a research study of children (aged 7-12) of mixed abilities in two English schools with the objective of discerning the children’s methods for solving simple
arithmetic problems. The children were interviewed and asked to do various addition and subtraction problems. The various strategies used by the children in solving the tasks provided the researchers with vital insights into the children’s understanding of and achievement in mathematics. Gray and Tall (1994) came to the conclusion that the ‘more able’ children are doing a different type of mathematics from the ‘less able’ and often the ‘less able’ are learning a more difficult subject. The authors illustrate this point well. Consider the task 16 - 13. ‘Below average’ children engaged in the ‘count-back’ strategy to solve this problem. Researchers agree that the cognitive complexity of this procedure is enormous and mistakes can be easily made (Gray and Tall 1994; Boaler, 2009). The children achieving at higher levels understood that number could be used flexibly and decomposed and recomposed numbers to solve similar problems. The research showed that lower attaining children did not change their method when they failed, instead they maintained that they needed to count more precisely. Thurston (1990) highlights the importance of compression in learning; once understood a concept is compressed and filed in the memory and used in the future to make associations with other concepts. Gray and Tall found that the lower attaining children were compressing ideas less – remembering methods and procedures was the focus of their learning. Boaler (2009) presents the learning of the lower-attaining children with an analogy of a never-ending ladder, with every rung of the ladder being another procedure to learn. Boaler, in agreement with Gray and Tall, believes that children who lack the ability to think proceptually often “cling to methods and procedures they are taught, believing that each method is equally important and must simply be remembered and reproduced carefully” (p. 142). Gray and Tall acknowledge that “the more able proceptual thinker has a simpler task than the less able procedural thinker” (p. 11).

THE STUDY

The research design of this study takes the form of a teaching experiment (Cobb, 2000) and involved the implementation of a programme of work over a five-week period in my classroom - one of two First classes in a reasonably large girls’ school in Dublin 15. There were 17 girls in the class and their ages ranged from six to eight years. The First class children received seventeen lessons in total. Four children comprised a focus group, chosen as a representative sample of the range of abilities in the class and were assessed and interviewed before and after the study, using the Mathematic Recovery Programme [1] assessments. The teaching experiment involved visual and audio recordings of the focus group at work and video-taped recordings of the share session at the end of each lesson. It also involved observational field notes, a researcher’s reflective journal and children’s work samples. Regular validation meetings were recorded with three participating teaching colleagues, Ms. Thomas, Mrs. Burke and Ms. Finnegan, for purposes of triangulating the data. The validation meetings allowed me time to talk about my plans and intentions, share the data generated by the programme, seek feedback, invite criticism of my interpretations and evaluations and communicate my future plans for the study.

The design and implementation of the programme were based on theories underpinning Realistic Mathematics Education (Van den Heuvel-Panhuizen, 2000) and Cognitively Guided Instruction (Carpenter, Fennema and Franke, 1996). Realistic Mathematics Education (RME),
first introduced by the Freudenthal Institute in the Netherlands in the early 1970s encompasses views on what mathematics is, how students learn mathematics, and how mathematics should be taught (Van den Heuvel-Panhuizen, 2000). Two of the principles upon which the curriculum of RME is based are the activity principle and the reality principle (ibid.). The activity principle refers to mathematics as learned by doing; as in this study, children are challenged with problem situations in which they learn to apply new and developed skills and procedures. The reality principle in RME involves putting mathematics into familiar, real life contexts which the learner can relate to. These real-life experiences and situations are used as a starting point for learning mathematics and pupils are active learners in a process through which they develop mathematical skills, strategies and intellectual insight.

In Cognitively Guided Instruction (CGI), a research-based model of children’s thinking that teachers can use to “interpret, transform and reframe their informal or spontaneous knowledge about students’ mathematical thinking” is presented (Carpenter et al, 1996, p. 5). In CGI classes, children learn to add multi-digit numbers as an extension of the methods they use to solve single-digit addition problems (Fuson et al., 1997). Word problems are used as the basis for instruction. Initially, children model the action and relations in the word problems using manipulative materials and gradually the activity becomes more abstracted and abbreviated as children begin to naturally use counting strategies, number facts and derived facts (ibid.). Teachers do not demonstrate solutions but a great deal of time is spent reflecting upon and discussing alternative solution methods (ibid.).

‘The Maths Fairy’

Each week of the study was considered a teaching episode and therefore each episode would inform practice for the subsequent episode following a cycle of planning, teaching and observing, and reflecting.

Planning

Prior to the commencement of the study, I appraised the children’s learning of mathematics as outlined by the PSMC. The concept of the Maths Fairy then evolved as a means of instructing the children on multi-digit addition in a fun and mysterious manner. The purpose of the Maths Fairy was to deliver appealing tasks and challenges to our class and for solving such, the children were rewarded with treasure hunts, games and prizes. Each teaching episode was assigned a theme – Treasure Hunt, Photograph Study, Watch the Box, Chill-Out Time and Mental Maths, and within each theme, a set of problems was created and specific learning objectives were outlined (table 1).
Treasure Hunt:
On completion of each task, the children receive directions to the location of a clue. After they gather the fifth clue on day 5, the children find their reward - hidden treasure.

Problems based on: School
Number of lessons: 5

<table>
<thead>
<tr>
<th>Theme</th>
<th>Overview of Lesson Objectives</th>
<th>Resources</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treasure Hunt:</td>
<td>That the child may be enabled to:</td>
<td>Lollipops sticks</td>
</tr>
<tr>
<td></td>
<td>- Engage with a task, realistic in nature set by the Maths Fairy.</td>
<td>Pegs</td>
</tr>
<tr>
<td></td>
<td>- Explore and invent methods for adding multi-digit numbers which do not require renaming.</td>
<td>Counters</td>
</tr>
<tr>
<td></td>
<td>- Orally explain her strategies to a partner or a teacher.</td>
<td>Cubes</td>
</tr>
<tr>
<td></td>
<td>- Record her work using a pictorial representation and a written narrative.</td>
<td>Copies</td>
</tr>
<tr>
<td></td>
<td>- Reflect on her work and the work of other children during a whole class share session.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Discuss the strengths and weaknesses of various strategies suggested by children.</td>
<td></td>
</tr>
</tbody>
</table>

| Table 1: Overview of lessons - Week 1 |

Teaching and observing
The children received the concept of the Maths Fairy with great excitement and enthusiasm. A typical lesson involved reading the problem, solving the problem in group settings and sharing invented procedures at whole class level. The children’s engagement with the tasks, choice of concrete materials used, strategies employed and general behaviours were observed and recorded. Conferencing with the children allowed me further insight into their thoughts and methods of practice. It was through observing the children at work on the tasks that I noted natural and somewhat unplanned differentiation take place: the children began the tasks each with her own level of understanding and progressed each at her own level of ability. The children chose concrete materials they felt comfortable using and used these resources in a manner that made sense to them. The children progressed by engaging in the tasks, and through observing and responding to the strategies put forward by other children during the share session. Each child developed new understanding according to her own individual learning pace. This is clearly evident as the children work on the same task but employ different strategies to solve the problem and present their work in different manners (figure 1).

![Figure 1: Using different strategies to solve the same task - week 1, task 4](image)

Reflecting
The share session afforded the children a forum to express their thoughts and ideas, and the strategies they had employed to solve the task. It also served as a means of developing
mathematical skills as outlined in the PSMC and granted me the opportunity to informally assess the children’s progress. The children were asked to recall previous methodologies used in solving the problems and demonstrate their understanding of such, communicate and express their work in front of classmates and teachers, offer reasons as to how they arrived at an answer and make connections between different strategies. As the children reflected upon the strategies used and engaged in discussion during the share session, it was my aspiration that they would implement and apply a new aspect they had learned, in the following problem-solving task.

FINDINGS

Theories of constructivism and guided discovery methods as embraced by this teaching experiment can help children to become proceptual thinkers.

Proceptual thinking is a combination of conceptual and procedural thinking which requires flexibility of thought on the part of the learner (Gray and Tall, 1994). My teaching experiment involved constructivist approaches and guided discovery methods and I found that this approach granted every child in my class an opportunity to develop herself as a proceptual/flexible thinker. The teaching experiment involved every child as an active participant in the learning process. Previous learning experiences were used to make sense of new experiences and situations. As a starting point the children used what they knew of single-digit addition and applied it to multi-digit addition. Initially, strategies such as count-all, count-on, ‘locking the bigger number in the head’ and counting on, doubles, near doubles etc. were used. Like most children in the class, Sarah began by counting-on from the larger number but quickly realised that this process was tedious so she counted out each number separately and then made groupings of ten counters and left over units (figure 2a). From observing strategies suggested by classmates, Sarah became confident in distinguishing between tens and units, and added the column values of numbers (figure 2b and 2c). During week three, Sarah adopted a method of splitting the numbers into their quantity values (figure 2c). She became very efficient at using this method and confidently explained her reasoning in front of the class. From the outset, Sarah, like every other child in the class used previously acquired knowledge to help her access new learning experiences. She continually observed others and built upon what she already knew in an effort to make her strategies more efficient.

As in a CGI classroom, no prevalent strategy was used by all children at one time so it was the role of the teacher to anticipate when new concrete materials were needed and to ensure that each child was using the materials correctly. Building on the children’s previous learning experiences, counters and cubes were the first manipulatives at the children’s disposal. Those who chose to use these materials quickly saw the difficulty involved and it was suggested that partitioning the numbers in tens and units was a more efficient strategy. The Dienes’ blocks and bead-strings allowed the children to develop their understanding of place value in a natural way through problem-solving and for most children it sped up the process of solving the task.
Figure 2 a), b), and c): Samples of Sarah’s invented strategy development

Every child began the study by modelling the action and relations in the word problems using manipulative materials and then gradually, for some children the activity became more abstracted as they began to invent methods without the use of manipulatives. Nicola and Sarah became extremely competent at solving the tasks using their copies and pencils solely, and applied number facts and derived facts in doing so (figures 3a and 3b).
Figure 3 a) and b): Demonstrating understanding of quantity value

Removing the possibility of using concrete materials encouraged the children to perform multi-digit addition mentally. The use of multiple-choice answers meant that children who were not entirely competent in adding the large numbers mentally, could reason which possible answer was the solution. As a consequence of the approach taken in this study, I believe that every participating child in some way became a proceptual/flexible thinker in respect of adding multi-digit numbers. Every child made substantial progress from the initial lessons in which basic count-on and count-all strategies were employed to developing a good understanding of place value as they explored multi-digit addition. As in CGI classrooms, this understanding of concepts emerged over time, as the children engaged in solving problems and exploring base-ten materials. Ms. Finnegan described this understanding as evolving “naturally” and not “imposed upon” the children. She sums up this study by saying that “a much deeper level of understanding is generated when it’s the children themselves who come to the understanding when they are ready.”

Communicating and expressing thoughts and ideas was a worthwhile but difficult task

A vital element of this research project was to enable the children to communicate and express their mathematical ideas. Initial assessments of the children in the focus group revealed that the children found it very difficult to articulate their mathematical work. Children benefit not just by learning to communicate mathematically but by communicating, they also learn mathematics (NCTM, 2000). Therefore, a fundamental aim of this teaching experiment was to enable and encourage the children to talk about their invented strategies.

The Maths Fairy left strict instructions with the children that they had only one opportunity to solve each task successfully so the entire class had to come to a consensus as to what the solution was. This important rule encouraged the children to work in collaborative settings and to discuss their procedures and solutions. The task of communicating and expressing procedures was even more challenging for seven children for whom English is not their first
language but the excitement that the Maths Fairy generated inspired every child to try her best.

Mrs. Burke expressed her concern that expecting the children to give a written account of their work was too difficult for this age group. She also feared that the written element to the tasks would take from the mathematics learning inherent in the programme. Initially, many children got side-tracked by presentation details, the accuracy of which was explained to the children as unimportant for the present task. Mary stood out among all of the children as having the greatest difficulty with recording her work. Mary was quite capable of presenting me with a verbal explanation of her work, but needed a lot of help to break down her ideas and sequence her steps on paper. In contrast, the written element of the tasks helped Nicola to focus and structure her thoughts. Ms. Finnegan also noted the different learning styles of the two girls:

The recording aspect of it was difficult (for Mary) and actually writing down each step was difficult and a challenge for her ... we went through it together and she explained to me how she had worked it out, she knew in her own mind how she had done it ... in contrast ... Nicola ... found the recording aspect actually very helpful in terms of her thought process and actually going through the stages.

Ms. Burke later in the study admitted that her earlier fears had subsided and that she was amazed at the progress the children had made. She highlighted the positive influence the teaching experiment had on the children’s writing skills as an added bonus:

I was astonished at how much the maths actually brought on the writing and how precise that age group is about ... getting the writing done properly ... they worked it out in their own heads and then were able to write it down.

Providing the children with a forum where they felt comfortable and supported in communicating and expressing their ideas, was an important feature of this study. During the share session all of the children were extremely eager to speak and share their work. In post-study interviews, all of the teachers agreed that children must be taught and encouraged to communicate and express their thoughts, ideas, strategies and concerns in mathematics from an early age. I believe that unless the ground-work has been laid in the early years, young adults will find it very difficult to verbalise their mathematical work.

CONCLUSION

Captivated by the concept of the ‘Maths Fairy’ and all it presented, the children wholeheartedly committed to the learning activity and forged positive associations with mathematics, keenly looking forward in anticipation to the next instalment. Realistically-based problem contexts presented by the Maths Fairy, captured the imaginations of the children, while the prospect of reward was equally intriguing. Within this approach to teaching multi-digit addition, the children’s understanding of number evolved naturally. I found that eliciting strategies from the children was extremely beneficial in developing their understanding of number and had longer lasting effects than if I had signalled to these strategies from the start. As the data shows, most of the children in my class emerged from the study with a deep understanding of the aspects of place value; column value and quantity
value. After the study, the children applied their understanding of place value to working with money, e.g. in making up 35 cent, the children informed me that there were 5 units which was 5 cent, and 3 tens which was 30 cent. It is this type of understanding and ability to apply in different contexts that we should strive to achieve in our classrooms.

On conclusion of the study, I instructed the children on the standard algorithm for the operation of addition, as per my school’s policy. I was delighted to see that due to their understanding of column value in the place value system, many children like Sarah, had already applied the standard algorithm for addition, unknown to themselves. Early in the study, so much work had been done with manipulatives in the process of inventing algorithms, that there was no need for whole class explicit instruction on the standard algorithm. This shows that it is very important to ‘lay the foundations’ correctly; with plenty of manipulative experimentation and invented algorithm exploration. A teacher must judge when his/her class is ready for the algorithm.

NOTES
1. The Mathematics Recovery Programme is an early intervention programme focusing on the area of number. A framework for intervention to meet pupils’ needs based on a detailed profile-based assessment is provided.

REFERENCES


POST-PRIMARY STUDENTS’ IMAGES OF MATHEMATICS

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**University of Limerick and NCE-MSTL

A questionnaire survey was conducted as part of a PhD study investigating post-primary students’ images of mathematics in Ireland. A definition of ‘image of mathematics’ was adopted from Lim (1999) and Wilson (2011). Students’ images of mathematics were proposed to include attitudes, beliefs, motivation, self-concept, emotions and past experiences regarding mathematics. A questionnaire was created with both quantitative and qualitative aspects. This paper focuses on the qualitative facet by reviewing students’ responses to the open-ended questions according to the five categories of image of mathematics found by Lim (1999). The qualitative data provides an in-depth insight into Irish students’ images of mathematics. Findings from the qualitative data afford an innovative insight into post-primary mathematics education from the student perspective, thus offering a means for mathematics educators to respond to students’ needs and encourage Irish post-primary students’ engagement with mathematics.

INTRODUCTION

Affective, cognitive and conative issues have come to the fore of mathematics education research in recent years. They have been examined in Ireland and abroad with studies by researchers such as McLeod (1992), Kelly and Oldham (1992), Ernest (1996), Lim (1999), Liston (2008), Eaton & O’Reilly (2009) and Wilson (2011) establishing a strong theoretical basis for examining the relationship between them and mathematics. Studies that have examined affective and cognitive issues (and, in some cases, conative issues) defined the combined issues as “mathematical identity” (Hill, 2008; Eaton & O’Reilly, 2009), “self-efficacy” (Tait-McCutcheon, 2008), “disposition” (Wilson, 2011) or “image of mathematics” (Kelly & Oldham, 1992; Ernest, 1996; Lim, 1999). This aspect of mathematics education had not previously been examined extensively in Ireland in relation to post-primary school students. Thus we identified an obvious gap in mathematics education research in the Irish context. As part of a PhD research study (Lane, 2013), we aimed to address these affective, cognitive and conative issues in Irish post-primary mathematics education by examining the image of mathematics held by students.

The general public’s view of mathematics, whether in Ireland or elsewhere, often depends on individuals’ experience of mathematics, particularly during school years. This is supported by Lim (1999) who, in her study of the public image of mathematics in the UK, found that most people did not distinguish between their image of mathematics and their image of learning mathematics. Therefore the process of teaching and learning mathematics plays a vital role in establishing a person’s image of mathematics. Mathematics education researchers have come to realise the significance of students’ attitudes, beliefs, emotions, motivation and self-concept regarding mathematics, with consequential effects on mathematical performance and
achievement. As Lane’s (2013) study began, a phased introduction of a new mathematics curriculum for Irish post-primary schools, Project Maths, was under way and thus there was a particular focus on post-primary mathematics education in Ireland. The current interest in innovation and change in mathematics education in Ireland and the relevance of affective, cognitive and conative issues to the mathematics education community both in Ireland and abroad provided strong motivation for the study.

We focused on 5th-year students in the Leaving Certificate cycle of post-primary school. It was hypothesized that these students would have formed a stronger image of mathematics than students who were in their first years of post-primary education. Furthermore, we chose to examine the images of mathematics of students of ordinary level mathematics specifically. It was expected that ordinary level students would have a more diverse range of images than students of the higher or foundation levels.

This paper focuses on the qualitative data findings from a questionnaire survey constructed to examine students’ images of mathematics. While most of the questionnaire consisted of quantitative fixed-response items from eight Likert scales (examining attitudes, beliefs, motivation, self-concept and emotions about mathematics), the qualitative questions provide important findings relating to students’ images of mathematics.

THEORETICAL FRAMEWORK

From the literature, it was clear that no universal definition exists for “image of mathematics”. However, for those researchers who define image of mathematics (either explicitly or implicitly), there appears to be a general consensus that it should include attitudes, beliefs, self-concept, emotions and past experiences regarding mathematics (e.g., Kelly & Oldham, 1992; Lim, 1999). For our definition of image of mathematics, we draw on the theories of Lim (1999) and Wilson (2011). Lim (1999, p.73) conceptualizes the term image of mathematics as:

A mental representation or view of mathematics, presumably constructed as a result of social experiences, mediated through school, parents, peers or mass media. This term is also understood broadly to include all visual and verbal representations, metaphorical images and associations, beliefs, attitudes, and feelings related to mathematics and mathematics learning experiences.

Lim divides the elements included in her definition of image of mathematics into two categories: the affective domain (attitudes, emotions and feelings regarding mathematics) and the cognitive domain (knowledge and beliefs regarding mathematics). Wilson’s (2011) theory of ‘disposition’ overlaps to a certain degree with Lim’s definition of ‘image of mathematics’. In his definition he proposes four components of disposition as follows:

- Beliefs / values / identities
- Affect / emotions
- Behavioural intent / motivation
- Needs
The first two components coincide with Lim’s definition of image of mathematics, while the fourth component ‘needs’ is similar to the factors of influence (school, parents, peers and media) included in Lim’s theory. The third component refers to the conative domain proposed by Ruffell, Mason and Allen (1998). Ruffell et al. suggests that a conative element works together with the affective and cognitive domains. We adapt the definitions of Lim (1999) and Wilson (2011) for our study, with ‘image of mathematics’ conceptualized as follows:

A mental representation or view of mathematics, presumably constructed as a result of past experiences, mediated through school, parents, peers or society. This term is also understood broadly to include three domains:

- The affective domain dealing with attitudes, emotions, and self-concept regarding mathematics and mathematics learning experiences.
- The cognitive domain dealing with beliefs regarding mathematics and mathematics learning experiences.
- The conative domain dealing with motivation regarding mathematics and mathematics learning experiences.

Self-concept is not included in the definitions of Lim or Wilson, but following our literature review which indicated the increasing significance of self-concept regarding mathematics (e.g., Gourgey, 1982; Tait-McCutcheon, 2008), it was decided that self-concept should be included in the affective domain of our definition for ‘image of mathematics’.

**METHODOLOGY**

The design for our research study was chiefly explorative. Ethical permission was sought and granted by the Ethics Committee in University College Cork prior to carrying out the investigative research. The main research question for the study was:

- What is the image of mathematics held by 5th-year post-primary students in Ireland?

This question was then refined through the literature review and the theoretical framework and broken down into nine sub-questions. These nine questions related to students’ (a) attitudes, (b) emotions, (c) self-concept, (d) beliefs, (e) motivation, (f) past experiences, (g) influences in mathematics, (h) causal attributions for success/failure in mathematics, and (i) prior achievement in mathematics.

A mixed-methodology was employed. The main method used to examine students’ images of mathematics was a questionnaire survey. The questionnaire contained both quantitative fixed-response items and qualitative open-ended questions. The quantitative aspect incorporated eight pre-established Likert scales to examine students’ attitudes, beliefs, emotions, self-concept and motivation regarding mathematics (see Lane, 2013). The Likert scales addressed the first five of the research sub-questions as well as the main research question. The qualitative element consisted of open-ended questions composed by the researcher and aimed to address the sixth, seventh and eighth sub-questions aforementioned as well as clarifying students’ overall image of mathematics. Both the quantitative and qualitative questions were used in examining the final (ninth) research sub-question.
This paper focuses on the qualitative element of enquiry. There were five open-ended questions in Section A of the questionnaire. The research purpose of each of the open-ended questions as well as research studies that support the relevance of these questions are shown in Table 1. Four of the five open-ended questions contained two parts with the first part consisting of a ‘tick the box’ option. The purpose of this tick the box option was, in part, to clarify questions which might otherwise be misunderstood by students, and in addition, to provide a means of analysing parts of the questions more quickly and efficiently using SPSS.

<table>
<thead>
<tr>
<th>Open-ended Question</th>
<th>Research Purpose</th>
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<tbody>
<tr>
<td>Q1: What is your earliest memory of mathematics?</td>
<td>To examine students’ past experiences of mathematics (Lim, 1999).</td>
</tr>
<tr>
<td>Q2: Have your past experiences of mathematics caused you to be interested/disinterested in mathematics? (a) Yes/No (b) Please Explain.</td>
<td>To examine students’ past experiences of mathematics and the influence of past experiences on current engagement in mathematics, (Lim, 1999; Eaton &amp; O’Reilly, 2009).</td>
</tr>
<tr>
<td>Q3: Who influences you most in mathematics? (a) Mathematics Teacher/ Parents/ Peers/ Media/ Other (b) Why?</td>
<td>To examine influences on students’ images of mathematics and students’ engagement with mathematics (Lim, 1999; Lyons, Lynch, Close, Sheerin &amp; Boland, 2003; Hill, 2008).</td>
</tr>
<tr>
<td>Q4: Which of the following contributed most to the grade you received for mathematics in the Junior Certificate? (a) Mathematics Ability/ Effort/ Luck/ Level of Difficulty of Exam/ Other (b) Please Explain.</td>
<td>To examine students’ causal attributions for success and/or failure in mathematics (Weiner, 1974).</td>
</tr>
<tr>
<td>Q5: Do you use mathematics outside of school and school work? (a) Yes/No (b) If so please give examples. If not, please give reasons.</td>
<td>To examine students’ image of mathematics in terms of the utility of mathematics in everyday life (particularly relevant given the new Project Maths curriculum) (Lim, 1999; Hill, 2008).</td>
</tr>
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Table 1: Qualitative questionnaire items and research purpose of the questions

In the analysis of the qualitative data, the constant comparative method of analysis was used (Conrad, Neumann, Haworth & Scott, 1993). The responses to the open-ended questions were sorted into categories and then integrated into broader categories. These categories were then coded and analysed using SPSS. In addition, individual responses were also reviewed qualitatively, bearing in mind previous research findings.

This paper focuses on findings from the open-ended questions that inform the main research question regarding students’ images of mathematics. The data was analysed qualitatively, bearing in mind the five images of mathematics found by Lim (1999), namely – the absolutist/dualistic image, the utilitarian image, the symbolic image, the problem-solving image, and the enigmatic image. Answers by students to each of the open-ended questions were reviewed by the researcher and were found to contribute to creating an impression of students’ overall images of mathematics in terms of the five images found by Lim. Not all students’ responses fitted into one of the categories by Lim. Some of the students’ answers were only a direct response to the question asked, providing only a one or two word answer. However, a considerable number of students gave more detailed comments and these were used to examine the existence of the five categories of image. These five images are not
strictly separate, and a student’s image of mathematics may fall into one, several or even none of these categories. The images of mathematics categorized by Lim do however provide a guideline for assessing the images of mathematics found in the study, by providing a theoretical basis for categorizing responses and assessing common or widespread images among students in our sample.

**Sampling methodology**

As mentioned above, no previous research has investigated the image of mathematics held by post-primary students. Therefore, we followed the sampling methodology used by Murray, McCrory, Thornton, Williams, Quail, Swords, Doyle and Harris (2011), in the ESRI’s study of nine-year-olds’ attitudes to mathematics in Ireland. The sampling objective of our research survey was to select a representative random sample of 400 5th-year ordinary level mathematics students in post-primary school in Ireland. It is widely accepted that non-response is a major issue in surveys and particularly in postal surveys (see Wiersma, 1991). In order to achieve a sample as close to the required 400 as possible, we decided to choose a sample with 3 times as many students as was required (that is, 1200 students).

Similar to Murray et al. (2011), post-primary schools were stratified according to type of school (secondary, vocational, or community and comprehensive), co-educational status (single-sex girls’, single-sex boys’ or co-educational), and school ethos (fee-paying or non fee-paying). A random sample of 60 schools was initially selected on a stratified systematic basis. As in the ESRI study, a maximum threshold of 20 students was set in any one school. Schools were provided with instructions for selecting a random sample of 20 students. An information sheet advised schools with fewer than 20 ordinary level mathematics students that all students should be asked to participate. For schools with more than 20 students, a random digit table with instructions was provided. Students and their parents were requested to sign an informed consent form prior to participation.

One of the limitations of our study was generalizability of findings. Ideally, a much larger sample size should be used as in the ESRI study, but given time and resource constraints this was not possible. A total of 356 students completed the questionnaires and while slightly below our sampling aim, this was sufficient to provide a preliminary examination of Irish post-primary students’ images of mathematics.

**QUALITATIVE DATA**

In this section we present findings from the qualitative open-ended questions. Students’ responses from the five open-ended questioned are considered in relation to the main research question: What is the image of mathematics held by 5th-year post-primary students in Ireland? The students’ answers and comments were reviewed for evidence of the five categories of image of mathematics found by Lim (1999). Not all of the comments made by students could be said to reveal their image of mathematics, but the qualitative analysis did reveal the existence of each of Lim’s image categories, with some categories found to be more prevalent than others. Due to the large amount of data collected in this study, not all of the students’ answers can be displayed in this paper. Rather we examine each of the five image categories
in turn, provide quotes from the students’ answers which fall into each category and discuss whether students’ images are positive or negative.

**Students’ images of mathematics**

In order to analyse students’ images of mathematics from the qualitative data, we refer to the five images of mathematics found by Lim (1999). Within each of the image categories, students’ comments also reveal their image of mathematics with respect to attitudes, beliefs, motivation, self-concept and emotions.

The **absolutist or dualistic** image of mathematics is that of a set of ‘absolute truths’ with only one right answer. The absolutist image has been fostered by the style of teaching that has been experienced by many students, whereby the emphasis is placed on the product rather than the process of mathematics (e.g., Hill, 2008; Tait-McCutcheon, 2008; Eaton & O’Reilly, 2009). On the one hand the absolutist view may result in a positive image of mathematics as there is a solution to each mathematical problem which, if found, cannot be disputed. On the other hand, the absolutist view can produce a negative image of mathematics, that of a subject lacking creativity and it can be frustrating when the one true solution cannot be reached. Examples of the absolutist image of mathematics held by students in our study can be seen from the following comments:

- “Some experiences make me interested – like when you work out a new sum … on your own it gives you a little kick and makes you interested to learn.”
- “If I get a hard question right I feel good to accomplish it …”
- “I knew all the answers when I looked at the paper.”
- “Getting frustrated if I couldn’t do it.”
- “When I got it wrong I didn’t want to do anything else.”
- “… used to get very stressed.”
- “Stressful and frustrating.”

The responses listed above illustrate the positive and negative effects of an absolutist image of mathematics. The importance of finding the right answer results in feelings of pleasure and achievement when the one true solution is obtained. On the other hand, when students were unable to find the answer they felt frustrated and stressed resulting in a negative image of mathematics. One student commented that he/she did not want to continue when they were unable to find the correct answer suggesting that the absolutist image of mathematics can lead to a lack of motivation in students. In the qualitative section of the questionnaire, students were asked in the first question about their past experiences or memories of mathematics – see Table 1. Although the majority of students (86%) simply stated that their earliest memories were of primary school or learning a particular type of mathematics at school, the remainder of the students (14%) gave detailed examples of their earliest memories, 24 students referring to negative experiences and 25 students recalling negative experiences. Of the students who referred to negative experiences, 15 spoke of finding mathematics difficult or struggling, leading to negative emotions or feelings about
mathematics, stating that they got “frustrated” and “upset”. The most explicit negative memories involved experiences with teachers. One student wrote that the teacher was “writing sums and answers on the board and we couldn’t learn anything”. A second student recalled a “teacher shouting at me when I got something wrong”. A third student remembered that the teacher was “angry at me when I couldn’t do a sum on the board and the class were laughing at me”. These descriptions by students are indicative of the culture of rote-learning which has been commonplace in our schools as found by De Corte, Greer & Verschaffel, (1996). Emphasis was placed on finding the right answer and negative memories held by students in our sample related to an inability to find correct solutions. As found in Ruffell et al. (1998), specific negative memories or experiences of mathematics can override more general, positive experiences of mathematics and in our study, the students who recalled negative memories of mathematics were found to have the most negative image of mathematics compared to all other students.

The utilitarian image views mathematics in terms of its usage and relevance. Usually, for those who like mathematics and have a positive image, mathematics is seen as a useful tool, while for those who dislike mathematics and have a negative image, mathematics is seen as irrelevant and not to be applied to everyday life. Some examples of the utilitarian image of mathematics can be seen the following comments by students:

- “Everyone uses mathematics in every situation if you think about it.”
- “The subject itself (mathematics) is a useful tool.”
- “You need maths for most things in life.”
- “I see maths everywhere. I will always need maths no matter what I choose to do in life.”
- “It’s a useful tool i.e. for budgeting and minding your money – it makes you think wisely.”
- “I am interested in the maths that would help me in the future not irrelevant maths.”
- “Maths is something you have to do there is no option, you will need it for the rest of your life.”
- “I love physics and see the usefulness of maths but think a lot of the school subject is unnecessary.”
- “Not everyone wants to become a scientist or engineer.”
- “I don’t see why we would need some of the maths course in the future.”
- “Nothing I do that makes me use it.”
- “Maths from post-primary school is not needed in daily life.”
- “I don’t know enough to be able to successfully use it elsewhere.”
- “I don’t know where it can be used.”
- “Nothing involves maths.”
When asked in Question 5(a) of the questionnaire whether they used mathematics outside of school and school work, 65.2% of students acknowledged that they did so. However, a considerable number of students – 118 (33.1%) – responded that they did not, suggesting that a third of the students sampled are unaware of the utility of mathematics in everyday life. Students’ responses to Question 5(a) correlated significantly at the 0.01 level with students’ overall image of mathematics. Responses regarding the utilitarian image of mathematics in our study were varied, similar to Hill (2008). While some students were found to have a positive image of mathematics as being useful and relevant in everyday life and necessary for the future, other students’ responses in relation to the utilitarian image of mathematics highlight a lack of awareness regarding the utility of mathematics. Students stated that they were unable to use mathematics as they didn’t know enough to use it while others claimed that mathematics is not needed in day-to-day life. This is similar to the findings of Picker and Berry (2000) who, in their research on students’ images of mathematicians, found a distinct lack of awareness of the utility of mathematicians. Referring to post-primary mathematics education specifically, a number of students in our study stated that they enjoy studying mathematics which is relevant but that much of the mathematics they learn is unnecessary. There appears to be a lack of awareness for some students regarding the application of school mathematics to real world situations. While the introduction of Project Maths aims to address this issue, one student, referring to the new Project Maths course, stated that not everyone wants to become a scientist or engineer. This suggests that while the importance of mathematics in these careers is highlighted to students, the value of mathematics in other areas is not commonly known to students. One student commented that “the problem is that teachers don’t explain what jobs use different topics”. There is clearly a need for a more widespread and detailed explanation to students of the relevance of mathematics in a variety of careers.

The symbolic image of mathematics is that it is a collection of numbers and symbols, or rules and procedures to be followed and memorized. This image has similarities with the absolutist image, with the two categories of image often going hand in hand. Students’ comments that are indicative of the symbolic image of mathematics include the following:

- “Reciting multiples off by heart.”
- “…found learning theorems pointless.”
- “Every year you begin maths it’s just a step up of the same question just with different rules.”
- “Teacher encourages me to learn formulas.”
- “Teacher made us repeat questions over and over until we knew them perfect.”
- “Learnt formulas and equations.”
- “… questions were different from previous years and confused me.”
- “Practised exam papers again and again.”
- “Learnt off all the theorems so knew one that came up.”
“Studied the formulas very thoroughly.”
“… don’t work well with numbers.”

There is an emphasis on memorising rules and formulae in mathematics teaching in post-primary schools (e.g., De Corte et al., 1996). This well-established ethos of teaching mathematics by focusing on learning off rules, formula and theorems has already been witnessed in previous studies such as Eaton and O’Reilly (2009) and Liston (2008). There is also a connection between this symbolic image of mathematics and exam preparation. Indeed one student stated that because the exam was different from previous years he/she was confused, thus blaming an ‘unpredictable’ exam for a low mathematics grade in the Junior Certificate. Rather than memorising the mathematics rules, students need to understand what these rules mean and to understand where the formulae and theorems come from.

Further evidence of the symbolic image and rote-learning can be seen from students’ earliest memories of mathematics (Question 1). A total of 51 students (14.3%) recalled learning or memorising multiplication tables. This was the third most frequent response category for this question and suggests that rote learning played a significant role in students’ learning of mathematics. Furthermore, in students’ causal attributions for success or failure in mathematics (Question 4), over half the sample of students in our study, 51.1%, selected ‘effort’ as contributing to the grade they received for mathematics in the Junior Certificate. In cases where students described the type of effort made, repetition and rote-learning were frequently cited.

The problem-solving image of mathematics sees mathematics as a set of problems to be solved. Students’ responses to the qualitative questions that illustrate this problem-solving image include:

- “Interested because I do my best to solve maths problems.”
- “It’s like figuring out puzzles.”
- “I like to be challenged.”
- “I like working with figures and problem-solving.”
- “I love figuring out problems for myself.”
- “Playing maths games in primary school.”
- “…encourage me to figure out things on my own.”
- “Use maths to apply logic – to sort out problems.”
- “Puzzle-solving and order to things.”

For the majority of students, the problem-solving image of mathematics was a positive one and solving mathematical problems was an enjoyable and rewarding experience. Students referred to mathematics as a challenge and appeared to enjoy figuring out solutions to problems on their own. This was also found by Hill (2008). Some students in our study mentioned problem solving when discussing the utility of mathematics, for example, in
Sudoku, brain-training, computer games, applying logic and creating order. The problem-solving image has particular relevance in Project Maths, which focuses on problem solving and the application of mathematics to real-world problems.

Finally, the enigmatic image of mathematics is of something seen as mysterious but yet something to be explored and whose beauty is to be appreciated. Lim (1999) found that a small minority of the sample, particularly those who liked mathematics and were directly involved with mathematics, viewed the subject as an enigma, something foreign and at times incomprehensible but elegant at the same time. Evidence of the enigmatic image of mathematics can be seen in the following statements made by students:

- “It makes me think outside the box.”
- “As you advance, the more you want to know where the maths comes from.”
- “A strange interest as the years have progressed.”
- “It feels like you need to be excellent at it and if not there’s no place for you in the mathematics world.”
- “The teacher guides us through maths.”
- “Maths seems like an elitist subject.”
- “Einstein struggled in maths too but discovered amazing things and I find that inspiring.”
- “…not got a flare for maths.”
- “I don’t have a natural mathematical ability.”
- “I’m naturally good at maths.”
- “I find some things easier than the other students and they come to me naturally.”

Each of the comments made by students listed above could fit into an enigmatic image of mathematics. One student wrote that the teacher “guides” them through mathematics, which is similar to the metaphor of mathematics as a journey that was found by Lim (1999). There was also the impression of mathematics as being strange or making one “think outside the box”, which ties in with the image of mathematics as something foreign. Two of the responses given by students referred to the elitist nature of mathematics, implying that mathematics is only for the very smart and only accessible to a few with a very high mathematical ability. This was a common image of mathematics found by Ernest (1996). Finally, a number of students mentioned having a natural mathematical ability or lack thereof. The notion of the mathematical mind or having a natural ability in mathematics is also seen in Picker and Berry (2000) who found that mathematics may be seen more as a magical power than as a subject that anyone can learn. This can create a negative image of mathematics as an unattainable skill, and impossible for those who do not possess a ‘natural’ mathematical ability.
CONCLUSIONS

The findings discussed in the previous section provide innovative information regarding the image of mathematics held by 5th-year, post-primary students in Ireland. Each of the five images of mathematics in Lim (1999) was found to exist among students in our study. The problem-solving image of mathematics was the most positively expressed image of mathematics, with a positive effect on enjoyment of mathematics. The qualitative data analysis highlights the need for more detailed and widespread explanation regarding the relevance of mathematics in daily life and in a variety of future careers. It is also necessary to emphasise understanding the mathematical process rather than memorising rules. Findings from this qualitative data are essential in giving students a voice. It is crucial to listen to students’ views in order to gain a better understanding of students’ needs and therefore to inform teacher education and the way we approach mathematics education in our schools.

REFERENCES


PROJECT MATHS ACADEMY: USING KHAN ACADEMY’S EXERCISE PLATFORM AS AN EDUCATIONAL AID IN A POST-PRIMARY MATHEMATICS CLASSROOM

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The focus of this paper is a First Year post-primary mathematics class in which Khan Academy’s online exercise platform was used weekly for an academic year. Interviews were conducted with the teacher, and students were surveyed about their opinions of the platform and its usage. Regular classroom observations took place to gain an insight into the context of these opinions. A subsequent survey compared these students’ attitudes towards mathematics classes with those of their peers who were not using the platform. Test scores were compared between three classes (one using Khan Academy; two not) to ascertain whether the platform had any effect on student performance. The platform was found to be an invaluable tool for class management: the teacher was able to provide the capable students with enough work while attending to students in need of support. Students enjoyed their time spent on the platform and the more capable students were able to work at their own pace and tackle more challenging exercises. Test results show that the platform may have a negative effect on student performance in the areas of integers and probability, but a positive effect in coordinate geometry. We comment on the evidence for statistically significant differences in the general results of those using the platform and those not.

INTRODUCTION

In his TED (Technology, Entertainment and Design) presentation in March 2011, Salman Khan, creator of Khan Academy (KA) (www.khanacademy.org), gave an introduction to his “exercise platform” (TED, 2011). This exercise platform supports a large range of instructional videos on mathematics which form the core of the KA. Both the videos and exercise platform are available online at no cost, and neither requires specialised software. Khan emphasised the importance of mastery, comparing the alternative to giving a unicycle to someone who is unable to ride a bicycle perfectly. Khan suggested that a student in a “traditional” classroom [1] may appear to fall behind the rest of the class and end up with lower grades but may in fact just lack understanding of a few concepts and consequently be held back in other areas. Mathematics builds on prior knowledge to a large extent, so these situations could occur on a continual basis. In the extreme case, a student at Leaving Certificate may be forced to take their maths exam at Ordinary Level due to having misunderstood some of the material covered early in their course of study.

In this paper, we consider the following research question:

How effective is the Khan Academy's exercise platform in facilitating the Project Maths Common Introductory Course for First year post-primary maths students?

The effectiveness of the platform may be interpreted in a number of ways. It may be effective from the point of view of the teacher implementing it: it may help with class management,
offer a superior learning experience to their students, facilitate a better learning environment and differentiated teaching, and provide details of student ability that are not available on the same scale, in the same quantity or with the same efficiency. From the perspective of students, it may provide a more individualised learning experience, allowing them to learn at their own pace. The approach of mastery learning and the interlinking of related exercises that are built into the platform may prevent students from tackling material that is too difficult for them due to not having covered or fully understood the prerequisite material for any given exercise. Lastly, the platform may be effective in leading to higher attainment levels as measured by summative assessment. As such, the specific aims of this research are:

1. to analyse the platform from the perspective of a teacher who has implemented it in his classroom;
2. to analyse the platform from the perspective of students who are using it in their classroom; and
3. to analyse the test results of students who have used the platform, and compare them to the results of control students who have not used it.

KA exercise platform

The platform is heavily influenced by the idea of mastery: students are not considered to have completed an exercise until they consistently answer questions correctly. This is evidenced by the leaf and stack system of grading questions. Each attempt at an exercise consists of a ‘stack’ of eight questions, represented as cards. Each card is imprinted with one to three leaves based on how well students answer the question on that card. Three leaves are awarded only when students answer questions correctly on their first attempt. If a student gives any number of incorrect answers before answering correctly, asks for a hint from the system, or asks for a portion of the worked solution, two leaves are awarded. Lastly, if a student asks the system for the full solution, the award is one leaf. The system is highly reactive – a student can attain mastery of an exercise by answering as few as the first four questions correctly, though a long string of incorrect answers may need to be followed by a string of more than two full stacks in order for mastery to be deemed attained. At the end of the stack, students are shown an animation of a progress bar filling up based on how well they performed. Too many one or two leaf cards will show a less-than-full progress bar, with the suggestion that another stack be attempted before moving on. A long string of three leaf cards is reflected in a progress bar filling up significantly, or even completely.

KA management tools

The exercise platform includes view for the teacher that displays a wealth of information about the students. Students may be grouped together into custom lists. The teacher can view the details of every exercise, showing the number of students at each stage of completion (not started, started, struggling, proficient, and review), allowing for fast overview of the state of the class for any given exercise. There are many benefits here in terms of discovering difficulty with an exercise (which may indicate that the material needs to be repeated and/or presented in a different way) or difficulty experienced by a student (which may indicate the student requires extra attention). Statistics related to that student's attempts at an exercise are
easily accessed: a bar graph is displayed, showing the number of problems attempted, the status of the questions, whether hints were used, how long each problem took, and each bar may be clicked to take the teacher to the exact problem attempted, as well as each action the student took, including time between actions, hints checked and answers given.

BACKGROUND

IT in mathematics education assessment

In a study by the University of Pretoria, a system of online assessment in a university level calculus course was implemented (Engelbrecht & Harding, 2004). Aside from paper assignments based on the use of software, the students completed a quiz online once a week. There is a time limit to the exercises given, but the system offers rapid feedback in a similar way to KA. There were also two online tests taken and one online final examination. Students answered a questionnaire, where the response was more in favour of the online assessment – the students liked the instant feedback, the lack of stress that came with the system, and the flexibility of the environment. On analysing the test results, there was found to be no significant difference between the online and paper forms of the test. The authors conclude that there is no reason to believe that performance standards cannot be maintained. They determine that there is a benefit to the teacher due to reduced grading, and to the diagnostic features that come with the online software. We note the difference in emphasis here, which is on creating a student-friendly and time-efficient summative assessment system.

A study by the University of Leeds investigated the effect of computer-based assessment on the performance of 260 ten-year-old students (Hargreaves, Shorrock-Taylor, Swinnerton, Tait and Threlfall, 2004). Two groups of 130 students of similar performance levels were formed, diminishing the need for student ability bias to be taken into account. Two tests were created, each of which was developed into a printed paper version and a software version – the content of the paper test and its corresponding software version were identical. Each group of students took one paper test, followed a month later by the opposite computer test, meaning each group acted as its own control. The mean of each test was higher for the computer assessment than for the paper assessment: by 10% for test 1 and by 2% for test 2. While each group performed better on the computer tests, there is evidence that one group was less competent than the other. Thus we note that the effects of the online assessment system should not be read in isolation.

Available exercise platforms

MyMathLab (MML) is a commercial online exercise platform developed by Pearson, focusing on mathematics instruction and assessment. Pearson have produced a document entitled ‘Making the Grade’ which contains 77 case studies in which its MyMathLab, MathXL, or other mathematics exercise platforms were used (Speckler, 2012). The studies were voluntarily submitted by instructors who had designed their own curricula involving the platform. They are primarily set in US schools ranging from high school up to four-year colleges, but also contain three international studies, two based in Canada and one in Singapore. A range of study types were conducted: observational; historical or retrospective, where MML module results were compared to previous years; longitudinal, where students in
MML modules were monitored as they progressed through further modules; and experimental, where students were divided into control groups and MML groups. Among the case studies, 18 (23%) showed increased final exam scores, 51 (66%) showed increased pass rates, and 28 (36%) showed increased retention and/or completion rates.

An independent case study of MyMathLab was carried out at Fayetteville State University in North Carolina (Kodippili & Senaratne, 2008). This study consisted of a module on college algebra, where the students' final grades were primarily determined by homework and test/exam results. 72 students were split into two groups for the duration of the module: one group was assigned traditional paper-based homework, whereas the other group completed homework on the MML platform. The final scores for the groups indicate that the MML group scored 6.3 percentage points (pp) higher on average than the traditional group. A significant result of the study is that the pass rate is 21pp higher in the MML group than the traditional group. However, the study cites small sample sizes and a high p-value as reasons why no definitive conclusions can be made about whether the platform significantly increases student achievement. The results attained are based entirely on final scores and do not indicate contextual learning or understanding of the material covered. The study indicates a common theme among users of exercise platforms: teachers/lecturers feel they have more time to interact with the students, since numerical assessment is taken care of by the platform itself, and that students are freer to learn at their own pace.

Critical analyses of Khan Academy's exercise platform

KA has been the subject of much debate over the last few years, primarily by teachers who are dubious about KA's styles of learning. Very little of this material appears in peer-reviewed journals or books, but is largely contained in newspaper articles and blogs. Much of this scrutiny relates to the videos on Khan's website rather than the exercise platform. Nevertheless, we consider these critiques to be relevant because similar issues may arise in relation to the exercise platform.

An article by Ani in the Washington Post (Ani, 2012) criticises Khan's preparation methods for making an educational video, as there is no prepared script involved. Given that each exercise in the exercise platform links to a related video on the same topic, there is a danger that students who are already experiencing problems with the material may watch a video that contains incorrect statements. This could hinder the students’ progress more than if they had never watched it. Ani (2012) mentions that instructional problems exist, e.g., in Khan’s Multiplying and Dividing Negative Numbers, where Khan implied that any number of negative numbers multiplied together result in a positive number. As this was an instructional error, it was possible that it might have been reproduced within the exercise platform itself in the hints or solutions to a problem (however, this particular issue is not present in the platform).

KA has been piloted in a number of schools since the exercise platform's inception and is soon to be piloted in nearly 50 grade schools in Idaho to earn credits for their mandatory online learning module. A pilot study took place at Oakland Unity High School (Castillo, McIntosh & Berg, 2012), using results from common tests given over several years. It
focused on the use of the KA exercise platform in a classroom and was not concerned with
the videos on the site. A pre-test was used as a means of finding a baseline to compare
classes. Over the course of three tests, the mean scores increased by approximately 20% in
each case. In the case of the final test taken, the mean doubled. Similarly, 40-50% of students
achieved 80% or higher in this final test, while only 5% did so previously.

Based on the studies reviewed, exercise platforms seem to be widely considered to be useful
to teachers. Every implementation studied shows improved results in some way, and teachers
appreciate the simplified class management that comes with using the platforms. They feel
they are able to focus on the students' needs, and that students appreciate being able to learn at
their own pace.

**METHODOLOGY**

A variety of qualitative and quantitative research methods were employed as appropriate to
the aims stated above. These included survey and interview methodologies, statistical analysis
of test scores and observations of the classes in which the exercise platform was applied.

The study focused on three first year post-primary maths classes from September 2012 –
April 2013. One of these classes (Class 1) implemented the KA exercise platform as part of
its curriculum, while Classes 2 and 3 used more traditional textbook-based exercises. The
students had not been streamed; however the results of the *Drumcondra Primary Mathematics
Test – Revised* (Educational Research Centre, 2007) taken in primary school were available,
and there was minimal difference between the mean scores for each class-group and so the
classes were assumed to have started the year with the same attainment level on average.

Each class had four 40-minute mathematics lessons per week, with 21 students in Class 1, and
20 students in each of Classes 2 and 3. Each had a dedicated mathematics teacher, except that
Class 1 was taught once a week by a student-teacher for the first term of the academic year.
The curriculum for all first year mathematics classes was designed so that the Common
Introductory Course (CIC) was completed, and they used a common textbook. The teachers
aimed to keep a similar pace in their lessons in order to be able to give common tests and
exams throughout the year. Apart from the KA classes, all three classes only experienced

Class 1 was examined in detail. These students typically attended their mathematics lessons in
the school’s computer room once per week, during which they completed exercises using
KA's exercise platform. Students were given a list of exercises to be completed in a given
topic. These exercises typically assessed the content students had been learning in their
traditional lessons and were structured so that the exercises considered essential were to be
attempted first, with more difficult versions available if students had time remaining. These
lessons were observed by the first author (SL). These observations provide a context for the
discussion below of the results of the interviews and surveys. They included marking progress
made on exercises, and listening to students discuss the platform, and the lesson material
among themselves. On days where lessons were not held in the computer room, the students' knowledge was assessed in a more traditional manner, with time during the lesson given over to answering textbook questions. Classes 2 and 3 were used as control groups for the purposes
of this study. In formal assessments, their knowledge was assessed solely in the traditional manner.

As the year progressed, Class 1's teacher was interviewed informally about the progress he was making with his implementation of the exercise platform. These interviews took place if the teacher was available following an observed lesson, which was about once every two weeks during term. A formal, recorded interview was conducted at the end of the study with prepared questions. Particular attention was paid to the teacher's changing methodology, as well as the process of linking the platform to the Irish curriculum.

Students were also surveyed over the course of the year using author-created surveys with questions of variable response type. The first survey was aimed at gauging student attitudes towards the exercise platform and website: only Class 1 was given this survey. The key areas assessed were student attitudes towards maths lessons in general, attitudes towards various aspects of the platform, as well as how much benefit students felt they got out of the platform. The second survey was distributed to all First year students, and looked more closely at how students felt after a mathematics lesson. The intention of the survey was to ascertain the differences between students using KA and students not using it in terms of their attitudes towards mathematics, and the quality of their mathematics lessons.

Over the course of the study, four common tests were taken by the classes on the topics of natural numbers, integers, probability, and coordinate geometry. Further tests were taken by each class, which were not common, and so were not analysed. The classes also took a common Christmas exam on the subjects of natural numbers, integers, fractions, decimals, and percentages. Analysis of these tests yields some interesting results, as shown in the next section.

RESULTS AND DISCUSSION

Teacher insights

The teacher of the KA class echoed the feelings of others who have implemented similar platforms in their classrooms. In his opinion, the platform is capable of holding the attention of all students, regardless of ability. The more capable students are free to work at a pace that suits them – they do not need to continue doing questions of the same type while the rest of the students catch up. The ability of KA to assess whether or not an individual student is ready to move on is an invaluable one in a classroom of 20 or more students. However, although the platform has been in development since 2006, it is still changing on an ongoing basis, which is a hindrance to planning usable lessons. As a result, the teacher's methodology changed from preparing an entire booklet at the beginning of the year to selecting exercises at most a few days in advance.

The main points made by the teacher who implemented the KA exercise platform in his classroom after the period of research concluded were:

General

1. KA is a free-gratis platform; this was a major consideration prior to implementing it.
2. After trialling the platform with a second year class, he decided in January 2012 to use it with his First year students the following year.

3. Prior to September 2013, he made connections between the curriculum he was implementing and the appropriate exercises on the platform, and developed a booklet to give to students containing the exercises required to complete a topic, as well as instructions for getting started with the platform (creating accounts, navigation, etc.).

4. It was worth the time and effort and would be implemented again the following year.

Successes

1. Students really seemed to love using the platform and were fully engaged.

2. He was able to spend more time with struggling students. More than half the class rarely (if ever) required assistance to answer questions.

3. Students were able to work at their own pace.

4. He did not need to think about getting students to focus or run out of things to do.

Issues

1. There were issues initially getting students set up with Google accounts and some students were not sufficiently familiar with using computers.

2. The platform is constantly in flux: exercises change names or content, or disappear completely. The booklet he initially created quickly became obsolete.

3. There was not enough content for the CIC.

4. The management platform was not that useful: students who were working more slowly than others might have been marked as struggling by the system when they were just working at their own pace.

Student surveys

Two surveys were conducted over the course of the study. The first was completed solely by Class 1 while the second was completed by students from each of the three classes. The first comprised eight short-answer open questions. The second contained 11 fixed-response items on a Likert scale. The number of students returning each answer was compiled and compared. Results and discussion of the main findings are given below.

Survey 1

The first survey was designed to discover how the students felt about their experience in computer room classes, particularly with regard to the nature of the implementation of the platform and their teacher’s methodology. It was distributed to 17 students present in Class 1 during a lesson mid-way through their academic year: they had been using the platform for approximately five months at the time. 12 of these surveys were returned and analysed.

Firstly, the students were asked where they ranked mathematics in terms of their favourite subjects. Two thirds placed mathematics in their top five and the remainder in their bottom five. Interestingly, no student ranked it as their favourite: the majority placed it second or
third, with two students stating it was their least favourite subject. Despite this, every student answered that they preferred lessons in the computer room to traditional classroom lessons. This preference for computer lessons reflects the feeling of students in the Pretoria study, though with even greater preference for the computer-based assessment.

Next, the students were asked whether or not there was much competition with regard to the platform’s reward system of points, badges, etc. There was a mix of answers here and it is unclear whether those students who felt competitive were driven to perform differently because of it. During observed lessons, there were only minor indications of competition: occasionally students would compare avatars available to them, or remark that they had just earned a badge, but it appeared to be more of a matter of pride to have earned these items rather than competition between students actually trying to get them.

No students stated that they had watched any videos on the website, showing that students received help primarily from their teacher. However, the hints and solutions offered in each question appeared helpful to those who used them. Nearly three quarters of students who used hints found that they were usually able to move on from the question they were stuck on. It was noticed in lessons that students typically would opt not to use these hints wherever possible, but instead rely on their teacher for assistance: it is possible that students were attempting to minimise their workload on each exercise by getting help from their teacher rather than the platform, since this does not affect their progress bar. It is also a good indication that the students felt that they could ask questions from the teacher without embarrassment, and may simply be the case that they found the teacher’s help more useful than that offered by the platform.

When asked if they have ever had to move on from an exercise because they got stuck for too long, most said they had not. This is despite the fact that in observed lessons students rarely managed to finish every exercise assigned in the available time. Reasons for this disconnect can only be speculated upon. Students may not consider this situation as “being stuck for too long”. Students also indicated that they tended not to go back to complete these unfinished exercises.

While five students believed that the platform was perfect as is, a further five suggested that it is too tough on mistakes and should require fewer questions to succeed in an exercise; that too much progress is lost when a mistake occurs, and that it should be possible to get three leaves on a question even after making a mistake. However, having seen students make mistakes even after getting several questions correct in a row, it is suggested that the platform handles student progress adequately within the purview of its mastery objective.

Survey 2

Survey 2 was administered to each of the three first year classes at the end of the year. It was returned by 20 students in Class 1, 19 students in Class 2, and 20 students in Class 3. The focus of the survey was a more general survey of students’ opinion on the classroom environment, and whether students felt sufficiently supported by their teachers. While some questions show a greater than 20pp difference in answers between the three classes, most do not show more than a 20pp difference between answers.
Students were asked about their attitudes towards their mathematics lessons. They were asked to respond to the statement “Maths is one of my favourite subjects in school”, choosing from the responses very/always true, a little/sometimes true or not at all/never true. In Class 1, only 15% chose the first option, compared to 63% and 45% respectively in Classes 2 and 3. Based on discussions with the three teachers, we have not been able to identify any significant difference in the pupils’ experience of mathematics during the school year that could underpin this variation in opinion. Equally, our observations of the KA classes do not lead us to conclude that these contributed significantly to a lack of enjoyment of the subject. The difference could relate to the different teaching styles of the teachers.

When asked if they can see how mathematics is useful in everyday life, only 45% of Class 1 thought this was very true, compared with 79% and 67% of Classes 2 and 3 respectively. KA is not particularly known to provide context in its exercises; in many cases, questions are presented as numbers and equations requiring a calculation. However, each class was using the same Project Maths textbook, so the additional context is likely not being provided by the textbook.

When presented with the statement, “My teacher shows us how to do questions, then we answer similar questions”, all three classes strongly agreed that there is some level of rote learning occurring in lessons, and most (70%, 74%, and 80% for Classes 1, 2 and 3 respectively) students believe that that is all they do. As KA’s platform typically is question-and-answer, there was not expected to be a significant difference in student attitudes between classes here. This was also reflected when students were asked if they fully understand a topic once the class finishes it – neither KA nor traditional rote learning appear to be more effective in terms of helping students’ understanding, from their own point-of-view, with 40%, 53%, and 55% reporting that they always fully understand topics when they finish them.

When asked whether they felt they could ask their teacher for help when they were stuck, a wider gap developed; while Classes 2 and 3 are no more than 10pp apart when answering strongly agree (79% and 75% respectively), students in Class 1 appear to be more reserved about asking for additional help with their work (55%). In contrast to the previous survey, KA students appeared not to want to ask their teacher for help. Given that this survey was more general, and the first survey focused on their KA lessons specifically, this disconnect may be in the nature of the lesson: students in Class 1 may feel as though they cannot ask questions during their traditional lessons, but can do so in the computer room. When asked if they get enough help from their teacher during lessons, the difference between Classes 1 and 2 fell below 20pp. Interestingly, while 25% of students in Class 3 felt they could not always ask for help when they needed it, 35% felt that they did not get enough help. Similarly, 21% of Class 2 felt they were not able to ask for help when they needed it, while 32% were not getting the help they needed. This suggests that there are students who feel comfortable asking questions, but do not act on it when they need to. The reasons for this are unclear. Class 1 had a similar, but smaller difference between responses to these two questions. It should be noted that this difference may be due to the different teaching styles of the individual teachers.

Class 1 had the highest percentage of students who felt they were getting a variety of explanations from their teacher. The explanations provided on KA may be different to those
that would be provided by their teacher in a traditional lesson, or possibly the teacher of Class 1 had a greater tendency to provide additional explanations than the other teachers.

Most surprisingly, when asked if they sometimes get help with mathematics from sources other than their teacher, Class 1 had the lowest proportion (15%) of students who felt this was very true, compared with 26% for Class 2 and 20% for Class 3. Almost the same proportion of each of the three classes (circa 40%) felt it was not true.

Comparing test results

We give here a brief quantitative overview of the comparative performance of the students in the three classes in the summative assessments undertaken during the school year. There were five common tests taken in all: these were completed as paper exams in all cases. Four of the tests covered specific topics (natural numbers, integers, probability and coordinate geometry). A fifth Christmas test covered a range of topics in the Number strand of the syllabus.

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<th>N</th>
<th>Mean</th>
<th>Sample Standard Deviation (SSD)</th>
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<th>Standard Error</th>
<th>Range</th>
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Table 1: Summary statistics for the common tests taken by each class

We consider first the pooled results of the four tests. Summary data are presented in Table 1. As can be seen here, the summary results are broadly similar for each class. Between Classes 1 and 2, the difference in means of 4pp is not statistically significant even at the 90% confidence level (t = 0.55, df = 39). Similarly, there is no statistically significant difference between the means of Classes 1 and 3 (t = 1.30, df = 39), or between Classes 2 and 3 (t = 0.76, df = 38), at these confidence levels.

Similar results arise for the results of the test on natural numbers, where Classes 1, 2, and 3 have means 77, 77, and 80; and SSDs 12, 12, and 12 respectively. No statistically significant difference is found between these results at the 90%, 95%, or 99% confidence level.

In the case of the test on integers taken by each class, the means were 70, 79, and 79; with SSDs 22, 14, and 16 for Classes 1, 2, and 3 respectively. Here, no statistically significant difference in the ability of Classes 2 and 3 was found (90% confidence). Class 1 appears to have lower results at 90% confidence, but there does not appear to be a statistically significant difference at 95%, or 99% confidence.

The probability test showed the largest differences between scores, with means 77, 88, and 89; and SSDs 15, 13, and 10 for Classes 1, 2, and 3 respectively. Class 1 performed worse than both Classes 2 and 3 at a statistically significant 99% confidence level (t = 2.46, and t = 2.97 respectively).
In the coordinate geometry test, students from each class performed at mostly similar levels, with means 68, 63, and 75; and SSDs 29, 27, and 18 for Classes 1, 2, and 3 respectively. The t-test suggests that Class 1 did not perform statistically significantly differently to either Classes 2 or 3 (t = 0.28, and t = 0.83 respectively), whereas Class 2 performed worse than Class 3 at 90% confidence (but not at 95%, or 99% confidence).

**Christmas Exam**

The common Christmas exam was delivered as a summary exam, assessing the retention of all the material covered by the classes in the first school term. The test covered some areas previously tested - natural numbers, and integers - as well as areas that had not been commonly tested - fractions, decimals, percentages. The results are summarised in Table 2.

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<th>Sample Standard Deviation (SSD)</th>
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<td>Class 3</td>
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<td>17</td>
<td>68</td>
<td>4</td>
<td>51</td>
<td>30</td>
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</table>

Table 2: Summary statistics for the common Christmas exam taken by each class

The difference in means of 8pp between Classes 1 and 2 implies that Class 1 performed less well than Class 2 at the 90% confidence level (t = 1.39), but this does not extend to the 95% or 99% confidence levels. Classes 1 and 3 did not have this performance difference at any significance level tested.

**CONCLUSIONS**

From a teacher’s perspective, the exercise platform is an effective educational aid. It allows for simplified class management: students can remain occupied with the platform through their own initiative while struggling students can be aided by their teacher. The teacher suggests that the platform is not entirely suitable for the Common Introductory Course, so is of limited use until algebra is studied in second year. The exercises are also in constant flux, showing care and growth for the platform, but this was a hindrance to preparing for lessons.

Students appear to enjoy the platform for the most part, but do not fully utilise it. They do not want to use the hints or solutions to increase their knowledge of the questions. A possible reason for this is that they do not like to lose progress in an exercise. Students in each of the three classes appear to have a broadly similar attitude to their lessons, although students in Class 1 seem to be more reluctant to ask their teacher for help.

The test results are broadly similar for the three classes: similar numbers of students achieved each grade letter. The only statistically significant difference arises in the test on Probability, where Class 1 students were outperformed by their peers in Classes 2 and 3. Insofar as differences can be ascribed to the use of the exercise platform, on a per test scale, KA appears
to be more effective for coordinate geometry, and less so for integers and probability. The extra graphical capabilities and interactivity of the platform might allow the students to develop an intuition for coordinate geometry that is not available in some other topics. The overall weaker performance of Class 1 is dissimilar to the results of other studies considered in the literature review. This is perhaps noteworthy, as previous studies appear to suggest that the KA exercise platform engenders increased levels of attainment. That said, these studies typically involve deeper engagement with the KA system, rather than using just the exercise platform. It is clear that further research is necessary to make definitive conclusions on the validity of the use of the Khan Academy system in Irish mathematics classrooms.

NOTES

1. The term “traditional” refers to a class where direct instruction and practice dominate the teaching and learning approach. Practice questions are drawn from textbooks, and there is an emphasis on rote learning.

REFERENCES


USING AN ON-LINE LEARNING FACILITY TO PERSONALISE
MATHEMATICAL LEARNING IN A LEARNING SUPPORT CONTEXT
WITH STUDENTS FROM SENIOR CLASSES IN A RURAL PRIMARY
SCHOOL

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St Patrick’s College, Drumcondra

Students from the senior classes in primary school who attend for Learning Support in mathematics have typically struggled to attain levels of mathematical skills and understanding which are commensurate with their peers. This experience of failure often has a detrimental effect on a student’s belief in their ability to succeed in mathematics, which in turn affects motivation. An action research approach was used to specifically focus on student attitudes, attainment and quantitative communicative competence before during and following an intervention designed to respond to their needs. One aspect of this research study involved freely available personalised learning provided by the on-line learning website Khan Academy, which was intended to allow students to build their own educational experience and assume responsibility for their own learning. However, the findings sound a cautionary note with regard to the capacity of on-line learning websites to support students attending for Learning Support in mathematics. The important role of the teacher in follow-up discussion of video tutorials in a co-operative learning context is evident and some difficulties with internet access in the homes of these students are also highlighted.

INTRODUCTION

The primary concern of this research was to raise achievement in mathematics by improving the attitudes, increasing attainment levels and developing the quantitative communicative competence of a cohort of students attending for Learning Support in mathematics in the final year of their primary education in a rural, multi-grade school. An intervention was designed which established a co-operative learning group and mathematical learning was presented in realistic contexts allowing students to evaluate the reasonableness of their answers and to experience mathematics with a greater degree of engagement (Wiliam, 2005). A discussion based approach infused the intervention in recognition that students should be enabled to communicate effectively, analysing, explaining and justifying mathematical thinking (The Organisation for Economic Co-operation and Development (OECD), 2010). This paper reports on student engagement with an on-line resource used in the intervention to support numeracy, advancing an aim of the national literacy and numeracy strategy which is not expected to be attained until 2014 [1] (Department of Education and Skills (DES), 2011, p. 22).

Computer-assisted instruction

Slavin and Lake (2008) argue that computer-assisted instruction (CAI) can assess students and provide individualised practice on skills that they have the prerequisites to learn but have not yet learned. In the United States, the National Mathematics Advisory Panel (2008) has
recommended that CAI should be considered as a useful tool in developing students’
automaticity and in developing specific mathematics concepts and mathematical problem-
solving abilities. In the Irish context, Shiel, Surgenor, Close and Millar (2006) have advised
that CAI should be used to develop computation skills and enhance mathematics reasoning
and problem solving skills. Project Maths in the Junior Cycle recommends using information
technology to access, manage and share knowledge (National Council for Curriculum and
Assessment (NCCA), 2010). Echoing the theme of individualised instruction, Whitby (2011)
argues that internet learning places children at the centre of their own learning by affording
them choices regarding what they want to learn and allowing them to receive instant
feedback. He cites Khan Academy as a website which freely provides this individualised
instruction (Khan Academy, 2011a).

Khan Academy

The Khan Academy Practice Platform (Khan Academy, 2011b) allows students to build their
own educational experience (Conway, 2011) offering video tutorials which are each shorter
than 10 minutes, with subject matter easily browsable by title (Hearst, 2011). Certificates and
badges earned by students on Khan Academy can be shared via email with fellow learners
(Khan Academy, 2011a). Tan and Pearce (2011) used videos to support and enhance learning
within the classroom. They highlighted the critical role of the teacher’s input during video
tutorials and in follow-up discussions. Students in their study appreciated the expertise of the
teacher and valued the opportunity to question and offer opinions.

Computer adaptive tests facilitate the collection of data, can help determine appropriate
instructional targets and can aid in the monitoring of student progress (Burns, Klingbeil, &
Ysseldyke, 2010). Adaptive achievement testing on the practice platform of Khan Academy
allows fewer questions to be used in assessing a student’s level of knowledge and determining
students’ skill level than in traditional assessment tests. This facilitates the optimum use of
prior knowledge and permits the personalising of the learning experience (Prineas & Cini,
2011). In education, ipsative assessment is the practice of assessing present performance
against the prior performance of the person being assessed. Ipsative assessment can be
particularly useful for children with learning disabilities and can improve motivation (Miller
& Lavin, 2007). Khan Academy encourages students to improve on previous personal scores,
removing the competitive element which can be detrimental to self-esteem. The Khan
Academy practice platform incorporates multiple feedback loops which allow for the rapid
collection, dissemination and analysis of student learning data (Khan Academy, 2011c).
Assessment data available to teachers allows for effective timely intervention (Bajzek,
Brooks, Lovett, Rinderle, Rule, & Thille, 2008).

As a freely available resource, Khan Academy can allow students to hone the skill of self-
guided, life-long learning and may provide a new model for differentiation using a method
known as the ‘Flipped Classroom’ (Bender, 2012). This is an approach to education in which
teachers assign work for students to review at their own pace at home thereby allowing for
increased opportunities for students to practice their skills and receive feedback from the
teacher in class. When students are required to use Khan Academy to target personalised
learning goals, time inside the classroom can be used for project-based learning. By
experiencing learning independent of the teacher, students develop a sense of responsibility for their own learning (Bender, 2012, p. X).

**THE INTERVENTION**

**Participants**

Three Sixth class students attending for learning support in mathematics participated in this study. Each has received all their primary schooling in multi-grade classes with up to three grades and thirty children in a class on occasion. When Frank was aged 5 years 6 months, he was assessed by a psychologist as functioning within the low range of ability. Scores across all aspects of mathematics have been low. In Fifth class he scored at the 13th percentile with particularly low scores on Measures and Word Problems in which he correctly answered 14% and 18% of questions respectively. Achieving the 18th percentile in Fifth class, Bernadette was invited to Learning Support following analysis of scores. Analysis indicated that Bernadette had scored consistently low on word-problem-solving in successive standardised tests. Although her scores in Number were historically average or above average, she scored particularly low on the standardised test in Fifth class, answering 30% of questions correctly. Similarly low scores were recorded for Measures (31%) and Word problems (32%) at this time. Denise scored at the 27th percentile in 5 Fifth class. Her scores on Word problems have declined on successive standardised tests and she answered 21% correctly on the Fifth class test. Number scores are significantly depressed in the initial year of teaching a new strand in mathematics (e.g. multiplication and division in Third class – 28% correct) but have improved with support (e.g. Fourth class scores – 62% correct). Scores on Measures have also been depressed (e.g. 33% answered correctly in Fifth class test).

**Procedure**

This study addressed mathematical word problem solving with three students from Sixth class who were attending for Learning Support in mathematics. A diagnostic criterion referenced test administered before and after the intervention assessed attainment. It was constructed to assess students’ abilities in dealing with word problems. All problems were drawn from the strand units in focus: Length, Weight, Capacity and Money. Students were individually monitored and encouraged to relate an explanation of the methods employed. These explanations were documented to assess communicative competence.

An intervention was designed based on statements from the content of mathematics curriculum for Fifth and Sixth classes (NCCA, 1999) particularly focussing on the strand Measures. Explicit links with strand units from the strand Number were made. A cooperative learning group was established and student learning was personalised through engagement with the Khan Academy website (Khan Academy, 2011a). Formative assessment strategies were embedded in the intervention. The learning intention was shared with students, effective questioning probed their understanding and corrective feedback was given as students worked to solve problems in realistic contexts.

Computation and fact based learning was addressed on the Khan Academy Practice Platform (Khan Academy, 2011b), as although each student had weak computation skills, practice
needed to be personalised to focus on individual needs. Data was gathered on each student’s engagement with both the practice and the video tutorial elements by the researcher who was the students’ regular ‘coach’ (Khan Academy, 2011c). Preparatory learning for each of the four cycles of the intervention was supported by video podcasts from both the Arithmetic and the Developmental Math playlists which were assigned throughout the academic year (Khan Academy, 2011d) with homework assignments during the intervention matched to prepare students for imminent work. Khan Academy uses a system of ‘Achievement Badges’ to promote engagement (Khan Academy, 2013). ‘Meteorite’ badges are common and easily earned in the initial stages of participation while ‘Moon’ badges are uncommon and represent an investment in learning. For the purposes of this intervention five of the meteorite badges, for persistence, perseverance and those which indicated attention to video lessons and five of the moon badges: ‘Awesome Listener’; ‘Hard at Work’; ‘Tenacity’; ‘Steadfastness’ and ‘Sticktoitiveness’ were considered particularly pertinent.

**FINDINGS AND DISCUSSION**

For the duration of this study, participant learning occurred across three distinct, yet overlapping contexts. Student-reported and observed attitudes to mathematics differed between the classroom context, the co-operative group context and the on-line learning context of Khan Academy, but only the latter context will be considered here.

**Khan Academy as a learning context**

Students had signed up to the Khan Academy website in September 2011. Data which had been collected over a period of seven months, and included both the pre-intervention and post-intervention phase, was captured and analysed on 1st April 2012 (Khan Academy, 2011c). Khan Academy encourages students to improve on previous personal scores, removing the competitive element which reportedly prevailed in the participants’ classroom context. During the intervention, students worked in this context individually at home and together co-operatively in the Learning Support class on both the Practice (Khan Academy, 2011b) and Watch (Khan Academy, 2011d) platforms (See table 1).

Table 2 indicates the relevant achievement badges earned both during the intervention and in the pre-intervention phase. As students failed to maintain their engagement with Khan Academy following the intervention, no badges were earned in the six week post intervention phase. 75% of all badges earned were awarded during the intervention. However, although both girls earned most of their badges during the intervention, Frank earned only half of what he had earned in the pre-intervention phase. Even though both girls availed of opportunities to work on Khan Academy during lunchtimes, Frank never did. Furthermore despite having access to the internet at home Frank rarely used Khan Academy outside of school even when work was assigned as homework during the intervention (Table 3). However, he did log on twice at home during the final week of the intervention.
Pre-Intervention Phase (07/11/11-18/11/11)

Regularly used in Learning Support class time with all 3 students engaged independently on separate computers: *This amounted to 15 minutes in each 45 minute lesson or longer if testing was in progress with one student.*

Intervention Phase (21/11/11-10/02/12)

Khan-Academy videos assigned as homework and encouragement to practise given at each lesson.

Post-Intervention Phase (20/02/12-30/03/12)

Students worked at home or in school during lunchtimes. Co-operative sessions in cycles 3 and 4 of the intervention: *15 minutes in two of every four weekly lessons.*

Table 1: Overview of student engagement with Khan Academy *Watch* and *Practice* platforms throughout the intervention

<table>
<thead>
<tr>
<th>Badges</th>
<th>Bernadette Pre-I Phase</th>
<th>Intervention Phase</th>
<th>Denise Pre-I Phase</th>
<th>Intervention Phase</th>
<th>Frank Pre-I Phase</th>
<th>Intervention Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Meteorite Badges</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nice Listener</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Great Listener</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Persistence</td>
<td>3</td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Perseverance</td>
<td>1</td>
<td>9</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Act I Scene I</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td><strong>Moon Badges</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Awesome Listener</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hard at Work</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tenacity</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Steadfastness</td>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sticktoitiveness</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total Badges Earned</strong></td>
<td>11</td>
<td>23</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Pertinent achievements badges earned by participants on Khan Academy
Reflecting the finding by Tan and Pearce (2011) which recognises the critical role of the teacher’s input during video tutorials and in follow-up discussions, badges awarded for achievement on the Practice platform tended to be earned during cooperative sessions in Learning Support Class time. In these sessions students engaged individually on separate computers, and learned cooperatively by consulting each other regarding progress, sharing badges earned and referring to the teacher for clarification and support. Despite logging in at home on more occasions than fellow participants, Bernadette earned her eleven badges only during these co-operative group sessions. Similarly, all of Frank’s four badges were achieved when working during these sessions. Denise earned all but one of her 23 badges in co-operative group time, achieving a badge for persistence on one of the two occasions when she logged in at home.

**Responsibility for learning**

If cooperative learning is to be effective, Bennett (2011) contends that students must develop internal standards for their work. Participating students had been familiarised with Khan Academy as a learning environment before the intervention and had engaged in many practice sessions as Bender has advised (2012). During the intervention students were encouraged to maintain their engagement with Khan Academy Watch and Practice Platforms as students needed to individually hone computation and basic procedural skills. Monthly homework assignments with hyperlinks to target videos were e-mailed to each student.

### Table 3: Engagement on Khan Academy during the intervention

<table>
<thead>
<tr>
<th></th>
<th>Bernadette</th>
<th>Denise</th>
<th>Frank</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Khan Academy Data November</strong> (captured 01/12/2011)</td>
<td>6 co-operative sessions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Logged on</td>
<td>9 occasions</td>
<td>7 occasions</td>
<td>6 occasions</td>
</tr>
<tr>
<td>Average Time spent</td>
<td>24 minutes</td>
<td>31 minutes</td>
<td>26 minutes</td>
</tr>
<tr>
<td><strong>Khan Academy Data December</strong> (captured 03/01/2012)</td>
<td>2 co-operative sessions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Logged on</td>
<td>4 occasions</td>
<td>Twice</td>
<td>1 occasion*</td>
</tr>
<tr>
<td>Average Time spent</td>
<td>9 minutes</td>
<td>7 minutes</td>
<td>4 minutes</td>
</tr>
<tr>
<td><strong>Khan Academy Data January</strong> (captured 29/01/2012)</td>
<td>4 co-operative sessions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Logged on</td>
<td>11 occasions</td>
<td>5 occasions</td>
<td>4 occasions</td>
</tr>
<tr>
<td>Average Time spent</td>
<td>17 minutes</td>
<td>39 minutes</td>
<td>32 minutes</td>
</tr>
<tr>
<td><strong>Khan Academy Data February</strong> (captured 29/02/2012)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Logged on</td>
<td></td>
<td>Twice</td>
<td></td>
</tr>
<tr>
<td>Average Time spent</td>
<td></td>
<td>11 minutes</td>
<td></td>
</tr>
</tbody>
</table>

*discrepancy explained due to absence from school on one occasion*
However, this aspect of the intervention was unsuccessful. In Denise’s family the computer was ‘in for repair’, in Bernadette’s family internet access was temporarily unavailable and Frank did not log on as he viewed it as ‘extra maths homework’. Following the first cycle of intervention students were given the option of accessing computers during lunchtime. This opportunity was infrequently availed of. On three occasions in early January both girls used the computers during lunchtime having received the results of a recent maths assessment test. Although commendable, this did not seem to be internally driven yet, but rather motivated by the results of a test.

During the third and fourth cycle of intervention opportunities were given for co-operative learning on Khan Academy and these sessions re-established the peer-support which students had been used to before the intervention and facilitated teacher input and follow-up discussions enhancing the personalised learning available on Khan Academy. This time represented 40% of Bernadette’s engagement with Khan Academy during the intervention. However for both Denise and Frank this figure was significantly higher: 75% and 90% respectively, indicating a lack of engagement with Khan Academy outside of the co-operative group and a reluctance to assume responsibility for their own learning in this context.

Perseverance

Ashby (2009, p.7) contends that, as a subject, mathematics requires “a considerable amount of perseverance from the individual in order to succeed”. This study therefore looked at levels of perseverance on the practice platform of Khan Academy. Badges for ‘Perseverance’ are awarded when more than 30 problems are answered mostly correctly in a skill before becoming proficient. Of the 25 badges generally indicating perseverance which were earned by participants during the intervention, Denise earned seventeen while Frank earned just one badge for ‘Persistence’. Moon Badges tended to be earned clustered with others on a limited range of skills, e.g. The badges ‘Perseverance’, ‘Steadfastness’, and ‘Tenacity’ were awarded to Bernadette for *Arithmetic Word Problems* and to Denise for *Numberline*. Denise also has ‘Persistence’ and ‘Sticktoitiveness’ badges for *Numberline 3* as she persevered, attempting 74 problems with the support of video tutorials and hints despite only ever correctly answering 9 consecutive problems.

Evidence on skill progress available on-line to the coaching teacher revealed that on occasion students failed to persevere with a challenge when not working in the group context. Of these skills which were started and quickly abandoned, Frank used the ‘hint’ help on only one occasion. Although Denise availed of the ‘hint’ option in three of four such attempts, she usually stopped immediately after taking the hint, indicating that the hint may not have given sufficient help. Despite availing of more hints and watching video tutorials related to the specific skill, Bernadette also seemed to find these supports insufficient (Table 4).
### Arithmetic Word Problems 2

Emily rode her bike for 7 miles on each of the past 2 days. How many miles did Emily ride her bike altogether?

### Fraction Word Problems

Luis ate 4 slices of pizza, and Omar ate 3 slices. If Luis ate 4/10 of the pizza, what fraction of the pizza was eaten?

### Ratio Word Problems

In geometry class, the girl to boy ratio is 6 to 2. If there are a total of 24 students, how many girls are there?

<table>
<thead>
<tr>
<th>Arithmetic Word Problems 2</th>
<th>Fraction Word Problems</th>
<th>Ratio Word Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Emily rode her bike for 7 miles on each of the past 2 days.</td>
<td>Luis ate 4 slices of pizza, and Omar ate 3 slices.</td>
<td>In geometry class, the girl to boy ratio is 6 to 2.</td>
</tr>
<tr>
<td>How many miles did Emily ride her bike altogether?</td>
<td>If Luis ate 4/10 of the pizza, what fraction of the pizza was eaten?</td>
<td>If there are a total of 24 students, how many girls are there?</td>
</tr>
</tbody>
</table>

### Table 4: Sample problems from Khan Academy on which Bernadette experienced difficulty

Bernadette experienced particular difficulty when working on word problems. On *Arithmetic Word Problems 2* where solutions necessitated either multiplication or division, Bernadette attempted three problems before giving up. Attempting *Fraction Word Problems* she watched a tutorial video and used hints on two of her three attempts but ultimately abandoned the exercise. Despite watching three minutes of video and using hints on all five attempts Bernadette failed to persevere with *Ratio Word Problems*.

Denise experienced similar difficulties on *Fraction Word Problems*, attempting three, availing of two hints but ultimately giving up. Once again, although she used hints she stopped after two attempts at *Prime Numbers* which posed questions like: Which of these numbers is prime? 87, 33, 46, 47, 9.

Working independently, although Frank often answered series of questions correctly he rarely persevered, never answering more than eleven before stopping. On *Multiplication 2* he correctly answered eight of his nine attempts and similarly on *Adding Decimals* he only failed to answer one of his eleven attempts correctly. Furthermore, despite answering all seven of his nine attempts correctly he failed to persist on *Subtraction 4* (Table 5).

<table>
<thead>
<tr>
<th>Multiplication 2</th>
<th>Adding Decimals</th>
<th>Subtraction 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>66 x 9</td>
<td>693 + 6.88 = ?</td>
<td>9106 - 5089</td>
</tr>
</tbody>
</table>

### Table 5: Sample problems from Khan Academy on which Frank failed to persevere

### IMPLICATIONS AND RECOMMENDATIONS

#### Limitations

Miles and Huberman (1994) have considered the challenges faced by the qualitative researcher working alone to define the problem, decide on sampling, design the instruments and collect and analyse the data and preparing a report. They have identified several criteria for assessing data quality. Mindful of these the researcher attempted to triangulate with several data collection measures and data sources. Findings were discussed with informants and codes were debated with a peer who had previously done similar research. However, having used purposive sampling, and used just three students in the intervention the results of this study cannot be considered as representative.
Elliott, (as cited in Cohen, Manion, & Morrison, 2007) argues that the style of action research reported here may neglect the wider curriculum structures, regarding teachers in isolation from wider factors. Indeed, although framed against the backdrop of a National Literacy and Numeracy Strategy which expects young people ‘to be able to think and communicate quantitatively’ (DES, 2011, p. 8) this was a small-scale intervention designed to address the particular needs of a cohort of students from a rural multi-grade school who were attending for Learning Support in mathematics in Sixth class. This context exerted a distinct influence on the students’ attitudes to mathematics which may not be reflected in general.

Despite raising attainment on the criterion-referenced test and successfully re-engaging Frank in mathematics learning, the hope that Khan Academy would be embraced by students as a support for independent learning was not realised. Nevertheless, this intervention established a small community of learners which developed a level of quantitative competence which enabled them to discuss mathematics and support each other’s learning.

Although there have been calls for teachers to consider using Khan Academy tutorial videos to ‘flip the classroom’ (Bender, 2012), i.e. set Khan Academy video tutorials as homework to create time in the classroom for higher order activities, this study has demonstrated that this is not a straightforward task. Whether for socio-economic (limited or lack of access at home) or motivational reasons (because it is seen as extra work) this may not be viable with a cohort of students attending for Learning Support. Furthermore, there are indications that the video tutorials and hints may not support new learning for these students outside of the immediate support of a teacher. Thus, although Khan Academy makes individualised learning freely available (Whitby, 2011), in this study it was insufficiently personalised for the students attending for Learning Support.

**Recommendations**

The National Literacy and Numeracy Strategy has charged the Department of Children and Youth Affairs, in cooperation with the NCCA and the Library Service, with the task of providing all parents with information and on-line resources regarding activities which can be used to support their child’s numeracy skills by 2014 (DES, 2011). Khan Academy may be considered as an on-line resource in this regard. However, these authorities must be cognisant of the difficulties faced by some students in accessing on-line resources. Students in this study were sometimes unable to access Khan Academy at home and findings indicate that internet access can be regarded as a luxury when family budgets were stretched. Although the national strategy recognises that “children from socially and economically disadvantaged communities are significantly more likely to experience difficulties in literacy and numeracy for reasons associated with poverty, poorer health, and a wide range of other factors” (DES, 2011, p. 9) there is a risk that in endeavouring to empower parents to support their children by making on-line resources available, some learners in these disadvantaged communities may remain marginalised.

**NOTES**

1. The strategy aims to provide all parents with information and on-line resources about activities that they can use to support their child’s oral language development, literacy and numeracy from birth.
REFERENCES


This paper presents the results of the initial phase of a two-phase exploratory mixed-methods study to gain the perspectives of students regarding the ‘importance value’ students place on mathematics. The ‘importance value’ is part of the Expectancy-Value Theory (EVT) and refers to both students’ perceived usefulness of mathematics and the importance students place on doing well in mathematics. This paper looks at the results of a survey questionnaire with the aim of gathering information relating to students’ interests and perceptions of how mathematics is used in careers.

The results contribute to the development of phase two of the study with the aim of using their interests and developing their perceptions in an effort to address the Leaky Science, Technology, Engineering and Mathematics (STEM) pipeline. The Leaky STEM pipeline refers to the decreasing amount of females in mathematical courses throughout their educational careers. The survey questionnaire was designed to find out students’ attitudes and beliefs about mathematics regarding mathematics future usefulness for their chosen career path.

Results from this preliminary study show that students do not agree that mathematics is useful in their daily lives and male students place a greater emphasis in doing well in examinations in order to get into the career of their choice than their female counterparts.

INTRODUCTION

This paper looks at the initial findings of a two-phase exploratory mixed-methods study to gain the perspectives of students with respect to the value that they place on mathematics and perceived usefulness of mathematics. Important questions are asked with regard to students’ perspectives of what careers require mathematics, what their least favourite topic in mathematics is, its future usefulness for their chosen career path and finally, their beliefs regarding mathematics using the survey items from the 2011 Trends in Mathematics and Science Study (TIMSS) (Mullis et al., 2009). In phase two of the study, Gesture-Based Technology (GBT) is developed based on the results of the survey questionnaire.

RESEARCH AIMS

The aim of this research is to gather information about students’ interests and perceptions regarding the use of mathematics, expectancy to succeed in mathematics and how mathematics is used in careers. This information is then used with the aim of developing these perceptions using a Gesture-Based Technology (GBT). The purpose is to examine the “importance value” and the impact GBT may have on students’ perceptions of mathematics. The aim of this paper is to discuss the results of the survey questionnaire as part of phase one of the study.
Phase one employed the Expectancy-Value Theory (EVT) to guide the student questionnaire items in an effort to determine the perceived usefulness that students place on mathematics along with their least favourite topic and expectation of doing well in mathematics. This is then used as an instrument to guide the design of the GBT from the students’ beliefs and attitudes towards learning mathematics. Watt, Eccles, & Durik (2006) observe the “Leaky STEM pipeline” and examine the attitudes that can impact students’ perceptions of the perceived usefulness of mathematics.

More specifically, phase two investigates the capacity for GBT to be used in the classroom by conducting classroom observations and post-intervention interviews with teachers. These interviews investigate the perceived usefulness of mathematics and the importance value students place on mathematics. The following sections outline phase one of the study in more detail and discuss the questionnaire items including those taken from both EVT and TIMSS.

**LITERATURE REVIEW**

The EVT originally formulated by Atkinson in the 1950s has, through the times, shown a prominence in education as a model for academic motivation (Watt, Shapka, Morris, & Durik, 2012). Watt, Shapka, Morris, Durik, Keating and Eccles (2012) explain that factors affecting students’ success in mathematics include the students’ belief about mathematics usefulness and the students’ perceived usefulness of Science, Technology, Engineering and Mathematics (STEM) with regard to their interests, goals and career aspirations. More specifically, the focus of using the EVT is to look closely at the alignment of students’ perceptions of STEM to their career interests and future usefulness of subjects such as mathematics.

The “attainment value” and “utility value” are two values that place importance on tasks. More specifically, the attainment value refers to how well an individual does on a task and relates that to the concept of his/her own self-identity; the utility value refers to how useful a task is. The importance value refers to a combination of both of the aforementioned utility and attainment values. The central argument for the EVT is that engagement in an activity can be predicted by the value a person places on doing well but also their expectancy to succeed (Wigfield & Eccles, 1992).

A strong link exists between the success of students and their beliefs and attitudes towards learning mathematics (Kislenko, Grevholm, & Lepik, 2005). Learners potentially engage in tasks differently depending on their beliefs and motives. Learners may engage in a task more deeply due to its usefulness to their career aspirations (Johnson & Sinatra, 2013). Johnson and Sinatra (2013) show that task values also play a very significant role in determining how students engage with a task if the task is related to a career or otherwise. Husman, Derryberry, Crowson, and Lomax (2004) investigated task values in undergraduate students - the values that students put on doing well on a task was investigated by two theoretical foundations, the EVT and the Future Time Perspective Theory. According to Husman et al. (2004), the utility value is measured without a time signature. While they also investigated other survey items, it is maintained that separating these values while being closely related would not be advisable. Johnson and Sinatra (2013) also ask whether attainment and utility values can be separated as they are closely related to each other.
They investigated the connections between engagement, task values and conceptual change. One hundred and sixty-six undergraduate students were randomly assigned to one of three task values. Johnson and Sinatra (2013) point out that motivation can vary from individual to individual based on different career pursuits while also a student could be motivated to maintain a positive image without considering their career pursuits. The students engaging in tasks to better their educational development will engage in a task differently to those who engage in a task to enhance their career prospects.

Interest, self-concept, self-identity and motivation all play an important role in determining success as explained above. Frenzel, Goetz, Pekrun, and Watt (2010) suggest that depleted interest is often due to a few varying factors such as curriculum change and a complexity of the academic content as students progress through the educational system. Parents and teachers do not significantly impact on the decreasing levels of interests in adolescents. Across Germany, America and Australian schools students show decreasing mathematics values. Frenzel et al. (2010) offer an explanation as to why there is a decreased interest, suggesting that it could be a curriculum change, factors inherent in age-related changes or increased complexity of academic content along with a consequent lower task intrinsic attractiveness.

Mathematics in second-level classrooms was reformed in Ireland in 2010 leading to a new curricula given the name, Project Maths (Brosnan, 2008). This transformed the curricula into the more practical, real-world nature of mathematics including areas where mathematics can be used in real life. There are also a number of practical applications given to teachers such as rolling dice activities, card selections and flipping a coin. An increase in communication in the classroom is emphasized with questions in state examinations requiring more verbose answers. This tends to suit females (Watt et al., 2012) and brings about a discussion relating to gender differences and mathematical ability.

**GENDER AND MATHEMATICS**

Sáinz and Eccles (2012) account for gender differences with respect to the self-concept of computers and mathematical ability. Girls tended to lean towards a stereotyped field. Sáinz and Eccles (2012) discuss the stereotype threat whereby women performed less well than their male counterparts not because they are less able but because of the stereotype threat. Watt, Eccles and Durik, (2006) explore the relationship between females and their involvement in mathematics-related careers. They examine the motivations influencing their choice; ability-related beliefs and different kinds of values that predict adolescents’ choice for their future occupation. Females are less likely to choose these mathematics-related careers. This pattern is known as the “leaky STEM pipeline”. Studies done more than thirty years previously examined the decisions of girls to opt out of these careers. They identify secondary school as being the best time to focus on young adults’ attitudes towards mathematics as this is the time that post-school decisions are made and individuals’ perceptions are shaped. Studies have continued to report on the leaky STEM pipeline and different measurements to capture the reasoning behind females dropping out of these careers. In a later study, two important measurements of career-related aspirations are: (a) the domain of study and type of
occupation aspired to and (b) the amount of prestige associated with the occupation, its social status and importance (Watt et al., 2012). They note that females are more likely than their male counterparts to engage in careers that have a social aspect.

Even though gender is an important factor in this research, the main research focuses on the connections made by both genders with regard to their expectancy to do well in order to get their course choice and students’ perceived usefulness of mathematics. The following sections explain the sample taken in this study, survey questionnaire items used and the preliminary results.

**RESEARCH METHODS**

**Sample**

For phase one, a representative sample of schools was determined based on all primary and post-primary schools in Ireland. Second and fifth year students were asked to fill out the survey at post-primary level, students aged between sixteen and seventeen. Students in Fifth and Sixth class at primary level (aged between 10 and 12 years) were asked to fill out an adapted version of the survey to suit reading age. This was in an effort to understand the students’ perceptions prior to entering post-primary level education and if there were any significant differences. The questions asked were adapted to reflect the different curricula at primary and post-primary level. Post-primary level is the main focus for both phase one and phase two as the post-school decisions regarding career aspirations are made at this stage (Watt et al. 2006).

The survey sampling strategy used was a clustered stratified sample strategy whereby the schools acted as the clusters. When selecting clusters for the sample, the relative size of each school was taken into account. The probability proportion to size was also considered when taking the nationwide survey sample.

Ninety clusters were taken and, to allow for a non-response rate, this was increased to 100 for post-primary. Post-primary schools were given the option to fill out the survey online only. As the end of the school year was approaching a sample of primary school children were also taken to make comparisons in terms of their attitudes and beliefs. A sample of 200 primary schools was taken. As the end of the school year was approaching, primary schools were given the option of filling out the survey online or on paper.

Those who participate in phase one of the study in post-primary, will be asked to participate in phase two of the study. In order to create a higher response rate a sub sample will be taken from the nationwide representative sample from phase one.

**Survey questionnaire**

The survey questionnaire looked at students’ perspectives regarding their career aspirations and their perceived usefulness of mathematics. The survey draws on two key frameworks, TIMSS and the EVT. Both of these frameworks were used to guide the survey design and construct validated questions to give to students.
As well as drawing on these, the survey asked students their most and least favourite topic in mathematics, perceived usefulness of mathematics in their daily lives and expectancy of doing well in the end-of-year mathematics examination. Students in the post-primary sample were given choices based on the strands and strand units in Project Maths such as Statistics, Probability, Geometry, Trigonometry, Number, Algebra and Functions. Students were also asked questions relating to mathematical careers and if they require mathematics in order to get into the course of their choice.

This paper focuses on a number of preliminary results from a few survey items. Findings are presented on questions relating to students’ favourite topic, their perception with regard to mathematics perceived usefulness, expectation of doing well in mathematics at the end-of-the-year and needing mathematics to get into their course choice.

PRELIMINARY RESULTS

Out of the sample taken from both fifth and second years, over half of second year male students chose probability as their favourite topic. This remained their favourite topic in fifth year. The chart below shows that second and fifth year females regarded probability as their favourite subject but a higher proportion of answers from females were distributed across all different topics. Both females and males decreased in percentages in second and fifth year although probability remained their favourite subject.

![Probability as a Favourite Topic](image)

Figure 1 - Probability figures for both male and females in second and fifth year

Overall nearly 80% of second year students expected to do well at the end of year examinations. In second year a combined percentage of 19% disagreed with the statement “I expect to do well on my end-of-year exams this year in mathematics.” The figure below shows the second year results from respondents showing the large portion of the respondents thought they would “do well” in the end-of-year exams in mathematics.
In fifth year, the expectation of doing well in the end-of-year examinations decreases. Sixty percent of respondents in the survey agreed *a lot* or *a little* that they would do well at the end-of-year examinations for both males and females combined. Thirty-one percent of students did not expect to do well in the end-of-year exams. This could be due to the course complexity and therefore a lower task intrinsic attractiveness (Frenzel et al., 2010).

There is a gender difference in results when it comes to the perception that mathematics is useful in daily life. Females tended not to agree that mathematics is useful in their daily lives. Males fared out similarly in fifth year and tended not to agree that mathematics will help in their daily lives.

The figure above illustrates the female results in fifth year. No female respondent in fifth year agreed *a lot* that mathematics is helpful in their daily lives. While 30% cent of males in fifth year agreed *a lot* that mathematics is helpful in their daily lives, however, 70% disagreed that mathematics is used in their daily lives.
Career options may act as a motivating factor with mathematics being required for entry into different courses. Eighty-three percent of males in fifth year stated that they agreed *a lot or a little* with respect to needing to do well to get into the course of their choice. Only 45% of females in fifth year stated that they needed mathematics for the course of their choice. It should also be noted that only 8% of females from the sample were taking higher-level Leaving Certificate mathematics.

**DISCUSSION**

The following conclusions can be drawn from the preliminary results.

Students in both second and fifth year stated that probability is their favourite topic in mathematics. Probability has been introduced to the new curricula with extra resources for teachers including technology resources that students can both use at home and at school. A curriculum change in mathematics has seen the introduction of rolling dice and other manipulatives in the classroom as a teaching and learning tool for probability (Brosnan, 2008). In a larger sample, it would be interesting to see if students achieve higher results in probability in state examinations than in other topics as a consequence of probability being their favourite topic. Frenzel et al (2010) suggests that curriculum change could be a factor in depleted interest. In these preliminary findings, the new resources provided by Project Maths particularly for probability, seem to have increased student interest.

As noted by the State Examinations Commission (2013) only 1.6% of females taking the Leaving Certificate higher level in 2012 obtained an A1 or 90% grade or above and only 4.5% of males. In the 2012 Junior Certificate higher level 14.9% of females obtained either 85% or above and 15.9% of males did so. While these figures relate to those taking the examinations in 2012, they give an indication as to how many students achieve a high mark in the Junior and Leaving Certificate examinations. While statistics suggest that more males take higher-level mathematics than their female counterparts, it appears that there is an increasing amount of females taking higher level mathematics (State Examinations Commission, 2013). This could be due to the curricula change with an emphasis on more social communication in the mathematics classroom. Watt et al. (2012) note that females are more likely to engage in careers geared towards a social aspect than males. This could also be true for the curriculum change with more emphasis given to greater emphasis on communication in the classroom and a requirement for more verbose answers to questions in examinations (Brosnan, 2008). In this particular sample, only 8% of the females were taking higher level mathematics at Leaving Certificate level.

Eighty percent of both male and female respondents state that they expected to do well in their end of year examinations when in second year in school and this decreases by 20% in fifth year. Thirty-one percent in fifth year did not expect to do well in their end-of-year mathematics exam. Fifth year is the year prior to state examinations and engagement in an activity can be predicted by the value an individual places on doing well in that task (Wigfield & Eccles, 1992). The perceived usefulness also plays an important role in determining the task value. A large percentage of respondents did not feel as though mathematics was useful in their daily lives; however, these students put a value on mathematics for course entry.
The results show that a greater number of males than females require mathematics for their career choice. Eighty-three percent of males place a value on requiring mathematics for their career choice and only 45% of females place the same value. Placing a greater emphasis on mathematics for their career choice could contribute to the disparity in higher level grades between males and females. This could be a motivating factor in the STEM pipeline showing that more males sit higher level leaving certificate mathematics than females and a greater number of males achieve 90% or more. This is where female students contribute to the leaky pipeline whereby students drop out of mathematical courses throughout the course of their educational pursuits (Watt, Eccles and Durik, 2006). Females place less emphasis on mathematics for their course choice than their male counterparts. By placing less emphasis on this importance, females may also place less emphasis on acquiring higher marks in this subject. This may relate to students engagement in a task relating to their career development versus students engaging in tasks to better their educational development (Johnson and Sinatra, 2013).

LIMITATIONS AND FUTURE RESEARCH

All studies have limitations and this study is no exception. This study adopts a pre and post intervention methodology in a select number of classrooms along with a representative sample for the survey questionnaire.

As with all surveys, the study would benefit from behavioural analysis to provide more evidence of students’ perceptions and student interviews may be beneficial in order to gain an in-depth understanding of the students’ views and attitudes towards mathematics. Taking an in-depth discussion with students in a group, interview or focus group may lend a deeper insight into the thoughts and attitudes of students and their perceptions with regard to task values. As with all survey questionnaires there are weaknesses. The survey questions that were taken from TIMSS were established by the authors to measure what that are intended to measure thus establishing content validity (Creswell, 2009). Other aspects of the survey have been piloted and have also measured the intended determinant for the questions regarding careers.

Further research could examine more in-depth investigations into the effectiveness of the task value and future career aspirations, a longitudinal study if the students do take up mathematical careers if they say they will at primary, junior or senior level and what changed their minds since. Furthermore, future research regarding the theoretical models of conceptual change, change in self-concept and understanding behaviours after school and its impact on mathematics education. Future research could also explore the impact that technology has to play in terms of the curriculum change and if different types of technology impact on students’ perceptions and their attitudes towards using mathematics in different careers.

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REFERENCES


VIDEO KILLED THE RADIO STAR: WILL THE UNIVERSITY MATHEMATICS LECTURE BE ITS NEXT VICTIM?

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During the 2012-2013 offering of a first year Maths for Business module, I produced videos of worked examples from the course, and made them available online. My aim in doing this was to provide additional worked examples as a learning support for students who may be experiencing difficulties in the module. In this paper I will describe my engagement in the following cycle of action research: reflecting (on lecturing practice in Maths for Business); planning the introduction of the online videos in the module; making the videos and observing their use and my students’ reactions towards them; reflecting on how the online videos were used by the students; and planning for how I will use this online resource in the 2013-2014 offering of the module.

INTRODUCTION

On a morning in September 2012, I sat at my desk, opened my iPad, and for the first time created a short video and uploaded it as an online resource for my first year Maths for Business module. The video involved me working aloud through a problem while simultaneously writing out the mathematics by hand on the screen. A series of specific events had led me to this point in time. As a self-confessed techno-phobic and a sceptic of all things e-learning, this was virgin territory for me. Here I was actually creating videos and uploading them to Blackboard, the university’s Virtual Learning Environment (VLE), with ease. There certainly was a novelty factor for me in creating these videos, and, over the next two weeks, I made a few more and uploaded them. That may well have been the end of it – the novelty would have worn off, and the few videos I had made would be condemned to a dusty corner of cyberspace - had it not been for the overwhelmingly positive response from my students towards them. As the semester progressed and I made more videos available online, the number of “views” per video kept increasing and the positive comments kept coming.

I began to feel that I had stumbled upon an educational goldmine. Unlimited possibilities opened up before me and very soon I was shocked to find myself thinking that for the next offering of the module in 2013-2014, I would make videos of the entire course, put them online, and leave attendance at “live lectures” optional for students. The reason I was shocked is because one of my core beliefs as a lecturer has always been that the (live) lecture is sacrosanct, it is the rock on which the rest of the module is built. Could a video of mathematics being explained really be a substitute for attending a live lecture? While the semester was in full-swing, I had little time to reflect in any meaningful way on this question or indeed on any of the other questions relating to the use of online videos that were bubbling to the surface of my mind. I gathered data (accounts of incidents from lectures, student feedback, and log files of the videos from Blackboard) and promised myself that I would take time out at a later stage to reflect, firstly, on the data gathered from the 2012-2013 offering of
Maths for Business and, secondly, to plan for how I will use online videos to support student learning in the next offering of the module. That time is now.

Kemmis and McTaggart (2005) describe participatory action research as generally involving a spiral of self-reflective cycles of the following: planning a change; acting and observing the process and consequences of the change; reflecting on those processes and consequences; replanning; acting and observing again; reflecting again, and so on … (p.563, italics authors’ own).

In this paper I will describe my engagement in the following cycle of action research: reflecting (on lecturing practice in Maths for Business); planning the introduction of the online video resources in the module; making the videos and observing their use and my students’ reactions towards them; reflecting on how the online videos were used by the students; and planning for how I will use the online video resources in the 2013-2014 offering of the module. This paper will be structured under these broad headings to reflect the cycle of action research, while bearing in mind Kemmis and McTaggart’s note of caution that the action research process “is likely to be more fluid, open, and responsive” that the “neat” cycle of planning, acting and observing, reflecting might suggest (p. 563).

Before proceeding, I will present some background on the Maths for Business module. It is a first year module that is core for a range of Business programmes in University College Dublin (UCD). The aim of the module is to familiarise students with mathematical techniques that they are likely to meet during their undergraduate degree and to demonstrate how these techniques can be applied to problems from Business. Over 500 students take this module in Semester 1 and each student is pre-enrolled to one of two groups: those who have achieved a minimum C3 in Higher Level Leaving Certificate (LC) Mathematics or equivalent, or those who have not. The majority of the latter class will have taken Ordinary Level LC Mathematics. For the remainder of this paper we will refer to these as Group A and Group B respectively. The content and assessment is identical for both groups, however Group B have 36 hours of lectures scheduled, while Group A have 24 hours. The rationale is that the lecturer for Group B can take more time with explanations and do more examples. Each group has one tutorial per week. (The semester consists of fifteen weeks comprising twelve weeks of lectures, one study week, and two examination weeks.) For the last two years I have taught Group B, while a colleague has taught Group A. In 2012-2013, there were approximately 320 students in Group A and 200 students in Group B. Lectures for both groups are held in tiered lecture theatres. The module is assessed as follows: 20% for a midterm multiple-choice question (MCQ) examination; 20% for a maximum of seven weekly quizzes that are administered in tutorials; and, 60% for a final two-hour examination.

REFLECTING

Since September 2010, four colleagues (all mathematics lecturers in Irish third-level institutions) and I have been involved in a project that involves reflecting on our teaching practice (Breen, McCluskey, Meehan, O’Donovan, & O’Shea, 2011). To do so we have engaged with John Mason’s Discipline of Noticing (Mason, 2002). As part of the reflection process we have each kept “brief-but-vivid accounts” (p. 46) of critical incidents or moments
from our lectures. Mason describes these as brief accounts that give an “account-of” (p. 40) a situation or incident, rather than an “account-for” (p. 40) it and explains:

To account-for something is to offer interpretation, explanation, value-judgement, justification, or criticism. To give an account-of is to describe or define something in terms that others who were present (or who might have been present) can recognise (p. 41).

Mason (2002, p. 95) summarises the collection of practices involved in noticing using four headings: systematic reflection, recognising, preparing and noticing, and validating with others. Systematic reflection involves not only keeping brief-but-vivid accounts, but also examining a collection of accounts to find “threads” (p. 65) or themes that perhaps highlight, or sensitise you to, some aspects of your practice.

One of the broad themes emerging for me from my Maths for Business accounts over the last two years has been “how to deal with information gathered during ‘walkabouts’ in lectures”. In order to gauge how my students understand a technique (and to give them a bit of a break), I usually give the class a problem to work on, either by themselves or in groups. While they work, I walk up the aisles of the theatre and look at the work of as many students as I can. These walkabouts act as a thermometer for student learning as they give me a sense of whether students are mastering the material. They also provide me with an opportunity to assess whether students might have any misconceptions or gaps in their knowledge, or any predispositions to mathematics, that may be preventing learning. The following two accounts highlight some issues that may arise:

I put up a problem. In it, the total revenue function, \( TR(q) \), and the total cost function, \( TC(q) \), are given and one is first asked to find and graph the profit function \( (TR(q) - TC(q)) \) and then to interpret information from the graph. In relation to the first part I announce: “We have done something like this already so I will give you a few minutes to find the profit function yourselves and graph it.” I walk up one of the aisles. One student asks: “What are the fixed costs?” I point out that in this example you don’t have to find them. Another student asks: “Do I multiply this by \( q \) to get total revenue?” “No”, I say, “you are given total revenue here”. “What is \( q \)” another asks. One has incorrectly subtracted total cost from total revenue, indicating a problem with basic algebra. A few have found the profit function, but then multiplied it by minus one, treating it as an equation equal to zero - again a problem with basic algebra. I realise most are unable to construct the profit function. I planned to finish this problem today, but end up explaining slowly to the class how to obtain the profit function and deciding that I will need to spend the next lecture going through the rest of the problem. (22 September, 2011)

As I walk around, a few students call me over. I sit down beside one fellow. He shows me his work. I can see instantly why he is stuck – it is some basic maths that is holding him back. I want to get a pen and explain it to him there and then, but know I can’t – it would take a few minutes and there are about 150 people here today. I bet others are having the same problem. When going over the problem with the whole class I take care to spend some extra time dealing with this issue. (11 September, 2012)
While walkabouts are invaluable in helping me to identify in real-time, difficulties that students may be having, they force me to make decisions in real-time too. What do I do with the information I have uncovered? In the two accounts above, I slowly went over the solution with the whole class, taking care to emphasise the “trouble spots”. Is this the best way to proceed? I regularly feel torn about how to proceed, and often feel frustrated at not being able to adequately support students in a forum such as the large lecture theatre.

Examples, examples, examples – some of the class need more examples, done more slowly, with more time taken on the basics. Even if I had double the number of lectures, I could not do any more worked examples for fear of boring half the class. While I do make all my notes and lecture slides available on Blackboard, I have concerns about written solutions. If students are struggling, particularly with the basics, I am not certain they will benefit from looking at four or five lines of a written solution. I also fear that out of desperation they may end up memorising the steps rather than understanding why each line works, or why a particular method is used. How do I give my students more worked examples? Whatever the answer to this question was, I was certain that it was not to do more worked examples in lectures, or to provide more written worked examples.

PLANNING

In September 2012 a colleague introduced me to an iPad app that made a recording of one’s workings on the screen in addition to capturing the audio. My investigations into the app coincided with the start of the teaching term, which in turn had prompted me to reflect on my previous year’s accounts for Maths for Business, as described in the previous section. The following is an account written a day after the third lecture:

I need to give my students more worked examples, but refuse to hand out written solutions. I decide to try to make a video of me working through the second supply and demand example that I didn’t get the chance to cover yesterday. (14 September)

After trying out a few apps, I ended up choosing one called “Explain Everything” (http://www.explaineverything.com). My motivation for using this app over others is that it allows you to upload the completed video as an mp4 file to Dropbox with ease. From there it is a simple step to upload it to Blackboard where it can be viewed with Quicktime or Realplayer.

Despite the plan to make online videos available being “spur-of-the-moment”, I was clear from the start that their aim was the following: To provide additional worked examples as a learning support for students who may be experiencing difficulties in the module. I was clear that I would only make videos of worked examples, as opposed to videos of theory, concepts or derivations from the module, as I intended them to be strictly supplementary to lectures. I was also clear that my target audience was those students in my class (specifically Group B) who were experiencing significant difficulties. My intention was to spend extra time in the videos explaining each step of a worked example carefully and highlighting any basic mathematics used, in a way that I felt I could not in lectures. I referred to the videos as “In-your-own-time examples” and stored all videos in a folder of this name in Blackboard.
As the semester had already started, I did not have time to read any research literature on the use of online video as a supplement, or indeed a substitute, for lectures. I was “flying blind” and needed feedback before I spent any more of my time making videos. On the Monday of the fourth week of term, I asked my students to take a look at the videos as I wanted to find out what they thought of them. Fifty-five out of the seventy-five surveyed at the next lecture had watched them and almost all responded positively towards them. Therefore I decided that I would continue making and uploading videos of this nature for the rest of the semester.

**ACTING AND OBSERVING**

By the end of the 2012-2013 *Maths for Business* module, I had made videos of 26 worked examples available online, amounting to just under thirteen hours duration in total. When making a video of a worked example using Explain Everything, I generally upload a pdf of the problem to the app from Dropbox. Usually I talk through the problem, annotating on the screen if necessary, before working through the solution on the tablet, explaining as I write. Since I have taught this type of module several times, the explanations come easily and I usually do just one take. I do not edit. If I make a mistake, I redo the video. For this reason, and the fact that the longer the video the larger the mp4 file, I try to limit each take or clip to no longer than seven or eight minutes. If a problem requires a longer solution, I break it into two or more parts. The videos are informal in nature. I describe them as being similar to how I would go over the solution to a problem with a student if he or she were sitting beside me. I always make the video after I have covered the material in lectures. This allows me to incorporate any information gathered on walkabouts into the videos – spending extra time on areas that students had difficulties with, or highlighting common errors that students made, when doing the problem in lectures. Some of the videos have been made specifically to provide students with feedback on common errors and misconceptions uncovered when tutors corrected the weekly quizzes.

In terms of observing the process I will describe feedback received from two sources: log files of videos from Blackboard, and student responses relating to the online videos from the end-of-semester online module feedback survey. Rather than postpone my reflections on the feedback to the next section, “Discussion – Reflecting Again”, I will present some here for the sake of continuity. As mentioned above, I also conducted a brief survey in the fourth week of term to get some initial feedback on the videos, but I will not discuss that here.

**Log files from Blackboard**

It is possible to enable “Statistics Tracking” on Blackboard so that log files for each video can be accessed. The log files give information on how many times each video was viewed and by whom it was viewed. It is also possible to extract information on the number of daily and hourly views. What we cannot tell from the log files is whether in any given view, the video was watched in its entirety or not, and this should be borne in mind in what follows. In this section I will examine, and reflect on, the log files of one particular video – a worked example on the following continuously compounded interest problem:

An investor places €10,000 in a special savings account. If it takes ten years for the principal to grow to €12,500, calculate the interest rate to one decimal place.
I have a few reasons for analysing the log files of this video. Firstly, it was made available online during the third week of term. Therefore we can examine the viewing pattern from the end of September 2012, to the 12 December, 2012, when the final examination was held. Secondly, the worked solution was recorded in a single video clip of length 5 minutes and 28 seconds, and is not spread over multiple clips. Finally, the pattern of views for this video is similar to the pattern found with most of the other videos.

In total, this video was viewed 466 times. The following is a graph of the number of daily views of the video for the first month of its availability:

The midterm MCQ test was held at 7pm on 24 October, which explains why there were 28 and 56 views of the video on 23 and 24 October respectively. To understand better the rest of the graph one must be familiar with the format of tutorials. With approximately 500 students in the module, 22 tutorials are offered each week, with eight scheduled on a Tuesday, ten on a Wednesday, and the remainder on a Thursday. Each tutorial starts with a 10-minute quiz on the topic(s) from the previous week’s tutorials. The remaining forty minutes is conducted in a workshop style – students experiencing difficulties with the problem set for that week’s topic(s) are invited to remain after the quiz to receive help from the tutor. Tutors reported that not many students availed of this service. The topic of continuously compounded interest was presented in lectures during the fourth week of term (1-5 October), a problem set on the topic was assigned for consideration in tutorials in the fifth week of term (Tuesday 9 - Thursday 11 October), and the topic was the focus of the quiz in the sixth week of term (Tuesday 16 - Thursday 18 October). Although the problem set on continuously compounded interest was assigned for tutorials in the fifth week of term, it is interesting to note that views of the video more than doubled during the sixth week when the topic was the focus of the quiz. Viewing patterns for other videos are similar to the above.

Focusing more closely on “quiz week” for the topic of continuously compounded interest, the following graph represents the number of views per hour of the day from Sunday, 14 October to Saturday, 20 October:
The total number of views for this period was 94, with 77 of these occurring between 16-18 October, the three days when quizzes were held in tutorials. These 77 views were by 57 people, 27 of whom were in Group A and 30 of whom were in Group B. It is worth noting that two tutorials are scheduled for 11am, with eight scheduled for 12 noon, four for 1pm, and the remaining eight for 2pm. This graph indicates that while the video is viewed in the evening times, there is evidence of peaks in viewing on the mornings of tutorials.

Apart from one view on 13 November, the next activity for this video was in the two weeks before the final examination. There were a total of 190 views from 29 November to 12 December, with number of views peaking at 59 on the day of the final examination.

The first thing that strikes me when reflecting on the above data is just how much assessment drives learning. In theory, students should have engaged with the topic in tutorials during the fifth week, yet the data and the feedback from tutors, suggests that they were more likely to engage with the topic the week it was being assessed. In fact one has to question how many students engaged with the topic in a meaningful way during the fourth week when it was discussed in lectures. It brings it home to me that it is exactly when the student decides to engage, that the supports need to be available. I start to question my rigid stance on “attendance at lectures is compulsory”. Taking (poor) attendance at lectures as an indicator of lack of engagement, then it is clear that not every student is ready to engage at 10am on a Monday, Tuesday or Thursday morning.

Secondly, reflecting on the number of views of this video, and indeed the videos in general, I take the view that the higher the number, the better. Firstly, the fact that this particular video was viewed 466 times indicates a level of satisfaction with it as a learning support. Secondly, this number of views indicates a level of engagement with the material by a significant number of students. I equate poor attendance at lectures with lack of engagement, yet clearly, at least some students are engaging. Taking as an example 5-12 October, the week the problem set on the topic was assigned, the video was viewed 59 times by 50 people. Of course not all may have watched the video in its entirety but even if half did, then it gives me a sense of supporting students outside the live lecture. Interestingly, 35 of these were from my Group B, while 20 were from Group A.

Finally, the viewing pattern around quiz-days does raise some concerns for me. Are the videos encouraging a surface approach to learning? Are students preparing for the quiz by just watching the video, rather than completing the problem sets? Recalling that 27 of the 57
people who viewed the video during the quiz days of 16-18 October were from Group A, I wonder if this behaviour is more prevalent in this group where the students have a stronger background in mathematics. On the other hand, is it not better they watch the video than do no preparation at all?

**Online module feedback survey**

In UCD, a student can provide feedback on any module taken by completing an online module feedback survey at the end of the semester. In the survey, students are asked to “Identify up to three aspects of the module that most helped your learning”. Of the 131 responses to this question in the *Maths for Business* survey, 71 identified the online videos as having helped their learning. As lecturer I can add some extra questions to the survey and used this opportunity to ask respondents if they had watched any of “In-your-own-time examples” videos and if they had to indicate in what ways (if any) they found them useful, and in what ways (if any) they could be improved. Of the 113 responses, 103 students indicated they had watched them and most of these expanded on their answers. I do not know whether a particular respondent is from Group A or Group B.

In relation to how they found them useful, a large number of respondents commented generally that the videos were “easy to understand”, “easy to follow”, or “explained everything clearly”, and stated that they “very useful” or “very helpful”, indicating a strong level of general satisfaction with them. Analysing the responses further, one theme to emerge is that of “pace”, referring both to the (slow) pace of explanations in the videos and to how the videos allowed students to learn at a pace that suited them. Five students commented that the explanations were at a “good pace” or a “steady pace”, and nine more specifically referred positively to the slow pace of the explanations:

> Generally just really helpful because they were so detailed and were explained really slowly, especially for people not that good at maths.

Eight students liked the idea of pausing or rewinding and replaying a video, with several commenting that this meant you could learn at your own pace:

> Helped me understand topics better especially because you could pause the video and do the problem out yourself at your own pace.

Another theme that emerged was that of the “videos supporting learning started in lectures”, with at least fifteen students making comments on how the videos clarified, or added to, what had been done in lectures:

> I always understand stuff in class and then at home a few days later I forget it so it was like having a permanent teacher on your laptop.

> Very useful when I didn’t understand things fully after the lectures.

It should be noted however that in contrast to those who said the videos supported lectures, four students said they were useful when catching up on missed lectures.
Over twenty students commented on how the videos were useful when revising in general or for the weekly quizzes and/or the final examination. The viewing pattern observed for the video presented in the earlier section, supports this.

As revision before class tests they were very helpful.

I was very happy with these videos and they really came into their own when studying for the final exam because it was like going into class again and getting extra lessons!

A few students made comments on the direct relevance of the videos to the module:

…they related directly to questions that we had to do so it helped a lot.

Also the style of the videos resembled my lectures and so reinforced what I had learned before in class.

In commenting on how the videos might be improved, nine students commented that more worked examples should be provided. In contrast to the positive comments about pace made earlier, four people suggested the explanations were a little slow.

On reflection, I am pleased with the positive, general comments that indicate a high level of student satisfaction with the videos. More specifically though, I am pleased that some of the comments support the assertion that the online videos are being used as intended: To provide additional worked examples as a learning support for students who may be experiencing difficulties in the module. Like the student above who commented that the videos were useful for those “not that good at maths”, I was pleased that a few students with self-declared weak backgrounds in mathematics expressed satisfaction with the videos. The fact that several students commented positively on the good pace or slow pace of the videos, indicates that they are benefiting from the deliberately slower, more elaborated explanations. Although four students say the videos could be improved if explanations were not as long, I suspect that they may not belong to my target audience. Finally, having intended these videos as an additional support to lectures, again I am pleased that a number of students made reference to the fact that they clarified or expanded on what they had already seen in lectures. While they were not meant to be a substitute for lectures, it is inevitable that students will watch them in an attempt to make up for having missed a lecture. On reflection, I feel that the videos achieved their stated aim.

**DISCUSSION - REFLECTING AGAIN**

Having already reflected on the log files and student feedback above, in this section I will focus more on the literature and illustrate how it is helping to frame my current reflections.

As mentioned in the “Introduction”, the videos started to open my mind to the possibilities afforded by online learning. At long last I started to pay attention to conversations about online learning and listened when colleagues debated about what this technology might mean for the future of universities, both economically and academically. And while I plodded along at one end of the online spectrum with my modest, home-made videos for *Maths for Business*, I learned that at the other end of the same spectrum were MOOCs (Massive Open Online Courses). Despite the perceived threat of MOOCs to teaching and learning as we know it,
John Mitchell (2013), the Vice Provost for Online Learning at Stanford, succinctly makes a case that “online learning isn’t one size fits all” in his article of the same title. He describes the diverse ways that Stanford use online tools to enhance learning. In addition to offering MOOCs, they share licenced online courses with other universities, offer distance education programmes with the help of online technology, and use this technology to enhance the education of their students on the campus. He describes how they flip-the-class, by making lectures available via online videos, and then using class time for interactions and discussions between lecturer and students. He is clear that “the highest quality education still requires this face-to-face contact between, and among, students and faculty” (para. 6).

Specifically in mathematics, Khan Academy dominates the online space. One cannot read about Salman Khan or Khan Academy and not be aware of the idea of flipping-the-classroom (Thompson, 2011; Tucker, 2012; Webley, 2012). Some describe this as moving the teacher “from being the sage on the stage to being the guide on the side” (Webley, 2012, para. 18). One article describes how a teacher of a class of fifth graders in a school in California is doing just this (Thompson, 2011). Rather than give a lecture on a topic in class, and then assign homework on it, she assigns Khan Academy videos for her students to watch at home, allowing her to use class time to support her students in problem-solving. Khan and others, including Bill Gates, believe that this approach can also address the problem of “middle-of-the-class-mediocrity” (Thompson, 2011, para. 6), by providing a more individualised approach to learning, allowing students to learn at their own pace.

I have reflected on whether I could flip the Maths for Business classroom? Initially I thought “no” for two reasons. Perhaps rather pessimistically, I felt students would not watch the videos in advance, and then, even if I freed up lecture time for problem solving, what good is that if all 200 students show up? But on further reflection, perhaps I have gone some way towards flipping the classroom this past semester. Having students work on a problem in class may take 20-25 minutes, whereas if I presented the solution, it may take 10 minutes. A few years ago I may have presented four worked examples in class and requested no student input, now I am more likely to present one worked example, ask the students to attempt another problem, and provide solutions to the other two problems via online videos. (I am much clearer on how I might flip my classes in my Analysis module, but that is another paper!)

I was also very keen to find what (if any) research had been carried out on using online videos or supports to, very specifically, teach mathematics at third-level. A paper by Trenholm, Alcock, and Robinson (2012) was extremely helpful in this regard. They performed an extensive review of the research literature with the purpose of examining tertiary mathematics lecturing practice, and of investigating the impact of e-lectures on this practice. On examining the research in this area, they found that articles describing reflection on individual practice or articles based on surveys were more common that articles describing empirical studies. And of the 23 empirical studies they reviewed, twelve used only self-report data, while the remaining eleven used one or more of the following as data: academic records; log file analysis; and, psychometric measure. The five studies that used log file analysis were of particular interest to me, and Trenholm et al. (2012) report on three of these (Cascaval, R., Fogler, K., Abrams, G., & Durham, R., 2008; Inglis, M., Palipana, A., Trenholm, S., & Ward,
J., 2011; Joordens, S., Le, A., Grinnell, R., & Chrysostomou, S., 2009). All three studies found at least a weak association with higher e-lecture use and academic performance.

Trenholm et al. (2012) caution that the studies based on reflections, surveys, and self-report data of students’ learning experiences are “in keeping with an institutional emphasis on student satisfaction, but may bear little relationship to academic achievement” (p. 709). They note that more empirical studies are needed to examine the impact of e-learning resources on academic achievement, and warn that without these, the other studies may create “a mirage of benefit” (p. 709).

If I decide to offer students the choice either to attend live lectures or access online videos, the study by Inglis et al. (2011) is perhaps the most relevant, as they specifically examined how students used optional learning supports. They recorded each student’s attendance at live lectures, access of online lectures, and visits to the mathematics support centre, and found four categories of student: those who primarily attended live lectures, those who primarily accessed the online lectures, those who primarily use the mathematics support centre, and those who made little use of any. They found that “those students who accessed online lectures had lower attainment than those who often attended live lectures or the support centre” (p. 490).

CONCLUSIONS - PLANNING AGAIN

I am certain that online videos will stay in Maths for Business. And I will continue to make my own. Although many online resources are available, I believe that a key feature of mine is that they are tailor-made, by me, for my students. (I also believe that I can make a video in the time it would take me to source a relevant one online.)

However I still have not decided how I want to use them, but after this exploration I am clearer about my choices. As I see it I have three options: firstly, the videos can remain as an additional resource to support learning; secondly, they can be used as a substitute for lectures; or, thirdly, they can be used in a more blended way to complement or “flip” the live lecture. The first is the easiest and safest option as I have done most of the work already. If I were to go with the second option, I would continue to give the live-lecture but offer students the choice to access whichever they wanted. I would also need to bear in mind that my target audience would change – I would be targeting the entire class, not just those that are struggling. For this reason, my videos would require more careful planning and thought than I have given them in the past. Also bearing in mind the analysis of the research literature by Trenholm et al. (2012), I feel that it would be essential to carry out an empirical study that monitors attendance at lectures, access to the online videos, and attendance at tutorials with a view to examining how use of the resources impacts academic achievement. The third choice is the most challenging for me as a lecturer but perhaps offers the greatest opportunity. Done correctly, I could use the online videos in a way that would enhance my practice and better utilise those 50-minutes when I am face-to-face with the class. Whatever option I choose I will continue to engage with the Discipline of Noticing and the action research cycle of acting, observing and reflecting again.
To end the paper I return to my original question: Will video kill the live-lecture? I am certain it will not as long as the live-lecture embraces the technology and adapts. And it is said that what doesn’t kill you makes you stronger!

REFERENCES


THE REASONS WHY STUDENTS DROP FROM HIGHER-LEVEL TO ORDINARY-LEVEL MATHEMATICS AT LEAVING CERTIFICATE

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In recent years, Irish policy makers, educationalists and employers have been concerned about the performance of Irish students and graduates in mathematics (McDonagh & Quinlan, 2012). Amidst this broad concern, the low uptake of Leaving Certificate higher-level mathematics has received increased attention. Low numbers in higher-level mathematics may have a negative impact on efforts to attract research and development to Ireland which build a knowledge economy (Engineers Ireland, 2010). While these concerns have resulted in the introduction of 'Project Maths' and a bonus point incentive in 2012, the number of participants in higher-level mathematics (22%) still contrasts greatly with the average of 70% of students taking the higher-paper level in all other subjects (excluding Irish). This study investigates the reasons that influence students' decisions to drop to Leaving Certificate ordinary-level mathematics. Data obtained from 80 senior cycle post-primary students who had dropped to ordinary-level mathematics at Leaving Certificate, having completed the higher-level Junior Certificate examination indicate that difficult content (including perceived difficulty) and time consumption are the most prevalent factors impacting upon their decisions. Parents and mathematics teachers were regarded as significant sources of influence in the decision-making process.

INTRODUCTION

"The number of high-achieving maths students in Ireland lags well behind that in other developed nations, raising serious concerns about our future competitiveness" (Flynn, 2012).

Amidst this broad concern in relation to mathematical attainment, recent reports in Ireland have highlighted the significantly low participation rates in Leaving Certificate higher-level mathematics (DES, 2010; McDonagh & Quinlan, 2012). More importantly, there has been a considerable decline in the number of Junior Certificate higher-level mathematics candidates who progress to completing higher-level mathematics at Leaving Certificate two or three years later. This study seeks to provide an insight into the reasons that influence students' decisions to drop to Leaving Certificate ordinary-level mathematics.

BACKGROUND - MOTIVATION OF STUDY

For a long time, there has been much concern about the performance of Irish students and graduates in mathematics among policy makers, educationalists and employers (McDonagh & Quinlan, 2012). In OECD PISA mathematics rankings, the performance of Irish 15-year-olds has dropped from 16th to 26th position in the three years between 2006 and 2009. Worryingly, the mean mathematics score of Irish students now lies considerably below the OECD average (Perkins, Cosgrove, Moran & Shiel, 2012). Amid these concerns, increased emphasis has been placed on both the low uptake of higher-level Leaving Certificate
mathematics and the high failure rates at ordinary level. In recent years, these concerns have resulted in the introduction of a new revolutionary Project Maths curriculum and a bonus points incentive for higher-level mathematics students, with the aim of enhancing participation and performance in Leaving Certificate mathematics. However, statistics reveal that only 22% of Leaving Certificate mathematics students still sat the higher-level paper in 2012, a figure which contrasts unfavourably with the average of 70% of students who took the higher-level paper in all other subjects, excluding Irish (McDonagh & Quinlan, 2012). Furthermore the high attrition rate from higher to ordinary-level mathematics is highlighted in a report by Smyth, Banks and Calvert (2011). Their results found that 66% of students in the sample who had taken Junior Certificate higher-level mathematics subsequently sat the ordinary-level paper at Leaving Certificate.

While concerns about the low levels of participation in higher-level Leaving Certificate mathematics have received widespread coverage, the reasons behind students' decisions to drop to ordinary level from higher level have received little attention. Of the previous research which has been carried out in Ireland, it was found that key reasons included prior attainment at Junior Certificate (Smyth et al., 2011), choosing subjects in a way that will maximise points (Malone & McCullagh, 2011), avoiding the risk of failing to meet matriculating requirements (DES, 2010) and the hard work and effort required (NCCA, 2005). Existing research has also established mathematics teachers and parents as the main sources of influence in the decision-making process (Smyth et al., 2011). Much of this research focuses on the views of educationalists or policy makers as to why students drop, rather than gaining an insight into students' own opinions. Thus little is known about the subjective experiences of students in relation to subject level choice in mathematics at Leaving Certificate. According to Cothran and Ennis (1999), the perspectives of students can greatly enhance the effects of educational reform as they can provide valuable insights needed into the complexities of teaching and learning. Therefore, this study, which is based on students' experiences, hopes to provide possible recommendations of how participation in Leaving Certificate higher-level mathematics might be enhanced.

Concerns about mathematics participation and performance are not solely applicable in an Irish context. Holton, Artigue and Kirchgraber (2001), for example, outline that this trend also affects other developed countries. In the same way, Tickly and Wolf (2000) suggest that there are international concerns about the mathematical skills of students emerging from second-level education. In the UK, where mathematics is not essential for matriculation, there are serious shortages in the number of people qualified in science, engineering and mathematics (Brown et al., 2008). Similarly in Australia, there are concerns over the quality and participation of young Australians in mathematics, particularly in higher-level courses at secondary schools (Chinnappan, Dinham, Herrington & Scott, 2007). However, in comparison, Kuenzi (2008) proposes that mathematical knowledge and skills, rather than declining worldwide, are merely patterned differently across the globe. For example, students in Hong Kong are amongst the highest performing in the world, with all students studying mathematics up to entry to university or the workplace.
Many studies which have taken place into students’ attitudes towards mathematics are from British and Australian students’ perspectives where, unlike Ireland, mathematics is not essential for matriculation purposes. In England, Wales and Northern Ireland, the General Certificate of Secondary Education (GCSE) examinations are sat by students of 15-16 years, similar to the Irish Junior Certificate. Subsequently, these students have the option of continuing their study of mathematics for two further years at GCE advanced level. Therefore these existing studies focus on the reasons for students not continuing their study of mathematics post GCSE (Brown et al., 2008) or the reason why students may drop out during the A-level course (Noyes and Sealey, 2011). Other relevant studies include profiling the reasons which lead to student disengagement in mathematics (Nardi & Steward, 2003) and evaluations of student participation in higher or advanced-level mathematics in Australia (Chinnappan et al., 2007), Wales (Jones, 2008) and England (Matthews and Pepper, 2005, 2007).

**RESEARCH METHODOLOGY AND SAMPLE PROFILE**

The research questions which this study set out to answer are:

- What are the most significant factors influencing a student's decision to drop to Leaving Certificate ordinary-level mathematics from higher level?
- What significant people impact upon students' decisions to drop to Leaving Certificate ordinary-level mathematics from higher level?

A questionnaire (Appendix A) was chosen as the principal data collection implement. For the two principal sections of the questionnaire, the people and the factors which influence a student's decision, Likert rating scales were used. An extensive review of the literature cited in the preceding section established the questions that were included in the questionnaire. Similar international studies were also explored in depth and key factors which related to students' decisions to drop higher-level mathematics were identified. For example, a principal factor influencing English students' decision to drop mathematics at advanced level was the perceived difficulty of the subject (Matthews and Pepper, 2005, 2007; Brown et al., 2008). Given that failure in English, Irish or mathematics prevents a student from entry to many third-level courses, Malone and McCullagh (2011) claim that students drop to ordinary-level mathematics because of the risks involved at higher level while effects of prior attainment at Junior Certificate on a student's decision have been highlighted by Smyth et al. (2011). The questionnaire was piloted with four students who had dropped to Leaving Certificate ordinary-level mathematics from higher level. Using feedback from this pre-test, minor changes were made to the initial questionnaire to improve its clarity.

A total of 80 students from three post-primary schools who were in either Transition Year, 5th or 6th year formed the research sample for this study. The sample was representative of all students who did Junior Certificate higher level and then subsequently dropped to ordinary level at Leaving Certificate. Sixty-two were female (78.5%). Of this cohort, 25 (31.6%) were 5th year students and 55(68.4%) were 6th year students. Eight of the participants (10%) received an A grade in Junior Certificate higher-level mathematics, 20 students (25%) received a B grade, 32 students (40%) achieved a C grade, 19 students (23.8%) attained a D
while the remaining student received an E grade. Table 1 illustrates the stage at which students dropped back to ordinary-level mathematics.

**Table 1: The stage at which students dropped to ordinary level**

<table>
<thead>
<tr>
<th>Stage</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>September of 5th year</td>
<td>43</td>
<td>53.8</td>
</tr>
<tr>
<td>Between September and Christmas of 5th year</td>
<td>13</td>
<td>16.3</td>
</tr>
<tr>
<td>Between Christmas and Summer of 5th year</td>
<td>4</td>
<td>5.0</td>
</tr>
<tr>
<td>September of 6th year</td>
<td>11</td>
<td>13.8</td>
</tr>
<tr>
<td>Between September and Christmas of 6th year</td>
<td>5</td>
<td>6.3</td>
</tr>
<tr>
<td>After the pre examination</td>
<td>4</td>
<td>5.0</td>
</tr>
<tr>
<td>Total</td>
<td>80</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Quantitative data were analysed using SPSS. Given that most of data the obtained from the questionnaire were categorical in nature, descriptive statistics were primarily used in the early stages of data analysis including frequency tables, cross-tabulation and suitable graphs. Non-parametric statistics were used to compare specific groups within the sample; Junior Certificate grade, school attended and the stage at which the student dropped to ordinary level. A Chi Square test for independence was used to investigate the relationship between categorical variables. Based on a cross-tabulation table, this test compares the observed frequencies occurring in each of the categories with the expected frequencies if there was no association between variables (Pallant, 2007). Relationships between factors were also explored using a Spearman rank-order correlation. This method was used as it is suitable for ordinal or ranked data, such as responses from the Likert scale utilised in the questionnaire (Pallant, 2007). Qualitative data from the open-ended responses was analysed using a 'general inductive approach' as described by Thomas (2006). The main purpose of this approach lies in the development of key concepts, themes or models through interpretations made from the raw data by the researcher.

**RESULTS OF STUDY**

The authors discuss the results obtained from the perspective of the research questions (RQ) previously identified.

**RQ1: What are the most significant factors influencing students’ decisions to drop to Leaving Certificate ordinary-level mathematics?**

When dropping to Leaving Certificate ordinary-level mathematics, there were two critical and interrelated factors which dominated students' decisions: difficulty of content and time consumption. Together these themes were referred to in 76% of the explanations given by students as to why they were happy with their decision to drop to ordinary level. These are factors which have been previously acknowledged both in Ireland and internationally (NCCA, 2005, Matthews & Pepper, 2005; Brown et al., 2008; Noyes & Sealey, 2011).
Difficulty of content

Over half of the participants (51.2%) ranked difficult content as their main reason for dropping from Leaving Certificate higher-level mathematics. This was the most prevalent reason given by students who achieved Junior Certificate grades B, C, and D. A relationship was established between dropping because of difficult content and the Junior Certificate grade attained, with difficult content having a greater influence on those students of lower Junior Certificate attainment. Nevertheless, 82% of prior attaining A or B students reported that this factor had a large or massive impact on their decision. Difficulty in higher-level mathematics is therefore an influencing factor for high-attaining as well as low-attaining students, a finding which reflects previous research (Matthews & Pepper, 2005; Brown et al., 2008).

For many students, dropping to ordinary level meant a much easier course and increased understanding of the content. The notion of ‘difficulty’ or 'struggling' with higher-level mathematics is referred to in a number of open-ended responses.

“Because the content is easier and higher level was too difficult.” (M-36-D) [1]

“Yes I am as I feel the ordinary level is more suited to me rather than struggling with the honours.” (F-34-D)

Interestingly, 93% of students who dropped to ordinary-level mathematics in September of 5th year reported that the difficult content significantly influenced their decision. Given that these students had not begun the higher-level course, it could be considered that it was the perceived difficulty which impacted upon their decision rather than their actual struggle with higher-level mathematics. Brown et al. (2008) propose that this perceived difficulty is often based on prior experience as learners or on information received from external sources. Likewise in this study, a number of students refer to their experiences at Junior Certificate as a contributing factor to their perceptions of difficulty:

“I found honours maths quite tough at J.C. level. Therefore I did not want to carry on with honours level at leaving cert.” (F-53-B)

“I struggled with the Junior Cert higher level” (F-39-D)

In a study of Year 9 (13/14 years) students' attitudes towards mathematics, Nardi and Steward (2003) found that mathematics was perceived by students as a demanding subject in which only an elite group of exceptionally talented individuals could succeed. Worryingly, eight students in this study share a similar perspective in relation to Leaving Certificate higher-level mathematics.

“I wouldn't be able for honours.” (F-67-D)

“Because I'm not good at maths and got a D in Junior Cert.” (F-23-D)

What these responses suggest is that the students view mathematical ability as innate. Similar to the findings of Brown et al. (2008), they perceive that there are 'fixed' boundaries in mathematics for each individual student, between what is possible and what is not. For the above students, it appears that they have dropped to ordinary-level mathematics as they feel the content of higher level extends beyond the boundaries of their mathematical ability.
Interestingly, the DES (2010) speculates that it is acceptable in Irish society for students to say they are 'no good at maths'. In line with this, it would seem that the students' perceptions about 'fixed' mathematical ability provide them with an acceptable rationale for dropping from higher-level mathematics.

**Time consumption**

Time consumption was ranked by 16% of participants as the main factor influencing their decision to drop to ordinary level. For students who had received an A grade at Junior Certificate, it was the time which higher-level mathematics consumed rather than the actual difficulty of the content which had the greatest impact on their decision to drop to ordinary level. A significant negative correlation (p<.05) between the influence of time consumption and Junior Certificate grade revealed that high scores given for this factor were associated with students of higher Junior Certificate attainment. For all other grades (except E), taking up too much time was still the second most prevalent reason given for dropping. This would support the view of the NCCA (2005) that much effort and hard work are thought to be required for higher-level Leaving Certificate mathematics.

Previous research in England acknowledges students' concerns in relation to the adverse effects which time spent at advanced level mathematics has on their other subjects (Matthews & Pepper, 2005; Noyes & Sealey 2011). Similarly, almost one third of students in this study revealed that dropping to ordinary-level mathematics meant that they now have more time to focus on their other subjects or as one student put it, they would not “fall down in other subjects.” (F-21-D)

As well as falling down in other subjects, some students perceived that the amount of time spent at higher-level mathematics was worthless and was not reflected in the grades achieved. Students were happier to give more time to their other Leaving Certificate subjects, given their perception that this effort would be rewarded with better grades. This supports the findings of international research carried out by Noyes and Sealy (2011) who conclude that students were prepared to take less enjoyable subjects in order to ensure success in examinations.

“It's (higher-level mathematics) not worth the time and effort.” (F-19-A)

“Basically the efforts I put into it wouldn't have produced a grade relevant to the amount of work done for it.” (F-48-A)

The above comments could be compared in some ways to Malone and McCullagh's (2011) perception that students choose Leaving Certificate subjects in a way that will give a greater reward for the effort they put in. Exploring this issue further, it is clear that the CAO points system, which governs entry to most third-level institutions in Ireland, contributes to students' perception that the effort applied at higher-level mathematics is 'worthless'. In fact the DES (2010) perceived this 'points race' as an influential factor on students’ decisions to drop Leaving Certificate higher-level mathematics. While a bonus points incentive was introduced in 2012 in an attempt to increase the uptake of higher-level mathematics, the findings of this study would suggest that some students seem to perceive that the mapping between the effort
applied and the points awarded in Leaving Certificate higher-level mathematics is still mismatched.

“I have more time to put more effort into all my other subjects and get more points.” (M-51-D)

“I feel I have a lot more time to spend at my other subjects in order to maximise my points.” (F-5-A)

The effects of the perception that it is easier to get better grades in other subjects can be seen elsewhere in the results. A significant percentage of students (61.3%) stated that “easier to get an A1 in other subjects to maximise points” had a large or massive influence on their decision to drop to ordinary-level mathematics. In fact, 11.3% of the participants ranked this at the greatest factor influencing their decision to drop to ordinary level. Again, this reason had a greater impact on the decision of those with higher Junior Certificate grades than their lower-attaining counterparts.

“Can get better results in other subjects so happier to spend more time on those.” (F-58-A)

This student received an A grade in Junior Certificate higher-level mathematics but dropped to ordinary level at Leaving Certificate so that she could spend more time on her other subjects in which she felt she could get better grades. It seems that students are playing the system with sophistication in the hope of maximising their Leaving Certificate points and thus gaining entry to the third level degree of their choice. Given that most students study seven or more subjects at Leaving Certificate, the Expert Group on Future Skills Needs (2008) speculate that mathematics is often perceived by students as a spare subject to be taken at ordinary level, a perspective which is shared by five students in this study. For example, on student stated,

“I'm already doing 7 honours so I don't really need it.” (F-48-A)

Fear of failing

According to Noyes and Sealey (2011), students in England drop mathematics at advanced level in order to minimise the risk of failing to meet entry requirements for third-level education. Given that failing English, Irish or mathematics in Ireland will prevent a student from entering many third level courses, Malone and McCullagh (2011) claim that students drop to ordinary-level mathematics because of the risks involved at higher level. A total of 55 students (68.7%) reported that fear of failure had a large or massive influence on their decision to drop to ordinary level. Furthermore, fear of failing was ranked by 13.8% of students as the greatest factor influencing their decision to drop, making it the third most prevalent reason.

The consequence of failing higher-level mathematics was highlighted in a number of the students' responses.

“If I fail higher maths, I wouldn't be able to go to college.” (M-42-B)

Subsequently, this risk can place extreme stress and pressure on students to pass higher-level mathematics. For most, this stress is relieved following the drop to ordinary level.
“I would be under major pressure to pass Leaving Cert higher.”  (F-39-D)

“I'm more comfortable doing the work and don't have the fear of failing.”  (F-51-B)

In line with the perceptions of Malone and McCullagh (2011), students are more comfortable dropping to ordinary level rather than running the risk of failing higher level and thus not meeting the entry requirements for their choice of third-level course.

**Prior attainment at Junior Certificate**

In England, Matthews and Pepper (2007) deduce that participation in advanced mathematics is skewed towards students with higher GCSE grades. The results of this study reveal that only three participants (3.8%) ranked the grade they received at Junior Certificate as the greatest influencing factor on their decision to drop to ordinary level. Not surprisingly, the three students had achieved either a C or D grade in their Junior Certificate examination. In fact, 73.7% of students who had received a D grade revealed that this had a large impact on their decision to drop. A significant positive correlation (p<0.05) established that students of lower Junior Certificate grades were increasingly influenced by their prior attainment. So, while prior attainment may generally not appear to have a dominant influence on students' decisions to drop to ordinary level, it had a significant impact on the decision of C and D-grade students,. This is further supported in the open-ended responses of five students, e.g.,

“Because I'm not good at maths and got a D in Junior Cert.”  (F-23-D)

The findings echo those of Smyth et al. (2011) who report that those who perform poorly in Junior Certificate mathematics are often discouraged from participating in Leaving Certificate higher-level mathematics.

The notion that prior attainment significantly influences the decision to drop to ordinary level is supported elsewhere in the results. A significant negative correlation (p<0.05) was established between the grade attained at Junior Certificate and the stage at which the student dropped to ordinary level at Leaving Certificate, with students of low attainment dropping at an earlier stage than their higher attaining counterparts.

**RQ2: What significant people impact upon students’ decisions to drop to Leaving Certificate ordinary-level mathematics?**

Measuring the extent of external influence on a person's choices can be a complex one, given that they are not always conscious of the degree of such influences. While students may feel that they alone are responsible for all their decisions, Ball et al. (2002) maintain that choice is in fact a complex sociological construct. According to Noyes and Sealey (2011), students' decisions will have been affected, consciously or subconsciously by parents, friends and teachers. Based on the results of this study, parents and mathematics teachers were identified as key sources of influence in the decision to drop to ordinary-level mathematics, thus supporting the findings of earlier research.

Parents were regarded as having the greatest influence on the decision with 70% of participants reporting that their parents had a significant impact on the decision-making
process. These findings could add some weight to McDonagh and Quinlan's (2012) claim that the decision to drop to ordinary level at Leaving Certificate is made by high attaining students who are influenced and advised by their parents. Although the results produced no suggestions as to how parents influence their child's decision, Noyes and Sealey (2011) report that parental influence is often in the form of guidance, advice and reassurance. Exploring the issue further, they speculate that parental influence in the decision is far more complex than the general advice of which students are aware. Given that a student's values and beliefs will have been shaped by their home environment, it could be considered that attitudes towards mathematics (which may influence the decision to drop higher-level mathematics) have been constructed before a student begins his or her Leaving Certificate studies. Thus students may not recognise the full extent of the influence which parents exert on their decision to drop to ordinary-level Leaving Certificate mathematics.

Almost half of students in the sample indicated that their mathematics teachers had a significant influence on the decision to drop to ordinary level. While Matthews and Pepper (2005) found that teachers only encouraged students who they felt were capable of studying advanced mathematics, particularly those of higher prior attainment, the findings of this study revealed no significant correlation between the extent of teacher influence reported by students and the Junior Certificate grade attained. Suggestions of how teachers impacted upon the students' decisions are revealed in a number of open-ended responses provided by the participants.

“… had bad honours teacher” (F-52-A)

“The teacher explains better and spends more time making sure people get it.” (F-77-B)

Interestingly these high-attaining Junior Certificate students referred to the standard of teaching at higher level rather than discouragement from the teacher. However it would be inaccurate to assume that these open-ended responses reflect internally sub-standard teaching, given that they do not take into account the experiences of those students who remained at higher level. It is also important to consider that these are students' reported perceptions rather than objective measures of good mathematics teaching. Nevertheless, further research in this area may provide valuable and more specific insights into the complexities of mathematics teaching and learning at higher level.

The findings of this study reveal that friends and siblings did not impact upon students' decisions to the same extent as parents or mathematics teachers. Only 12 students (15.1%) revealed that siblings had a large influence on their decision, while 11 students (13.8%) reported that their friends largely impacted upon their decision. Whether or not these results accurately demonstrate the extent of sibling influence may be questionable, given that they do not take into account the number of participants who do not have siblings. According to Brown et al (2008), students' perceptions of difficulty are informed by older students who had or were presently taking the course, including older siblings. To gain a more in-depth insight into the influence of siblings, it would be necessary to identify the students in the study who have older siblings who have previously completed the Leaving Certificate.
CONCLUSION AND RECOMMENDATIONS

Based on the findings of this investigation, the researchers pose possible recommendations for policy makers, teachers and parents to consider in relation to Leaving Certificate higher-level mathematics:

• As high attaining mathematics students in this study perceived that higher-level mathematics is too time consuming and had an adverse affect on their other subjects, the workload could be split over two separate modules (the equivalent of two Leaving Certificate subjects). Given the vital importance of mathematics in society, this may encourage more students to take the subject at Leaving Certificate and thus increase levels of mathematical attainment in Ireland.

• The results from this study would support the idea of a repeat mechanism, previously referred to in an article by Bielenberg (2011). In order to relieve the stress and pressure associated with failing Leaving Certificate mathematics, students who fail higher-level mathematics in June would have the opportunity to take the ordinary-level paper in September of that year and thus meet the entry requirements for third-level education.

• One eighth of students in this study are not satisfied with the standard of teaching at higher-level Leaving Certificate mathematics. In order to gain a greater insight into this 'poor teacher' issue, it would be necessary to carry out further research into students' experiences of teaching and learning at Leaving Certificate higher-level mathematics.

• The results of this study support similar international research (Brown et al., 2008; Noyes & Sealey, 2011) in highlighting parental influence on students' decisions, although results provide no suggestions as to the nature of this influence. This is another area that could be further explored in order to determine how parents influence the decision of their children with regard to subject level choice.

NOTES

1. Students' open-ended responses are identified by gender, the student's identification number and the grade attained at Junior Certificate respectively.

REFERENCES


**APPENDIX A- QUESTIONNAIRE (ADAPT ED)**

1. Are you male or female?
2. What year are you currently in?
3. What grade did you receive in Junior Certificate Higher Level mathematics?
4. At what stage did you drop from Higher level to Ordinary level?
5. On a scale of 1 to 5 rate the influence the following people had in your decision to drop to Ordinary level. (1 being no influence and 5 being massive influence): Maths Teacher, Parents, Siblings, School Principal, Career Guidance Teacher
6. On a scale of 1 to 5, rate the influence the following reasons played in your decision to drop to Ordinary level: Difficult Content, Taking up too much time, Fear of failing, Grade received at Junior Cert, Do not need maths for 3rd level, Taking up 8th subject, Easier to get A1 in other subject to maximise points.
   Other reason? __________________________________________________________
7. Place these reasons in order from 1 to 7 by placing the number in the appropriate box. (1 being the greatest influence and 7 being the least influence): Difficult content, Taking up too much time, Fear of failing, The grade I received at Junior Certificate, Not needing maths for 3rd level degree, Taking up an 8th subject, Easier to get A1 in other subject to maximise points
8. What option subjects are you studying for the Leaving Certificate?
9. Are you happy with your decision to drop to ordinary level?
   Please explain: ___________________________________________________________________
A MATHEMATICAL DISCOURSE COMMUNITY IN AN IRISH PRIMARY CLASSROOM

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St Patrick’s College, Drumcondra

In a mathematical discourse community, all members are positioned as mathematical authorities capable of evaluating thinking and developing and explaining their own ideas. Despite a curriculum focus on discussion-based, constructivist approaches (DES, 1999), it is likely that mathematical discourse communities are scarce in Irish classrooms which often follow textbook-based traditional formats. This paper presents details of a teaching experiment which aimed to create a mathematical discourse community in an Irish primary classroom using the Math Talk Learning Community Framework (Hufferd-Ackles, Fuson & Sherin, 2004) as a design for learning. In conjunction with Boaler and Brodie’s (2004) teacher question categories, this framework was also used to analyse teaching experiment lessons. Details and analysis of one lesson will be presented here.

INTRODUCTION

A mathematical discourse community is one which engages in discourse of a mathematical nature. The power of this deceptively simple concept is obvious when considered in the context of school mathematics. It seems that opportunities for classroom discourse focused on mathematical thinking may be somewhat limited (e.g., NicMhuirí, 2011a) and despite calls for a full implementation of the “constructivist, discussion-based approaches” outlined by the curriculum (Shiel et al., 2006, p. 155), mathematics lessons in Ireland have been found to revolve around textbook activities (Eivers et al., 2010) and to use traditional approaches of teacher exposition followed by pupil practice (Lyons et al., 2003). In traditional approaches, students rarely have opportunities to argue positions, justify these arguments and engage in refining their ideas through discussion (Boaler, 2009). It is suggested that it is this type of discourse, sometimes involving dialectical conversations (Bereiter, 1994), that differentiates mathematics from other forms of discourse. Bereiter (1994) links such discourse to an underlying community commitment to the improvement of ideas and argues that science can be understood as progressive discourse. Sfard maintains that “becoming a participant in a mathematical discourse is tantamount to learning to think in a mathematical way” (2001, p. 5, original italics). This raises questions about discontinuities between school mathematics and mathematics at domain level and how classroom discourse may facilitate or constrain students’ mathematical thinking.

Researchers who have trialled non-traditional, discussion-orientated approaches have documented both the opportunities for the development of student thinking and the challenges for teachers (e.g., Lampert, 1990). Creating opportunities for students to share their thinking may involve refraining from explicit teacher explanations at times to allow students to take on the role of explaining mathematical ideas. The difficulties for a teacher in facilitating such discourse are many including negotiating to what extent he/she is willing to air student misconceptions uncorrected and when it is necessary or desirable to “tell” or take
responsibility for explaining mathematics (Dooley, 2011). My research aimed to investigate the teacher and student experience in a mathematical discourse community. In a teaching experiment carried out in my own classroom, I attempted to facilitate a discourse community where students had opportunities to discuss their mathematical thinking and refine it in the course of classroom discussion. This approach stands in contrast with traditional approaches and raised challenges for my students and for me as teacher-researcher. In this paper I will present analysis of one lesson that occurred during the teaching experiment.

METHODODOLOGY

Perspective

I adopted the sociocultural perspective described as cultural, discursive psychology by Lerman (2001, p. 87). Lerman describes this research approach as “a particular focusing of a lens, as a gaze which is as much aware of what is not being looked at, as of what is” (2001, p. 90). This point is pertinent as the complexity of describing and analysing the discourse of a classroom community through time necessitates a certain ‘focussing of the lens’ that foregrounds some issues while acknowledging contextual detail.

Learning as transformation of participation in social practices

The notion of learning as evolving participation in social practice has much support in the literature (e.g. Rogoff et al., 1996). Learning mathematics can be understood as becoming a participant in the discourse of mathematics (Sfard, 2001) and mathematics can be understood as progressive discourse (Bereiter, 1994). In this way, it is possible to conceive of learning mathematics as becoming a participant in progressive discourse. Combining these notions, the ‘transformation of practices’ involved in learning mathematics in the discourse community can be thought of as students’ increased participation in authentic mathematical practices which facilitate progressive discourse. In this sense, my examination of learning within the discourse community was not focussed on the specific content of a lesson or a sequence of lessons. Instead my examination of the student experience in a discourse community had a more holistic focus on what might be learned from the discourse community approach.

Teaching experiment

Dooley (2011) details how design research has been applied to the classroom in the form of the “classroom design experiment” (Cobb, Gresalfi & Hodge, 2009) which has its roots in the teaching experiment approach to research. Steffe et al. note that teaching experiments provide a means of crossing “the chasm between the practice of research and the practice of teaching” (2000, p. 270). In my own research, the aim of the experiment was both the facilitation of a discourse community and the study of this instructional design.

Design for learning

The Math Talk Learning Community (MTLC) framework is an example of research that has successfully explored a discourse community in an urban Latino third-grade classroom in America (Hufferd-Ackles, et al., 2004). A year-long study followed a class teacher as she successfully changed from traditional teaching to more reform-orientated practice. Tracking the progress of the classroom community led to the development of the MTLC framework
which describes developmental trajectories for both teacher and student actions across the areas of questioning, explaining mathematical thinking (EMT), source of mathematical ideas (SMI), and responsibility for learning (RFL). These trajectories track changes in teacher and student actions as the classroom community moved from operating as a traditional community to a discourse community. Table 1 shows an overview of the levels of the MTLC framework. There are four levels of teacher and student actions ranging from level 0 to level 3. Level 0 describes a traditional teacher-directed classroom and level 3 describes a mathematical discourse community. The descriptions of teacher and student actions at levels 2 and 3 of the framework are consistent with a discourse community (see table 2) and although I could not specify student actions, by attempting to follow teacher actions at the higher levels of the MTLC framework, I hoped to facilitate and develop a discourse community. The mathematical topics of fractions, decimals and percentages were chosen as focus topics as these have been identified as problematic for teachers and students (Eivers et al., 2010).

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 0</td>
<td>Traditional teacher-directed classroom with brief answer responses from students.</td>
</tr>
<tr>
<td>Level 1</td>
<td>Teacher beginning to pursue student mathematical thinking. Teacher plays central role in the math-talk community.</td>
</tr>
<tr>
<td>Level 2</td>
<td>Teacher modelling and helping students build new roles. Some co-teaching and co-learning begins as student-to-student talk increases.</td>
</tr>
<tr>
<td>Level 3</td>
<td>Teacher as co-teacher and co-learner. Teacher monitors all that occurs, still fully engaged. Teacher is ready to assist, but now in more peripheral role.</td>
</tr>
</tbody>
</table>

**Table 1: Levels of the MTLC framework (Hufferd-Ackles et al., 2004, p. 88-90).**

<table>
<thead>
<tr>
<th>Level 2</th>
<th>Questioning</th>
<th>Explaining Mathematical Thinking (EMT)</th>
<th>Source of Mathematical Ideas (SMI)</th>
<th>Responsibility For Learning (RFL)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher Actions</td>
<td>Teacher asks probing and open questions. She facilitates student-to-student talk.</td>
<td>Teacher elicits multiple strategies and supports detailed descriptions of student thinking.</td>
<td>Teacher builds on explanations by asking students to compare and contrast. She uses errors as opportunities for learning.</td>
<td>Teacher encourages student responsibility for understanding the ideas of others. She asks whether they agree or disagree and why.</td>
</tr>
<tr>
<td>Student Actions</td>
<td>Students ask questions of one-another’s work, often at the prompting of the teacher.</td>
<td>Students give information as it is probed by the teacher. They begin to stake a position.</td>
<td>Students exhibit confidence about their ideas and share their thinking even if this is different from their peers. Student ideas sometimes guide the direction of the lesson.</td>
<td>When the teacher requests, students explain other students’ ideas in their own words. Helping involves clarifying other students’ ideas for themselves and others.</td>
</tr>
</tbody>
</table>
### Table 2: Teacher and student actions at levels 2 and 3 of the MTLC framework.

<table>
<thead>
<tr>
<th>Level 3</th>
<th>Questioning</th>
<th>EMT</th>
<th>SMI</th>
<th>RFL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Teacher Actions</strong></td>
<td>Teacher expects students to ask one another questions about their work. The teacher’s questions may still guide the discourse.</td>
<td>Teacher follows student descriptions of their thinking closely. She stimulates students to think more clearly about strategies.</td>
<td>Teacher allows students interrupt her explanations and students explain and ‘own’ new strategies. Student ideas are used for lessons or mini-extensions.</td>
<td>The teacher expects students to be responsible for co-evaluation of everyone’s work and thinking. She supports students as they help one another sort out misconceptions. She helps and follows up when needed.</td>
</tr>
<tr>
<td><strong>Student Actions</strong></td>
<td>Student-to-student talk is student-initiated. Many questions require justification. Students repeat their own or other’s questions until they are satisfied with answers.</td>
<td>Students justify their answers with little prompting from the teacher. Students realise that they will be asked questions from other students when they finish. Other students support with active listening.</td>
<td>Students interject their ideas as the teacher or other students are teaching, confident that their ideas are valued. Students spontaneously compare and contrast and build on ideas. Student ideas form part of the content of many math lessons.</td>
<td>Students listen to understand then initiate clarifying other students’ work for themselves and for others during whole-class discussions as well as in small group and pair work. Students assist each other in understanding and correcting errors.</td>
</tr>
</tbody>
</table>

**METHODS**

The teaching experiment was carried out in a disadvantaged boys’ school with 17 Fifth class students (10 -11 years). Lessons were recorded using a digital voice recorder based on their perceived potential for interesting classroom discourse. ‘Interesting’ in this case should be understood to mean relevant to the research because of predicted participation patterns of students in whole class discourse. Some of the limitations of audio recording were mitigated by collecting other sources of data. The classroom was equipped with an interactive whiteboard which runs Smart Notebook software. This allows a digital record of board work to be saved. These files were useful when attempting to understand student contributions as many of the representations that students refer to have been recorded. These files also contain samples of students’ work and students’ own mathematical representations as it was common practice to have students present their work on the board. In total, 31 recordings were collected on different days. The ethical issues of teacher-research in one’s own classroom are nontrivial but the mandated guidelines of St. Patrick’s College were followed at all stages.
Data analysis

A descriptive synopsis was written for all recordings and 14 recordings were transcribed with a view to being representative across mathematics topics and over time. Five of the transcribed lessons were analysed in depth (NicMhuirí, 2012) and some have been discussed elsewhere (NicMhuirí, 2011b). Not all teaching experiment lessons could be considered successful in terms of the lesson objectives or the overall goals of the experiment and the lesson presented here is not intended to be a model of exemplary practice. Instead it was chosen for the insights it may provide into what it means to learn mathematics in a discourse community and what it means to attempt to facilitate such a community.

MTLC framework analysis

Examination of the nature of student experience at group level was conducted using the MTLC framework (Hufferd-Ackles et al., 2004). As mentioned above, the MTLC framework has four components; questioning, explaining mathematical thinking (EMT), source of mathematical ideas (SMI) and responsibility for learning (RFL).

Questioning

The feature that differentiates level 1 teacher questioning from level 0, is a focus on student thinking rather than answers. At both of these levels, it is unlikely that students will ask questions. At level 2, the teacher may encourage students to ask questions of each other. The level 3 descriptors for questioning have implications for RFL and imply that students are expected to listen to and question the explanations of others. Because different types of questions may provoke different responses, it was considered worthwhile to investigate the nature of questions and how these linked with other components of the framework. For example, the type of questions posed by a teacher may determine the manner in which a student explains his or her mathematical thinking, e.g. with a single word answer or with a fuller explanation and possible justification. Thus teacher questions have implications for the EMT and SMI components. Student questions can also be linked to the other components of the framework and have particular relevance to considerations of RFL.

Teacher questions were counted and coded using Boaler and Brodie’s (2004) teacher question categories. Category descriptions and examples are given in table 3. Repeated questions were only counted once. Type 1 questions aimed at gathering information or leading students through a method tend to be associated with traditional teaching. While they are also commonly used in reform classes, teachers in reform classes are more likely to use a larger range of questions (Boaler & Brodie, 2004). Boaler and Brodie’s question types can also be linked to teacher and student actions in the MTLC framework. For example, a teacher is more likely to use type 4 questions probing student thinking at levels 1 and above than at level 0. Similarly, one might expect a higher proportion of type 5 questions aimed at generating discussion in a discourse community. Student questions were also counted and coded. Off topic student questions were not considered. The remaining student questions were coded as “questions seeking clarification about mathematics being discussed” or “questions seeking organisational clarification.”
Explaining mathematical thinking (EMT)

As a classroom community moves through these levels of the MTLC framework, a certain amount of responsibility for learning and mathematical authority is devolved to students and while the teacher may tell answers and provide explanations at level 0, students share this responsibility at higher levels. This is linked with the idea of students, rather than teacher or textbook alone, as source of mathematical ideas. The descriptors at level 3 imply classroom norms where students are expected to explain and justify their reasoning.

Source of mathematical ideas (SMI)

As with the EMT component, the progression through the levels for SMI, reflects the devolution of responsibility for mathematical ideas from teacher to students. The level 3 descriptors imply a norm of comparing solutions to check for mathematical similarity or difference. They also have implications for patterns of interactions within classroom discourse and students may interrupt the teacher to interject their own ideas. Explicit reference is made to teacher exploitation of student errors as opportunities for learning.

Responsibility for learning (RFL)

Progression through the MTLC levels is characterised by the devolution of responsibility for evaluation of mathematical thinking from teacher to students. At level 0 and level 1 the teacher is the ultimate arbiter of mathematical truth and is the sole mathematical authority of the community. However at levels 2 and 3 this authority is shared with students, and students are expected to be responsible for evaluating their own work and the work of others. Students demonstrate RFL by asking clarifying questions when they do not understand. The descriptor for level 3 implies classroom norms where students are expected to explain their mathematical thinking in a manner that can be understood by others. The other students are expected to actively listen to explanations and ask clarifying questions or state why they disagree.

RESULTS

The analysis of one lesson will be presented here. It was carried out in Spring and students had been engaged in teaching-experiment lessons from the start of the school year. It was the fifth in a series of lessons on percentages and the recording was about 25 minutes long.

Descriptive synopsis

Students were presented with a copy of ‘Ryan’s Spelling Test’ where 3 out of 10 spellings were incorrect. They worked in pairs to decide what fraction and percentage of spellings were correct before sharing their ideas with the class. Conor explained that 7 were right and Jake explained that this is the same as 70%. I introduced the idea of repeating the test ten times to see how many would be correct out of a hundred. Darragh noted that Ryan would have to do 90 more spellings and Andrei noted that he would get 30 wrong. I then presented students with the problem shown in figure 1 which asked them to consider the fraction and percentage of our class that were present and absent. The students were given some time to consider this problem in pairs before I called the whole class back together again. Anthony suggested \( \frac{22}{25} \) present. Steven asked if the fraction absent was \( \frac{3}{25} \). I asked for suggestions on what
percentage this might be. Alan suggested 70% present and justified this by saying, “’cause if you just imagine a hundred people and you take away three and they count as tenths that’d be seventy and then there’s thirty people out”. I asked what the other students thought of his answer. Darragh said, “No”. I asked if they would like to hear it again. Many said that they would and Aidan suggested that Alan should not do it “speedy.” Alan repeated his suggestion and this time Darragh agreed. I asked Anthony if he agreed with Alan or if he had an idea of his own. Anthony suggested 97% present and 3% absent, “’cause like if the whole class was in that’d be a hundred per cent and three people are out so that’s taking like three off.” Jonathan stated that he understood now and Alan suggested that both his answer and Anthony’s answer might be right. At this point I told the class that neither answer was right.

Jake then suggested: “Twenty one per cent and ten-tenths are in and three per cent are out.” I was not sure how to represent his suggestion on the board with the previous suggestions and asked him about it. He agreed that it should be written as 21 $^{10}_{10}$ % present and 3% absent as shown in figure 1. He explained the 3% was for the three children who were absent and when I asked him if 21 $^{10}_{10}$ was another way to write 22, he agreed. I asked Andrei what he thought and he replied with the correct answer of 88% present and 12% absent. Michael said “Oh!” and Darragh laughed. Alex, sounding like he might not believe it, asked “Eighty-eight in the class?” As I wrote Andrei’s suggestion on the board some of the other students made comments. Darragh said, “Oh, yeah it’s by four” and “Three by four is twelve.” Alan said, “Oh, yeah I get it.” I asked Andrei to explain it. He said, “’cause it’s twenty five so count in four up to a hundred … And since that’s up in fours you put twenty two up in fours and you get eighty eight.” I restated his suggestion using multiplication and the formal multiplication equivalence procedure. Students joined in as I explained and I asked the class what they thought of it. Edward said that he thought it was “brilliant” but another unidentifiable student said he “sort of” understood it. Alan said that it was a good way of figuring it out but Alex said that it was a bit confusing and Michael said that he did not understand.

I suggested that maybe somebody else would like to explain Andrei’s idea. Darragh volunteered to explain and said, “Twenty two by four is eighty eight and three by four is twelve. Add them together and you get a hundred per cent.” I suggested that he might have skipped one of the most important things and asked him how he knew to multiply by four. He
said that he just guessed. I asked Andrei the same question and he said, “Four times twenty five is a hundred.” Darragh said that he understood and asked for a chance to explain again. He said, “By four is a hundred and that’s you’re whole hundred per cent so eh … Then what you do the bottom you do the top so twenty two by four and then what’s left out of a hundred per cent so eighty eight take away a hundred is … twelve.” He then appeared to ask either Steven or Michael if they understood his explanation. Steven indicated that he had not understood and Michael added that Darragh had spoken too quickly. Darragh then gave a more elaborate explanation: “Right you have to turn the twenty two- twenty fifths into hundredths to find what per cent it is and if four twenty fives is a hundred … So you multiply by four. What you do to the bottom you have to do to the top so you have to multiply twenty two by four which is eighty eight. So that’s eighty eight hundredths so that’s eighty eight per cent.” During this phase of explanation Michael interrupted him and added his own comments. Then I set the students some written work to do from their textbooks.

<table>
<thead>
<tr>
<th>Question Type</th>
<th>Description</th>
<th>Examples</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Gathering information, leading through a method</td>
<td>Requires immediate answer. Rehearses known facts/ procedures. Enables students to state facts/procedures.</td>
<td>Right, could you count up again which ones, how many has he right? *</td>
<td>5</td>
</tr>
<tr>
<td>2. Inserting terminology</td>
<td>Once ideas are under discussion, enables correct mathematical language to be used to talk about them.</td>
<td>What is this called? Can you tell me another word for that? *</td>
<td>0</td>
</tr>
<tr>
<td>3. Exploring mathematical meanings and/or relationships</td>
<td>Points to underlying mathematical relationships and meanings. Makes links between mathematical ideas and representations</td>
<td>How did you know it should be seven tenths because I saw some people who had just written down seven? *</td>
<td>4</td>
</tr>
<tr>
<td>4. Probing, getting students to explain their thinking</td>
<td>Ask student to articulate, elaborate or clarify ideas</td>
<td>What did you, how did you come up with that fraction? *</td>
<td>16</td>
</tr>
<tr>
<td>5. Generating discussion</td>
<td>Solicits contributions from other members of class</td>
<td>Now what do you think of that solution? *</td>
<td>8</td>
</tr>
<tr>
<td>6. Linking and applying</td>
<td>Points to relationships among mathematical ideas and other areas of study/life</td>
<td>Where else have we used this? *</td>
<td>0</td>
</tr>
<tr>
<td>7. Extending thinking</td>
<td>Extends the situation to other situations where similar ideas may be used</td>
<td>Would this work with other numbers? *</td>
<td>0</td>
</tr>
<tr>
<td>8. Orientating and focusing</td>
<td>Helps students focus on key elements in order to enable problem solving</td>
<td>What is the problem asking you? *</td>
<td>0</td>
</tr>
<tr>
<td>9. Establishing context</td>
<td>Talk about non-mathematical issues to enable links to be made with mathematics.</td>
<td>Have you shared pizzas with your family?</td>
<td>0</td>
</tr>
</tbody>
</table>

^denotes examples from focus lesson, *denotes examples from Boaler & Brodie (2004)

Table 3: Boaler and Brodie (2004) teacher question category analysis with examples.
Table 4: Analysis of student questions by type and number with examples.

**MTLC Analysis**

**Questioning**

Tables 3 and 4 show the results of the analysis of teacher and student questions. There was a range of teacher questions including questions focussing on thinking and generating discussion. Like other teaching experiment lessons, some question types did not occur perhaps due to weaknesses of my own teaching approach (NicMhuirí, 2012). In particular, there was a lack of type 2, inserting terminology questions. These questions could be considered particularly important given the emphasis on developing mathematical discourse in a disadvantaged school setting. My approach was to value thinking regardless of how this was expressed. I felt that if I insisted on precise mathematical vocabulary, it may have inhibited students’ willingness to share their thinking. However, I should have attempted to balance this consideration with efforts to develop more precise terminology. Type 3 questions, exploring mathematical meanings and relationships, were explored in great depth with multiple student contributors to the same conceptual question. In this way, the relatively low number of type 3 questions does not reflect poor attention to mathematical meanings. Instead it relates to the methodology of not counting repeated questions and the sustained discourse on individual mathematical meanings. Not unusually, there were few student questions (Chin & Osbourne, 2008). However, the student-to-student query unprompted by me as teacher, where Darragh questioned if his peers understood his explanation shows high levels of RFL on his part.

**Explaining mathematical thinking and source of mathematical ideas**

The lesson activities for this recording consisted of a preliminary starter activity that was intended to prepare students for the harder challenge of finding the percentage present and absent in the whole class. I had hoped that the first activity, finding the numbers of spellings correct out of ten and discussing how this was expressed in hundredths, might help students make this same connection in the second task. However, most students failed to make the connection and even when Andrei explained his solution, Darragh who was an able student, still seemed to miss the importance of the conversion to hundredths step. However with further input from Andrei, Darragh appeared to develop his ideas and in turn took on responsibility for explaining the mathematics to his peers. In this and other instances, I

<table>
<thead>
<tr>
<th>Question type</th>
<th>Example <em>(Notes on context)</em></th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seeking clarification about mathematics being discussed.</td>
<td>Darragh: Do you get that? <em>(Darragh asked if his peers understood his explanation.)</em></td>
<td>6</td>
</tr>
<tr>
<td>Seeking organisational clarification</td>
<td>Student: What page? <em>(the student asked which text book page to go to)</em></td>
<td>1</td>
</tr>
</tbody>
</table>
refrained from providing explanations and students took on this role and were the primary source of mathematical ideas throughout the lesson. In general, there were low levels of teacher EMT and high levels of student EMT and students appear to be positioned as SMI.

Responsibility for Learning

It was the students rather than me as teacher who played the largest role in evaluating mathematical reasoning. The strongest evaluative move I made was when I stated that neither of the first two solutions suggested by Alan and Anthony was right. This move was intended to motivate students to offer other solutions or at least to generate discussion. Andrei’s central role here should not eclipse the active participation of many other students. Students like Anthony displayed confidence in explaining and defending their mathematical ideas even when these were incorrect. The students also showed a willingness to take risks as Jake’s unexpected suggestion indicates. There was a notable ease with the possibility of multiple possible solutions and Alan suggested (incorrectly) that this was a possibility. Acceptance of multiple solution strategies was not a feature of earlier teaching experiment lessons (NicMhuirí, 2011b). Students who did not actively suggest solutions to the problem still participated by asking questions and admitting that they did not understand. The fact that students struggling to understand admitted this and criticised the speed of the explanations of their peers suggests an expectation that student contributions should be understandable by all. This in turn suggests an obligation on the contributor to do his best to communicate his ideas and suggests high levels of RFL within the community.

Summary

Much of the student-to-student discussion that characterises level 3 of the MTLC framework was present in this discussion. Student ideas were central to the development of the lesson and many students showed high levels of responsibility for learning by attending to and commenting on the contributions of their peers. Other students displayed RFL by questioning the ideas they did not understand and demanding clearer explanations from their peers.

CONCLUSION

This paper focused on classroom discourse at group level, but it is clear from the brief lesson synopsis that the nature of the participation of individual students varied considerably. This is true both when the participation of individual students was compared across lessons or compared with peers. Considering the temporal aspects of classroom discourse and tracking the participation trajectories of individual students was a significant further part of my own research (NicMhuirí, 2012). The experience of learning mathematics in a discourse community is very different from the student experience in a traditional classroom. In traditional classrooms the extent of expected student RFL is often diligently “copying methods that teachers demonstrate and reproducing them accurately again and again” (Boaler, 2009, p. 2). Student RFL in a discourse community is more complex with students playing a role as source and explainers of mathematical ideas. A central feature of the discourse community is the expectation that students should “be responsible for co-evaluation of everyone’s work and thinking” (Hufferd-Ackles et al., 2004, p. 90). It is this democratic feature that leads to devolution of mathematical authority from teacher to students and
provokes genuine mathematical discourse. This also has implications for the teacher experience in a discourse community. A major step toward the creation of a classroom discourse community is the postponement or abandonment of a teacher evaluative move to create opportunities for students to take an evaluative role. Tensions exist around respecting and promoting student agency and when it is appropriate to ‘tell’ or explain mathematical ideas (Dooley, 2011). In my own experience, I found these concerns to be related to the issue of lesson coherence (Fernandez, Yoshida & Stigler 1992). Fernandez et al. note that students may not always comprehend the relationship between different lesson events and highlight the teacher’s role in bridging the gap, usually by talk, when students are unable to make the requisite links independently. Maintaining coherence in lessons and across lessons proved to be a central challenge of the teaching experiment (NicMhuirí, 2012).

What has been presented in this paper is not exemplary practice but an “example of practice” (Ball & Lampert, 1999). Like Ball and Lampert, I wanted to use my classroom as an example of “a serious effort to teach elementary school mathematics for understanding and as a site for developing new ways to investigate teaching and learning” (1999, p. 374). I do not claim that the resulting teaching practice is exemplary just that it may be an interesting, perhaps even a useful example of practice for other teachers and researchers (Erickson, 1986).

REFERENCES
Erickson, F. (1986). Qualitative methods in research on teaching. In M. Wittrock (Ed.), *Handbook of research on teaching* (pp. 119-161). New York: MacMillian.


NOTES ON GROUP-WORK BASED TUTORIALS IN A LARGE SERVICE TEACHING MODULE

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We report on a teaching project that involved the use of peer-supported group-work tutorials in a large (n = 414) service teaching module in Dublin City University in the academic year 2010-11. We describe the background and motivation for the project, and its design and execution. This includes a corresponding tutor training element. We report on feedback on the tutorials obtained from students and tutors, and discuss the students’ performance on the module assessments in the light of the group-work tutorials. We found little evidence of success in the project, and attempt to relate this to existing conceptual frameworks describing the effective implementation of group-work.

INTRODUCTION

The performance of third-level students in mathematics continues to be a cause for concern. We report here on a teaching project that sought to address these concerns by drawing on a teaching approach that has a firmly established conceptual framework. Likewise, the effectiveness of this approach – complex instruction (Cohen and Lotan, 1997) – has a strong evidence base. The project involved a new approach to the tutorial system for a large service teaching module taught in Dublin City University in the academic year 2010-11.

The module in question (MS136 - Business Mathematics) is taken by first year undergraduate students from a variety of degree programmes in business and economics. Roughly 400 students take this compulsory module each year (414 in 2010-11).

Our primary concerns relate to student achievement, in particular the persistently high failure rate. We also had concerns regarding the low rate of participation in tutorials, and speculated on the connection between the two, although there was no clear statistical association. (This is perhaps not hard to understand. Students may attend but not participate in a tutorial. Likewise, students could have passed the module without needing to attend tutorials.)

Our decision to implement a group-work format in tutorials was motivated by these concerns. This approach to cooperative learning is a well-established teaching method (Cohen, Manion and Morrison, 2006) that, crucially, was felt to present students with the opportunity to learn, in other words, circumstances that allow students to engage in and spend time on academic tasks (Hiebert and Grouws, 2007)

Of particular importance to this project was the keynote lecture of Boaler at MEI3 (Boaler, 2009) in which the complex instruction approach to cooperative learning was described. This strongly informed the conceptual framework underpinning this project. This framework is described below. We describe the development and implementation of the group-work tutorial system. We present data on the effectiveness of the group-work tutorials from different perspectives and summarise student feedback on the tutorials. We conclude with some reflections on the project, seen in the light of these results and the conceptual framework.
CONCEPTUAL FRAMEWORK

In third-level mathematics teaching, a distinction is frequently made between service teaching – the teaching of mathematics to students whose programme of study is not primarily mathematical in nature – and other mathematics teaching (that is, the teaching of mathematics to students enrolled on a programme that is primarily mathematical in nature). The importance of service teaching to mathematics departments in Ireland, both in terms of mission and sustainability of those departments, is reflected in the amount of associated research and development (e.g. Burke, Mac an Bhaird and O’Shea, 2012; Hoban, Finlayson and Nolan, 2011; Faulkner, 2009; Ni Fhloinn, 2009; Cleary, 2007; Gill and O’Donoghue, 2005). It has been claimed that the field of mathematics service teaching generally is under-theorised and under-researched, but efforts have begun in order to address this deficit (Gill and O’Donoghue 2007). We report here on a service teaching project. The work we describe is developmental in nature and seeks to improve the outcomes of the teaching of a particular module.

The project we describe involves the development and implementation of group-work based tutorials in a large service teaching module. We consider the group-work described below to be an example of cooperative learning: a structured, systematic instructional strategy in which small groups work together toward a common goal (Cooper and Mueck, 1990). Cohen (1994) specifies further a key part of this definition, that is, a situation in which students work together in a group small enough that everyone can participate on a collective task that has been clearly assigned.

The call for a move towards cooperative learning in mathematics can be traced back to at least the 1980’s (Springer, Stanne and Donovan, 1999). As described in that paper, there is no single theoretical base for group-work as a pedagogical strategy, and a variety of conceptual frameworks exist that draw on a wide range of fields including philosophy of education, cognitive psychology, social psychology and humanist and feminist pedagogy (Springer et al., 1999, p. 24). We will appeal principally to the conceptual framework described by Cohen (1994) and elaborated further by Cohen and Lotan (1997).

Thus group-work is now well established as a teaching approach (Cohen, Manion and Morrison, 2006), and its credentials as an approach that underpins successful teaching and learning can be said to be equally well established. In relation to the use of group-work in primary and second level education, “meta-analyses have consistently reported that cooperation has favorable effects on achievement and productivity, psychological health and self-esteem, inter-group attitudes, and attitudes toward learning” (Springer et al, 1999, p. 23). This meta-analysis reports similarly positive outcomes in relation to cooperative learning of science, mathematics, engineering and technology at third level: “The magnitude of the effects reported … supports more widespread implementation of small-group learning in undergraduate SMET” (Springer et al, 1999, p. 21).

A list of advantages of group-work in teaching is offered by Cohen et al (2006, p.199): nineteen different features are identified. Group-work is noted as a characteristic of mathematics teaching in countries rated as high achieving in the 1995 and 1999 TIMSS
mathematics assessments (Conway and Sloane, 2005). It noteworthy that group-work was not found to be a feature of Irish mathematics classrooms in a study carried out before the introduction of Project Maths (Lyons, Lynch, Close, Sheerin and Boland, 2003).

Cooperative learning is at the heart of the work of Elizabeth Cohen and colleagues that seeks to embed an equitable approach to teaching in heterogeneous classrooms. An important element of this work has been the drive to establish the conditions for productive learning in small groups (Cohen 1994). This has led to the notion of complex instruction, the nature of which is outlined in (Lotan, 1997). This overview article begins with the following definition:

Complex instruction is a pedagogical approach that enables teachers to teach at a high intellectual level in academically, linguistically, racially, ethnically as well as socially heterogeneous classrooms (p.15).

This approach entails a combination of a specialised curriculum based on open-ended tasks, cooperative student groups and a set of organisational arrangements that seek to maximize the benefits of cooperative learning for students. These arrangements include the structuring and assigning of roles within the groups, but relate more importantly to the role of the teacher. A crucial part of this role is to maximise the number of interactions between students. Given the sociological origins and intent of complex instruction, it is important to realise that this refers to all students, and particular teaching strategies are described that seek to meet this aim by addressing issues of status within student groups. These strategies are the adoption of a multiple-ability orientation, and assigning competence to low-status students. In adopting the former, teachers “widen their own and their students’ conception of ‘smarts’” (Lotan, 1997, p.23). Teachers assign competence to low-status students when they draw particular attention to the contributions of such students and thereby elevate the status of these students within the group (Lotan, 1997, p.23).

Drawing on work of sociologist Charles Perrow on organisational structure, Cohen, Lotan and Holthuis (1997) posit three propositions that form a framework describing conditions for productive cooperative learning. The first of these is that when working on open-ended tasks, “…the extent to which students talk and work together will be related to organisational effectiveness” (Cohen et al, 1997, p. 33). Thus the teacher should seek to maximise task-related interaction between the students, described as lateral relations. Recognising that open-ended tasks lead to uncertainty on the part of students, the second proposition asserts that “…the more frequently the teacher uses direct supervision, the lower will be the rate of lateral relations among students” (Cohen et al, 1997, p. 34). The notion of direct supervision, and its obverse, delegation of authority, is important here. Direct supervision refers to such teacher actions as informing, instructing or defining; disciplining; asking a factual question. Actions such as stimulating higher-order thinking, making connections, talking about multiple abilities and assigning competence lie outside the domain of direct supervision. Thirdly, the authors note the link between the number of different tasks on which different student groups are engaged and the opportunity for the teacher to delegate authority.

Combining these propositions allows the authors to present a summary theoretical framework:
differentiation → delegation of authority → lateral relations → effectiveness (Cohen et al, 1997, p. 35)

This article concludes by presenting evidence that supports this framework in terms of the existence of the appropriate correlations based on observations of a total of 50 different classes. Further discussion of the effectiveness of complex instruction is presented in the four chapters of Part V of Cohen and Lotan (1997). See also (Boaler and Staples 2008), and an account of the effectiveness of this approach to teaching in (Boaler 2009).

Next, we seek to draw connections between the notion of delegation of authority as described above and a teaching and learning strategy that emphasises questioning and prompting of students (Watson and Mason, 1998). This strategy seeks to engender mathematical thinking capacities in students. These capacities are listed, and include such activities as exemplifying, generalising, justifying and explaining. The teaching approach involves the use of explicitly given questions (generally open or leading questions) and prompts designed to lead the student to engage in a particular form of mathematical thinking. For example, for the capacities of “Exemplifying/Specialising”, we find

Give me one or more examples of .. / Is .. an example of ..? / Find a counterexample of … (Watson and Mason, 1998, p.8)

We take the view that in asking such questions and offering associated prompts, the teacher is not engaging in direct instruction, and so is delegating authority. Thus, appealing to the framework described above, this approach to teaching should support effective group-work.

Finally, we note the importance of ground rules (or cooperative norms) for group-work and of assigning roles within groups (Cohen et al 2006, Cohen 1994).

THE TEACHING PROJECT

Overview

The project involved developing and implementing a peer-supported group-work tutorial system for the first year module MS136 (Business Mathematics). Post-graduate tutors, peer tutors and the module coordinator/lecturer (BN) provided the teaching for these tutorials. Students were assigned to a specific group in a particular tutorial, and were given a different role each week. The majority of groups contained four students; a small number contained three, with at most seven groups in each tutorial. Preparatory work was assigned and attendance/participation was compulsory: continuous assessment marks were awarded for participation in the tutorials. A training workshop for tutors was developed and delivered, and the curriculum for the module was adapted to the new tutorial regime.

The module

MS136 is an introductory level calculus module, closer to Ordinary Level Leaving Certificate (LC) mathematics than Higher Level. The module includes procedural content in the context of business and economics, and an element that seeks to develop students’ mathematical thinking capacities. This is a service course taught to some 400 students on 11 different programmes. The mathematical prerequisite is grade D3 or higher in either Ordinary or
Higher Level LC mathematics. Students attend two lectures and one tutorial for each of the 12 weeks of Semester One. Assessment is in the form of two multiple-choice in-class tests given in weeks 7 and 12 (7.5% each), and one terminal written exam (85%). The module suffers from persistently high failure rates (30-35%) and low levels of lecture and tutorial attendance.

A peer-supported group-work tutorial programme was introduced in 2010-11. This included an amendment of the assessment schedule: 5% of the total for the module was awarded for attendance and participation in the tutorials, leaving 5% for each of the two multiple-choice in-class tests. Many key aspects of the module were unchanged: the same lecturer, syllabus, examination and class-test structure and content remained in place from previous years.

**Tutor training**

The importance of teacher preparation for the implementation of group-work is emphasised by both Cohen, Manion and Morrison (2006) and Lotan, Cohen and Morphew (1997). In developing a training workshop for the group-work tutors, we were cognisant of the principles of effective group-work as laid out above. Thus we sought to address the key issue of encouraging participation on the part of students during the tutorials.

The training workshop was developed by a project team comprising the authors and three experienced post-graduate mathematics tutors from DCU. A SWOT analysis on group-work was carried out in July 2010 which led to basing the training workshop on the five elements discussed below. These were developed over July/August 2010 and delivered to the group-work tutors at the beginning of the academic year 2010-11.

*Introduction to group-work tutorials*

This element of the training workshop involved a discussion of the basic principles of and rationale for group-work. We discussed how group-work benefits students and outlined the role of the tutor in group-work tutorials.

*The first tutorial*

This part of the workshop was designed to enable the tutors to introduce their students to group-work. This was done by having the tutors themselves experience the activities scheduled for the first tutorial. In order to translate the principles of the *Introduction to Group-work Tutorials* to a meaningful experience, the tutors engaged in three activities: an ice-breaker that allowed students to introduce themselves to one another; a communication task that sought to build an appreciation of the importance of working cooperatively and a personalities task that sought to build an appreciation of the need for awareness and respectful communication with other group members. (In the tutorials themselves, the first tutorial also included an exposition of the ground rules for group-work.)

*Case study of group-work*

In this session, tutors read and took part in a structured discussion based on a fictional account of a (mildly) dysfunctional group. This built on previous work of one of the authors using case studies in tutor training (Nolan, 2008). This seeks to confront prospective teachers with some of the issues that they may encounter in the classroom. The students discuss these issues
and potential resolutions, and thereby prepare for similar situations that arise in their teaching. The issues in this case relate to the group failing to adhere to the ground rules for group-work.

**Ground rules, conflict resolution and grading**

The fourth element of the workshop involved a discussion of the ground rules for the tutorials devised by the project team. Advice on resolving difficulties between group members was also given, and post graduate tutors were instructed on their role in grading students. Grading was based on a ‘tick’ system. Students were instructed to (i) carry out preparatory work for the tutorial; (ii) arrive on time and (iii) engage with the worksheet and their group during the tutorial. A failure on the part of the student in any one of these resulted in a tick against the student. Full marks were awarded if a student had no ticks; half marks if there was one tick, and no marks if there were two. Leniency was promoted within this system.

**Questioning skills for group-work**

The final part of the workshop involved a session developing tutors’ questioning skills. This was based on the work of Watson and Mason described above (Watson and Mason, 1998). Included here was a video case study that allowed the tutors to critique the questioning approach taken by a tutor in a fictional setting.

**Ground rules**

The following ground rules for group-work tutorials were devised by the project team. These were informed by the SWOT analysis of July 2010, which in turn was informed by the research literature on group-work, in particular the discussions of Cohen, Manion and Morrison (2006) and Cohen and Lotan (1997). In summary, these rules stipulate: (i) active participation; (ii) mutual respect; (iii) talking and listening equally; (iv) no-one is finished until everyone is finished; (v) giving answers is not helping – give explanations when helping; (vi) call the tutor for group questions only; (vii) arrive on time; (viii) carry out assigned preparatory work; (ix) adhere to your role.

The guiding principal of these ground rules was that they must engender discussion on the part of the students. Likewise, we see these ground rules (cooperative norms in the language of Cohen (1994)) as an effort to adopt the perspective of complex instruction that organisational arrangements are central to effective cooperative learning.

**The peer tutors**

There is an extensive literature on peer tutoring in third level mathematics which is beyond the scope of the present article. Here, peer tutoring describes the involvement of second year students in tutoring their first year peers. The project was advertised in April 2010, and the tutors were recruited in August 2010. The only qualification was that the prospective tutor must have passed the module MS136 in the academic year 2009-10. Twelve second year students participated in the project. They were paid the hourly undergraduate demonstrator rate for this work. The tutors took part in the training workshop in September 2010.
Tutorial structure

Each of the 414 students enrolled in the module was assigned to a one-hour weekly tutorial based on their programme timetable. They were then randomly assigned to a group of four within their tutorial. (A small number of groups had only three members.) There were either six or seven groups in each tutorial. Teaching was provided by one postgraduate tutor or academic staff member, and one or two peer tutors. Each student was assigned a colour code: red, yellow, purple, green (R, Y, P, G). Tutorial worksheets were available on Moodle at least one week before tutorials. Each question was assigned to two of R/Y/P/G: these questions were to be attempted by those students in advance of the tutorial. In tutorials, students were asked to discuss questions and to work together and with tutors to develop complete solutions. The first tutorial, as described above, formed an important part of the teaching.

The group roles comprised chair, recorder and ordinary members, and rotated weekly. The role of the chair was to ensure that the group adhered to the Ground Rules and kept to the tutor’s time-keeping guidelines. The recorder was asked to keep a legible version of the group’s work on each tutorial sheet question. They also had the task of providing the other members of the group with a copy of these solutions within one day of the tutorial. Ordinary members were given the task of co-operating with the recorder in providing solutions to their assigned exercises. All students had the task of working cooperatively on the worksheets.

We note that this structure was significantly different to that of previous years, where attendance was not compulsory, and tutorials focussed on students working individually or in ad hoc groups with the assistance of the tutor.

Curriculum

The syllabus of MS136 had been devised in cooperation with client Schools in DCU over the course of several years. Thus the project did not entail any revision of the syllabus. For the group-work tutorials, questions that sought to promote student interaction were included. Thus the following four question types were included: definitions; example generation; true or false questions; procedural questions. (All four question types appeared in the terminal examination: the example generation and true or false questions formed a compulsory question worth 25% of the total.) The first type question type could be described as “take out your notes and look something up”. The second and third question types were of crucial importance to the project. It was with these that we hoped to engender meaningful lateral interactions on the part of the students. Watson and Mason (2005) have written extensively on the use of example generation in learning mathematics. We claim that questions of these two types had not previously been encountered by the students in the context of mathematics. We base this conclusion on our knowledge of the nature of textbooks, assessment material and the approach to teaching and learning in Irish secondary schools. See for example (O’Keeffe and O’Donoghue, 2009), (Lyons et al., 2003) and (State Examinations Commision, 2005). We anticipated difficulties with these questions by spending lecture time on a discussion of their nature and strategies for answering them. We note that while the more conceptual questions played an important role in relation to engendering cooperative work, the procedural questions played an equally important role in terms of the module learning outcomes.
RESULTS

We note two points. First, our implementation of group-work sought to draw on the ideas of complex instruction, but we do not claim that it provides an example of this teaching approach. This has implications for the data gathered in the course of the project. We did not seek to carry out classroom observations or measure the correlation between the instances of lateral relations and effectiveness. Second, our principal aim with this teaching project was to increase the effectiveness of the module in the narrowly defined terms of assessment marks. The results we present seek to address this, and seek to understand the impact of the group-work tutorials through the use of a student survey, and through the authors’ reflections on the project seen through the lens of the conceptual framework described above.

Attendance and participation

We recall the grading system used in the tutorials. Each student attended ten tutorials during the course of the module, and was awarded a mark of 0, 0.5 or 1 for each. These marks were totalled, and scaled to the 5% of the module total that was awarded for attendance and participation in tutorials. The spread of marks is summarised in Table 1.

These results indicate a high level of attendance and participation in the tutorials, and show what can be taken to be success in this regard. Nearly 63% of students earned a mark of 80% or higher. However it is of concern that even with the reward of marks that are not contingent on procedural ability in mathematical tasks, some 9% of students failed to earn marks in even one of the ten tutorials. It was exceptionally rare that a student who attended a tutorial did not earn a mark, so these marks correspond to students who did not attend any tutorials.

Examination results

There was a significant decline in exam performance in the module. Summary statistics are represented in Table 2, comparing the examination results with those of the corresponding examination from the previous academic year. The sharp decline is evident.

<table>
<thead>
<tr>
<th>Tutorial Mark (Max = 5)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attendance and Participation</td>
<td>36</td>
<td>35</td>
<td>28</td>
<td>56</td>
<td>115</td>
<td>144</td>
</tr>
</tbody>
</table>

Table 1: Summary of marks for tutorial attendance and participation
<table>
<thead>
<tr>
<th>N</th>
<th>Mean</th>
<th>Median</th>
<th>Mode</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2009-10</td>
<td>387</td>
<td>51</td>
<td>52</td>
<td>43</td>
</tr>
<tr>
<td>2010-11</td>
<td>379</td>
<td>40</td>
<td>38</td>
<td>26</td>
</tr>
</tbody>
</table>

Table 2: Summary statistics for examination results.

Correlations

We recall the breakdown of marks for the module: 5% for tutorials, 10% for in-class multiple-choice tests and 85% for the written examination. In Table 3, we present the correlation coefficients for the following pairs of marks: tutorial mark (“tutorial”) and overall module total (“total”); tutorial mark and written exam mark (“exam”) and tutorial mark and class test mark (“test”). We note a positive correlation in all three cases. There appears to be a strong short-term gain: successful participation in the tutorials is strongly associated with the class tests that take place during the teaching period of the semester. However this association falls off sharply when we compare the tutorial mark with the exam mark.

<table>
<thead>
<tr>
<th>Correlated Marks</th>
<th>Tutorial vs Total</th>
<th>Tutorial vs Exam</th>
<th>Tutorial vs Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pearson’s $R$</td>
<td>0.52</td>
<td>0.39</td>
<td>0.77</td>
</tr>
</tbody>
</table>

Table 3: Correlation of tutorial and other assessment marks

Survey results

Students who had taken the module were asked to complete an online survey on the tutorial system in the second semester of 2010-11. Following a familiar pattern, participation in this survey was very low (10% response rate). The survey contained ten statements with responses called for on a Likert scale offering options from **strongly agree** to **strongly disagree**. We summarise responses from the five questions we found to be of most interest.

*I found the maths tutorials helpful in terms of learning the course material:* 50% of students gave responses in the **disagree** categories.

*I found the maths tutorials helpful in terms of passing the exam:* here, 55% of students gave a response in the **disagree** categories.

*The tutors in my tutorial were helpful:* There was a more equal split here, with 40% disagreeing or disagreeing strongly and 43% agreeing or agreeing strongly.

*Apart from learning maths, there are advantages to having group-work tutorials:* Here, we found that 78% of students either agreed or agreed strongly with this statement.

*Overall, the group-work tutorials for maths are a good idea and should be continued.* Again, an even split, with 48% in the **disagree** categories and 50% in the **agree** categories.

DISCUSSION

As we have seen, the outcomes of the project were very mixed. Inasmuch as the principal aim was to address the high failure rate in the module, the examination results point to a failure of
the project. In fact the decline in the examination results was so extreme that other sources of an explanation for this drop were sought. These were found: we hypothesised that the decline was due to examination questions that may have been considered ‘unpredictable’ by students.

We also see mixed results in relation to the students’ attitudes to the group-work tutorials. However, one positive outcome was the students’ recognition of benefits other than learning maths of the group-work tutorials. This is perhaps reflective of the wider sociological and personal development benefits of cooperative learning (Springer, Stanne and Donovan, 1999). One area of success of the project was the high rate of participation in the group-work tutorials. This high rate associated with short-term procedural competence as seen in the correlation between participation in the group-work tutorials and marks in in-class tests.

Based on feedback from the tutors, and on the first author’s experience working as a tutor, we note that this success reflects the observation that a significant amount of cooperative learning did indeed take place in the tutorials. Likewise, the predicted difficulties arose in relation to withdrawal from the group by individuals, or its domination by others. Tutors also reported the difficulty of adhering to the principles of complex instruction, mainly in terms of acting in ways that embodied delegation of authority and refraining from direct instruction. This is perhaps unsurprising given Cohen and Lotan’s (1997) description of the intensive professional development needs of teachers who seek to undertake complex instruction.

The tutors as a whole also reported that students spent a disproportionate amount of time on the example generation and true/false questions. This may have been at the cost of developing their procedural competence. This observation is reinforced by data from the Maths Learning Centre (which provides a drop-in service that students can attend for one-to-one assistance with mathematics) which shows that 33% of the students in MS136 attended a drop-in session during the semester, making a total of 352 visits between them. These students primarily asked for help with the “example generation” and “true or false” questions. Anecdotally, with the former, their greatest struggle seemed to be the fact that there could be more than one correct answer when asked for an example; with the latter, the true/false questions demanded a fuller understanding of the material than many students appeared to have. In terms of the operation of the tutorials, frequent complaints included the fact that other group members were “slowing them down” so that their group did not finish the full tutorial sheet by the end of the tutorial; that they felt embarrassed and under pressure when they were unable to contribute anything to the group for the questions they were meant to have attempted in advance; and that not all group members were pulling their weight, regardless of the structures in place. This speaks to a shortcoming in terms of the tutors’ ability to adopt a multiple ability orientation and to assign competence to students in the appropriate way (Lotan 1997).

We speculate that the mathematical thinking exercises may have ‘distracted’ students from what could be described as more pragmatic and strategic examination preparation. Asking students to engage in mathematical thinking is asking them for a long-term commitment (Mason, Burton and Stacey, 1982): a 12 week, one hour per week programme that follows (for many students) 12 years of direct instruction is likely not sufficient for students to adopt the dispositions that underpin the benefits of mathematical thinking.
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ARE WE THERE YET? AN EXAMINATION OF ISSUES RAISED BY THE ICMI STUDY SCHOOL MATHEMATICS IN THE 1990s AND OTHER PUBLICATIONS FROM THE 1980s

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In this paper, questions raised in the 1980s with regard to mathematics education are discussed, and ways in which the corresponding issues are regarded in 2013 are considered. The extent to which the ‘1980s’ questions are still relevant to Ireland at present, in particular during the rollout of the curriculum initiative ‘Project Maths,’ is examined.

INTRODUCTION

In the course of browsing through some older mathematics education books recently, I was struck by the extraordinary relevance of those dating from the 1980s to the current situation in Ireland. The books addressed issues that, if not necessarily new, were of great interest at the time; typically they offered suggestions that might shape directions to be taken by mathematics education over the following decades. Of especial note was School Mathematics in the 1990s (Howson & Wilson, 1986), henceforth referred to this paper as School Mathematics. In only a hundred pages of text, Howson and Wilson highlighted a range of themes, many of which are germane to current Irish mathematics education.

At much the same time as I was re-reading the works from the 1980s, I obtained a copy of the newly-published Third International Handbook on Mathematics Education (Clements, Bishop, Keitel, Kilpatrick & Leung, 2013). With over a thousand, rather than just a hundred, pages at its disposal, the breadth and depth of discussion in the Third Handbook are naturally greater than in School Mathematics; however, many of the same themes can be identified. A chapter on curriculum, of which Howson is a co-author (Cai & Howson, 2013), provides one obvious sequel to the 1986 publication.

Taking these two books as a starting point – or rather as starting and finishing points – this paper examines a number of issues, both in general and with specific reference to Irish developments. It deals especially with those pertinent to the revised mathematics curriculum, Project Maths, currently being rolled out in Irish second-level schools. Additionally, in focusing (though by no means either exclusively or exhaustively) on the writings of Geoffrey Howson, it pays a tribute to one of the great characters contributing for decades to the field of mathematics education worldwide.

The following section of the paper, identifying trends and landmarks in mathematics education from 1980 to 2013, provides a context for the discussion. It outlines some of the trends prominent in the 1980s; against that background, it gives more details on the three main sources for the paper: School Mathematics, the Third Handbook, and Project Maths. The next four sections focus on themes identified as being of special interest for Irish mathematics education. In these sections, the general approach is as follows: issues are identified in School Mathematics and other relevant literature from the 1980s; for each,
equivalent themes in the Third Handbook are located, where appropriate; and the application to Ireland is then examined. The concluding section emphasises the view that discussion of these issues will and should continue.

TRENDS AND LANDMARKS: FROM 1980 TO 2013

Trends and Landmarks in the 1980s

In the 1980s, several important trends in mathematics education can be noted, for example with regard to curriculum content, teaching and learning, and the resources available in the classroom or school. They provide a backdrop for the issues discussed in this paper.

As regards curriculum content, at the start of the decade the ‘modern mathematics’ period had faded, and the reaction in the form of ‘back to the basics’ was perhaps also past its peak. Around that time, curricular data for 21 education systems were documented systematically for the Second International Mathematics Study (SIMS), conducted by the International Association for the Evaluation of Educational Achievement; analysis of these data was an important focus of activity especially in the early 1980s. SIMS introduced to educational discourse the three-level curriculum model with its terminology of intended curriculum (specified in national syllabuses or curriculum guidelines), implemented curriculum (taught by teachers) and attained curriculum (learnt by students) (Travers & Westbury, 1989). The Study took more account than did other comparable ones, before or since, of the scope and variation in participating systems’ intended and implemented curricula as contexts for interpreting scores in the Study’s attainment tests. However, the process was slow, and final results were not published until the end of the decade (Travers & Westbury, 1989; Robitaille & Garden, 1989). By that time, Howson (1991) was carrying out a smaller-scale study, primarily of European countries’ curricula. It showed that by then – unlike, for example, in the period notable for ‘modern mathematics’ and other approaches from the 1960s and 1970s (Howson, Keitel & Kilpatrick, 1981; Travers & Westbury, 1989) – the curricula tended to be eclectic in provenance rather than reflecting any one philosophy.

By contrast, with regard to teaching and learning, certain key ideas tended to dominate. The decade witnessed an explicit focus on problem solving, discussed below. Increased attention was paid to constructivism and hence to teaching approaches that facilitate such learning. Among the resources available to support learning, calculators were becoming widely available, if not necessarily widely used in schools. Attention was turning to the more versatile microcomputer. The potential of computers, not only to change mathematics itself and to add to the topics that could be addressed at school level, but also to support many different ways of teaching and learning the subject, made their role a matter of special interest (Howson & Wilson, 1986; National Council of Teachers of Mathematics, 1989).

As well as through identification of these trends, the decade can be viewed through the lens of four landmark publications. First, in 1980, the American National Council of Teachers of Mathematics (NCTM) published its Agenda for Action, in which the initial recommendation was that “problem solving be the focus of school mathematics in the 1980s” (NCTM, 1980, p. 1). This prefigured much work on curriculum, teaching and learning in the thirty years that followed. The second landmark document was the Cockcroft Report on mathematics
education in England and Wales (Cockcroft, 1982). It too emphasised problem solving, for example as one of six approaches to teaching and learning that are listed in its famous Paragraph 243. At a time at which mathematics was being taught for longer to more students than would have been the case twenty years earlier, the report was critical of tendencies to offer diluted versions of advanced courses to the lower-achieving students. Instead, it proposed a ‘foundation list’ of rather basic content that might be addressed by all students, and in effect specified three nested curricula, building up from the foundation list, that might cater for the full range of abilities. The impact of the report can be judged by the extent to which it was discussed at the Fifth International Congress on Mathematics Education (ICME 5), held in 1984; it was the subject of an invited address and presentation (Carss, 1986), and Sir Wilfred Cockcroft himself was a notable character at the congress. The third landmark publication is School Mathematics (Howson & Wilson, 1986), described in the following paragraph. Fourthly, at the end of the decade came the NCTM’s Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). Identified by Cai and Howson (2013, p. 957) as “monumental,” this perhaps displaced the Cockcroft Report as a point of reference for international discussions. The Standards ended the decade, as the Agenda for Action had begun it, by highlighting the importance of problem solving.

**A landmark in 1986: School Mathematics in the 1990s**

The decade that saw these trends and landmarks also saw the publication of several volumes in the extensive series produced by the International Commission on Mathematical Instruction (ICMI). School Mathematics was the second in the series. The book was the outcome of a process that started with circulation of a document to ICMI national representatives. This was followed by discussion at an international seminar, held in Kuwait and attended by an invited group of mathematics educators. School Mathematics provided a synthesis of the views of these ‘experts,’ and was designed to provoke and stimulate discussion rather than to end it. The format of much of the book involves identifying important questions and listing possible ‘alternative’ answers, together with the likely consequences of each alternative. As the authors emphasised, “In considering [the issues] in the light of their own circumstances, teachers, students, curriculum developers, and other mathematics educators working in different educational systems will – and should – sometimes come to different conclusions” (Howson & Wilson, 1986, p. 5). Underlying this contention was a belief that the aims of mathematics education will – and should – vary from country to country and indeed from time to time. That belief, henceforth referred to for convenience as the ‘variety principle,’ constitutes a recurring theme throughout this paper as well as in School Mathematics.

Mathematics education issues do not fall into tidy discrete packages, and many themes are woven throughout the book. The ones picked out for discussion here, because of their perceived relevance to current Irish mathematics education, are:

- the role and status of mathematics in the curriculum for all students, and allied issues of curricular differentiation
- the structure and content of the mathematics curriculum
- specific topics in the curriculum
• the processes of curriculum change.

They provide the four main sections in the rest of this paper. Before considering them, however, attention is turned to the present: to the Third Handbook and Project Maths.

**A landmark in 2013: Third International Handbook on Mathematics Education**

The Third Handbook obviously is a successor to two previous such handbooks, but is much more than an update of its predecessors. It is divided into four major sections that reflect current issues: social, cultural and political dimensions; mathematics education as a field of study; technology in the mathematics curriculum; and international perspectives on mathematics education. The prominence given in the final section to international studies of attainment was not what the authors initially intended, but came about in recognition of the strong influence that such studies have on mathematics education today. The various chapters were written by experts in the relevant fields, and each chapter was reviewed by at least two readers. It is impossible to do full justice here to the scope and depth of the volume. However, as mentioned above, many themes observed in School Mathematics can be identified; selected ones are noted and discussed below.

**A landmark in Irish mathematics education: Project Maths**

Project Maths aims to introduce a wide range of changes in the practice of mathematics education in Ireland (National Council for Curriculum and Assessment [NCCA], 2013; Project Maths, 2013). It was piloted in 24 schools, starting in 2008; the rollout nationwide began in 2010 and is continuing on a phased basis, with different content ‘strands’ being introduced in successive years. Changes have been made in the intended content, while the role of contexts and applications has been enhanced. With regard to implementation, the introduction of Project Maths is being supported by a drive to broaden the paradigms of teaching and learning familiar in Irish classrooms, especially in encouraging approaches that facilitate constructivist rather than passive or rote learning. Moreover, major changes are being made in the State examinations, to assess the aims of the curriculum more fully by emphasising solution of problems set in contexts, and in particular to counteract the strong culture of ‘teaching to the (predictable) test’ that unfortunately permeates Irish education.

At the time of writing, Project Maths is a work in progress. Up-to-date details can be retrieved from the NCCA and Project Maths documentation (cited above) and from links to research that the associated websites offer. A summary for people unfamiliar with the Irish system, together with a flavour of the local culture and issues, is given by Lubienski (2011).

Against this background, consideration is now given to the four themes listed above.

**MATHEMATICS FOR ALL AND CURRICULAR DIFFERENTIATION**

The issue addressed in this section is the role of mathematics in the education of the entire cohort of school students. With regard to the question “Should mathematics remain at the heart of the school curriculum for all?”, School Mathematics offered four alternatives:

1. No; ‘real’ mathematics cannot be taught to everybody.

2. Yes; mathematics must be planned so that it can be effectively taught to all.
3. Yes; but it is accepted that although taught to all it will not be understood by all.

4. Yes; but students will be taught different types of mathematics or will be taught the same mathematics at different rates, dependent on their ability/attainment (condensed from Howson & Wilson, 1986, pp. 8-9).

Consequences of the alternatives included differing degrees of challenge for students and teachers and (for alternatives 1 and 4) the problems of course selection at the point(s) at which differentiation is introduced. The authors noted that alternative 4 attracted most support at the time, but they judged that differentiation of this type had generally been unsuccessful. The perceived difficulties perhaps chiefly reflected experiences of students who either terminated their mathematics education early or were presented with unsuitably challenging material.

The recommendations of the Cockcroft Report, identified above as a key publication of the era, are in line with Alternative 4. School Mathematics, while acknowledging the attempt to obviate providing weak versions of advanced curricula for lower-achieving students, was rather critical of the somewhat impoverished diet that its foundation list of content offered.

In the Third Handbook, Clements, Keitel, Bishop, Kilpatrick and Leung (2013) address the issue of ‘mathematics for all’ in the first chapter, and note that it is still problematic. In line with the ‘variety principle,’ they conclude by asserting that “‘mathematics for all’ should generate forms of mathematics that arise out of, and are obviously related to, the needs of learners and the societies in which they live” (Clements et al., 2013, p. 33). The chapter by Cai and Howson on curriculum also refers to the issue; however, the authors focus less on the problems for lower achievers – a notable theme in School Mathematics – and more on the need to consider mathematics education for the gifted (Cai & Howson, 2013).

What are the messages here for Ireland? Mathematics is offered to all or almost all students throughout their school career, in a format that again reflects Alternative 4 above. In the 1980s, the introduction of a third, less challenging, mathematics syllabus for lower-achieving students in the second-level junior cycle represented differentiation along the lines indicated in the Cockcroft Report – though not necessarily because of it; at that time there was much dissatisfaction among Irish teachers at the perceived mismatch between the needs of at least some students taking the then Lower Intermediate Certificate course and the content they were trying to address (Oldham, 1992a). Such differentiation was carried through to the senior cycle with the introduction of the Ordinary Alternative – later, Foundation level – syllabus in the 1990s. That syllabus, in line with though again not necessarily because of the Cockcroft recommendations, was built from the bottom up, rather than being a watered-down version of the existing Ordinary course (Oldham, 1992b). Interestingly, Project Maths has reverted to offering two syllabuses (though, at least at present, three examinations) for the junior cycle; it is not yet clear how this differentiation will work out in practice.

As regards higher achievers, results from the Programme for International Student Assessment (PISA) over the years have raised concerns about the extent to which potentially higher-achieving Irish students are served by the system (see for example Perkins, Cosgrove, Moran & Shiel, 2010). Perhaps the much less predictable examinations – that key aspect in
the implementation of Project Maths – will encourage these students to develop higher-order strategies, rather than focusing excessively on honing their skills in executing procedures.

MATHEMATICS CURRICULUM: STRUCTURE AND CONTENT

The theme encompassing the structure and content of the curriculum is entwined with the previous one. However, the focus in this section is on the broad areas of mathematical content included at least somewhere in the curriculum, rather than on the target groups at which particular topics are directed. In addressing the subject, School Mathematics used the SIMS three-level curriculum model, described above, and drew heavily on the ongoing work of that study. For the intended curriculum internationally, a “considerable degree of uniformity” (p. 39) was identified for the 13+ age-group (more exactly, the grade level containing the modal number of 13-year-olds) with regard to the five specified topic areas: arithmetic, algebra, geometry, measurement and statistics, though emphasis differed as regards statistics and especially geometry. For the implemented curriculum, reported by participating teachers, the authors noted an even greater degree of similarity in a focus on arithmetic, algebra and measurement – this occurring despite varying amounts of time given, cumulatively, to mathematics in the curricula. Correspondence with the Cockcroft foundation list and other such minimal specifications of curriculum was highlighted.

Given that the ‘modern mathematics’ reforms of the 1960s were still quite recent, it is not surprising that outcomes and issues related to that period were discussed:

… not all the innovations have been abandoned. We have already mentioned the earlier introduction of coordinate geometry…. Probability and vectors (but not vector spaces) seem to have become established in the curricula in many countries…. These examples, though, only serve to emphasise that it has proved easier to introduce new techniques that can readily be used on collections of exercises, than abstract concepts: a fact that should cause no surprise, but which should not be overlooked when planning for the 1990s.

We also note attempts … to cut down on the amount of mathematics taught. (It was a feature of the 1960s reforms that not only was content changed, but it was usually increased in extent….) …. Yet … there is still a great temptation to present a macho-mathematical image to the world. (Howson & Wilson, p. 47).

More generally, again in line with the ‘variety principle’ – “the goals of school mathematics will not be identical everywhere” (p. 21) – the authors warn against any attempt for national curriculum developers to look for uniform solutions or to import practices uncritically from other countries.

In the Third Handbook chapter by Cai and Howson, that warning is repeated. Echoing the earlier work, the authors consider the notion of an (intended) ‘international curriculum,’ and overall reject it on the grounds that “an international curriculum equally suited to the educational traditions … finances and aspirations of all countries, is a chimera that should not be chased” (Cai & Howson, 2013, p. 960). They do, however, look at common influences for curriculum change. These include: pressures imposed by the major cross-national studies of achievement, the Trends in International Mathematics and Science Study (TIMSS) and PISA,
currently prominent as noted above; common learning goals, including emphasis on higher-order processes such as problem solving; and – paying attention to what they claim is an under-researched area – the influence of public examinations.

For Ireland, there is much of relevance. First, in the 1960s, the country had adopted ‘modern mathematics’ to such a notable extent that Westbury (1980, p. 513) referred to the dangers of importing ‘lock, stock and barrel’ a curricular approach that did not have roots in the national mathematics community. However, for many years from 1980, Ireland was perhaps guilty of paying too little, rather than too much, attention to movements taking place and issues being examined elsewhere (Oldham, 2001). This has changed over the last 15 years, so it is again pertinent to heed the ‘variety principle.’ The themes highlighted in the paragraphs quoted and issues cited above are evident in discussions surrounding the intentions and implementation of Project Maths. They include emphasising problem solving and other higher-order processes, especially through assessment in public examinations; keeping up with the international Joneses in PISA and in pre-university syllabus content; and, more generally, introducing, maintaining, de-emphasising or deleting topics, as discussed in the following section.

MATHEMATICS CURRICULUM: SPECIFIC TOPICS

As well as considering the overall scope and structure of mathematics curricula, School Mathematics focused on five particular so-called content issues: geometry, probability and statistics, applications, calculators, and computers; it also raised issues with respect to calculus. The discussion of applications is somewhat different in style from that of the other topics, which are therefore considered together here; the discussion of applications follows.

Geometry, Probability and Statistics, Calculators, and Computers

According to School Mathematics, no topic “arouses so much concern amongst mathematicians as does [synthetic] geometry, the teaching of which has undergone a total transformation in the last thirty years or so” (Howson & Wilson, 1986, p. 58). Building on the SIMS curriculum analysis finding with regard to variety in geometry curricula, and using the ‘alternatives and consequences’ format, the authors listed three alternatives:

1. The idea that geometry should or can be treated in school as a system of knowledge (organised deductively or not), where concepts and facts have to be known simply because they belong to the system, is abandoned.

2. An axiomatic or pseudo-axiomatic school geometry course is offered.

3. ‘Local’ deductive systems, such as angle properties of the circle, are taught (Howson & Wilson (1986), p. 60, condensed and paraphrased).

Consequences noted for the first alternative included a break with a long-standing tradition and a loss for higher-achieving students in encountering proof and the challenge of attempting ‘riders’ (‘cuts’ in Irish parlance). The second was deemed to be unduly hard for most students; “few are likely to be able to cope with the intellectual demands … even fewer will appreciate the significance of the ‘game’ which they are playing” (Howson & Wilson, 1986, p. 60). The third might appear peripheral in the curriculum and so be overlooked by teachers.
The other topic highlighted by SIMS with regard to non-uniform coverage was statistics, examined here along with probability. Unlike geometry, this is a comparatively young topic, both in mathematics itself and in school curricula. School Mathematics made the case for inclusion of both probability and statistics at school level. The authors suggested that “simple statistical ideas should be part of every secondary school student’s education, to help to develop a critical attitude to numerical information presented by the mass media” (p. 12) and that a sense of probability needed to be developed to counteract naive notions leading to quantitative misjudgments. However, they argued that, for all but a few students, “it is inappropriate to pursue statistics at a school level beyond [the] descriptive stage…. [whereas] the teaching of probability would seem to offer far more opportunities at a school level for fostering mathematical thinking than does statistics” (pp. 57-58).

With regard to technology – calculators and computers – discussion focused on the way that these tools can change the scope and type of topics addressed in the school curriculum, as well as affecting the way in which they are taught and learned. For calculators, the authors displayed a rather uncomplicated acceptance of their value and emphasised aspects of their role yet to be fully developed. Three quotations illustrate the points: “the potentiality of the calculator for helping children come to terms with arithmetic has not been greatly exploited; neither has its use for the teaching of more sophisticated mathematical ideas and concepts” (p. 66); “Further up the school, the question is … ‘how much time can be spared to explore the many possibilities opened up by its use?’” (p. 67); and “It is essential that more emphasis be laid on studying how best they can be used…” (p. 68).

The discussion of computers (notably ‘micros’) was more exploratory. An interesting aspect is the likely effect on teaching calculus: “the teaching of the calculus must, surely, change…. We cannot afford to neglect such a powerful teaching aid…. “ (p. 71). More generally, the extent to which content, processes and pedagogical approaches should remain in the curriculum, even when they could be at least partially replaced by use of a computer program or package, was discussed in some depth.

In their chapter in the Third Handbook, Cai and Howson (2013) echo the themes from the earlier discussion. They note that “Seeking international agreement on a geometry curriculum could be impossible, for since the ‘fall’ of Euclid, countries have largely gone their own way” (p. 968); “Statistics at an elementary level has … reached a canonical degree of acceptance, although there are differences in how far one might develop the teaching of that and probability at a higher school level” (p. 969); and:

Significant work also remains to be done to explore the way in which new technology could affect both the mathematics included in the curriculum and how it could more effectively contribute to the teaching and learning of mathematics…. The influence of technology has still to be felt on the actual mathematics taught in schools (p. 969).

In general, however, the authors do not emphasise individual mathematical topics. These are considered chiefly in the section of the Third Handbook focusing on technology. Two issues are of particular relevance to this paper. One is the use of dynamic geometry environments and the research issues they open up, with regard both to teachers (for example, in their
scaffolding of learning by modelling the dragging process and highlighting its features) and to students (for example, investigating how the tools mediate learning about proof). The other is statistical reasoning, especially with the aid of appropriate computer packages. However, a further issue is notable by its absence; calculators receive comparatively little attention.

For Ireland, it is appropriate to take the issues almost in reverse order. The plea in *School Mathematics* for emphasis on investigating how calculators can be used to enhance learning is still all too relevant; for various reasons, discussion of which is outside the scope of this paper, opportunities in this respect were missed when calculators were incorporated formally into the primary and junior cycle curricula – somewhat tardily – around 2000 (Close, Oldham, Surgenor, Shiel, Dooley & O’Leary, 2008). (The continued reference to mathematical tables, rather than calculators, in the Irish junior cycle syllabuses introduced in 1987 provoked a wry comment from Howson (1991) in his study of curricula around that time.)

Likewise, the discussion in *School Mathematics* on the extent to which technology, or other developments, might lead to a reduction in emphasis on calculus in the school curriculum is of current relevance. Project Maths actually contains only a little less compulsory calculus than does the course it is replacing. However, most Higher course students took an additional calculus module; this masked the fact that, already in the early 1990s, there was an attempt to rebalance topics in the senior cycle, reducing the emphasis on calculus and introducing elementary probability for all students rather than only some (Oldham, 1992b). In Project Maths, curriculum content for probability and statistics is greatly enlarged: perhaps more so, or more so for more students, than envisaged in *School Mathematics* as described above. However, vectors – in *School Mathematics*, listed with probability as among ‘new’ topics that had remained in curricula after the 1960s – have been removed. The changes with regard to all these topics have not met with universal approval, notably from the Irish Mathematics Teachers Association (IMTA) (IMTA, 2013) as well as from some third-level constituencies (see for example Grannell, Barry, Cronin, Holland & Hurley (2011)).

While technology can be used in the teaching and learning of probability, statistics and calculus, its most emphasised role in Project Maths is in geometry. Use of a dynamic geometry environment is strongly encouraged, and research on classroom experience will be important. A more contentious issue, especially for junior cycle students, is the actual system of geometry set out in the syllabus. It comes under the heading of alternative 2 above, in being unusually formal: perhaps more formal than the system it replaced, in the most recent of a long line of changes in this area. Higher course students, who may be asked to produce proofs of theorems in State examinations, are expected to observe the rules of the ‘game.’ Thus, at least some teachers are feeling pressure to introduce the proofs exactly in the forms given in the curriculum document, as other – perhaps more intuitive – versions are likely to be inadmissible because of breaching those rules. This again has led to controversy, with the IMTA pleading for a more flexible approach (IMTA, 2013). It is worth noting that – mirroring the wider international situation, as described by Howson and Wilson (1986) – there has been tension for years among the curriculum bodies, the mathematics community and the mathematics teacher community over geometry in the junior cycle (Oldham, 1992a). The degree of success obtained in implementing the new system will be of great interest.
Applications

*School Mathematics* included an extensive discussion (relative to the length of the book) of the role of applications in the mathematics curriculum. The authors noted that applications have featured in curricula for many years, for example in addressing utilitarian aims. References were made to the rise of modelling. A discussion of the use of contexts is especially interesting; the authors provided examples of contexts that might be practical but inappropriate for school students, or not very practical but engaging. Many of the issues are relevant to Realistic Mathematics Education (RME) (Freudenthal, 1991), which had already developed in the Netherlands at the time, but became more prominent internationally in the 1990s. RME has since gained further attention from its association – in diluted form – with PISA, which focuses on problem solving in real-life contexts, rather than those that are realistic in the sense used in RME and *School Mathematics*: real to the students. It is refreshing now to read an account that predates current familiarity with the issue.

The teaching of applications is not an explicit focus of the *Third Handbook*. However, the topic – or its modern equivalent in terms of solving problems set in real-life contexts – looms large as a current issue in Ireland; the focus on problem solving that emerged internationally in the 1980s is still regarded here as (dare one say) problematic. Grannell and others (2011) point to difficulties for teachers in implementing intentions with regard to problem solving. The IMTA (2013) is particularly anxious about the effect of new ‘contexts and applications’ sections in the State examination papers; these, probably more than other innovations, have pushed teachers and students outside their comfort zones. This raises issues about processes involved in curriculum change. One aspect is discussed briefly in the following section.

**THE PROCESSES OF CHANGE**

Change can be uncomfortable. Pushing people outside their comfort zones may be very necessary from time to time; however, it is not good to destroy their confidence and ridicule the work that they have done for years. In writing in *School Mathematics* of the implementation of change, the authors stated:

> On many occasions the strategy employed for persuading teachers to change has been to pour scorn upon current practice – existing procedures have been derided and innovation offered as a panacea. The results have usually been disastrous. To deride current practice is to discount the teacher as a judge and a professional…. (Howson & Wilson, p. 94).

Unfortunately this is relevant in Ireland. At times there has been a missionary zeal in some of the literature, and more especially the media publicity, around Project Maths that has led to the inference that mathematics teaching up to now has been uniformly unsatisfactory. This is not only false, but demoralising for good and dedicated teachers. The current relevance of the quotation above, together with the aptness of the arguments regarding synthetic geometry, were perhaps the most important factors in prompting the present writer to work on this paper.

**CONCLUSION**

Are we there yet? If this means “Have we – the international mathematics community, or its equivalent here in Ireland – resolved all our problems?” then clearly the answer is “no.”
Fundamental questions, such as those raised in the 1980s and discussed in this paper, address issues at the heart of mathematics education; they do and should persist. However, the answers, in line with the ‘variety principle’ formulated above, do and should change to suit different situations and times. Some of the differing answers have been outlined in the paper.

It can be argued that there should be ‘an Irish solution to an Irish problem.’ The answer provided by Project Maths may not be perfect (few things are!), and only time will allow us to determine its main strengths and weaknesses. However, the debate around its introduction – addressing issues such as those raised in the 1980s but overlooked in Ireland for too long – has been a much-needed event. In effect, we in Ireland have been compressing 30 years of questioning and development into just one decade. An earlier paper by the present writer (Oldham, 2001) ended by saying that there were exciting times ahead; we are now experiencing those exciting, or perhaps even over-exciting, times.

For this writer, revisiting the work of the 1980s has contributed to the excitement. School Mathematics in the 1990s, in particular, was seminal in re-igniting her recognition of the importance of the period and also the contributions of Geoffrey Howson over the years. It is crucial that we continue to ask the questions raised, and always aim to find answers that – giving the final word to Cai and Howson (p. 970) – “provide engaging and powerful learning experiences for students whatever topics in mathematics they may be studying.”

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GETTING RATIO IN PROPORTION? AN EXAMINATION OF THE
EMPHASIS ON RATIO-RELATED CONCEPTS IN IRISH
MATHEMATICS CURRICULA

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This paper offers a discussion of the attention given to ratio and related concepts in Irish mathematics curricula. Examination of Irish and other data from comparative international studies, and of the place given to the topics in Irish curriculum documents and textbooks, leads to a suggestion that they are under-emphasised. A small-scale study of the knowledge of ratio for teaching possessed by Irish and other prospective teachers provides some support for the contention. A call is therefore made for discussion and further research.

INTRODUCTION

The topic ratio, together with the related area of proportional thinking and reasoning, is important in students’ mathematical development. Typically, ratio figures widely in countries’ mathematics curricula at primary or second level (or both); moreover, the topic is addressed extensively in research on teaching and learning mathematics. Unfortunately, the research indicates that the topic is problematic for many students, and recent work indicates that it may be problematic for teachers also. This points to a situation in which the problems are perpetuated, as some teachers may not be able to help their students to develop appropriate understanding of the ratio concept and proportional thinking (henceforth referred to, where appropriate, as ‘ratio-related concepts’).

It is the contention of this paper that the ratio concept may have been insufficiently addressed in Irish mathematics education: that the topic has, so to speak, slipped beneath the radar. Through opening up a debate on a possible ‘ratio problem,’ it is hoped to bring the topic to prominence in Irish mathematics education discourse. This may help either to refute the contention, or – if it is supported – to pave the way for ameliorating the situation. Encouragingly, some such amelioration may be under way already; the newly introduced second level curriculum (the ‘Project Maths’ initiative) and its teacher support materials give ratio-related concepts a higher profile than has been the case heretofore. However, at least in the experience of this writer, the improvement has yet to be reflected in discourse among mathematics teachers and educators. The MEI conference is perhaps the most appropriate forum for starting such discourse.

In the second section of the paper, a brief overview is given of research on teaching and learning ratio-related concepts and skills. The third section examines the place of ratio in mathematics curricula internationally, as reported in cross-national studies such as the Trends in International Mathematics and Science Study (TIMSS). Against this background, in the fourth section, Irish curriculum documents and textbooks from the 1960s are examined to trace the emphasis, or lack of emphasis, on the ratio concept and allied topics. The fifth section of the paper describes a project, currently being run by a research group in the Association for Teacher Education in Europe, on prospective teachers’ knowledge of the ratio
concept. It was through participation in this project that the author’s attention was drawn to the existence of a possible problem for Ireland (though not unique to Ireland) in the area. Finally, suggestions are made for addressing the issues identified.

**SETTING THE CONTEXT**

In this section, a brief literature review is provided. It focuses initially on seminal work from the 1970s and 1980s on students’ errors in dealing with ratio problems, and then refers to the very substantial research undertaken on ratio-related issues since that time. Finally, work in relation to prospective teachers’ knowledge of ratio-related concepts is considered.

According to studies undertaken by Kathleen Hart and others in the 1970s and 1980s, ratio was a problematic area (Hart, Johnson, Brown, Dickson and Clarkson, 1989). Students tended to see ratio as an additive operation rather than a multiplicative one, and in addressing ratio problems essentially replaced multiplication by repeated addition. In the Concepts in Secondary Mathematics and Science (CSMS) study carried out in the 1970s, many students who answered questions correctly did so by using intuitive strategies (such as doubling or halving) that they had not been taught, rather than adopting the strategies that had featured in lessons. The latter would be more likely to be usable in advanced mathematics as well as for comparatively simple arithmetic problems. The follow-up study Strategies and Errors in Secondary Mathematics (SESM) investigated the incorrect ‘addition strategy’ for enlargements – adding a fixed amount to enlarge a figure, rather than multiplying by the scale factor – and found the same avoidance of multiplication. Later work with primary school children confirmed that there are difficulties in teaching basic concepts for the topic.

Subsequently, two major handbooks of research on teaching and learning were produced under the aegis of the American National Council of Teachers of Mathematics. Both contain substantial syntheses of ratio-related research, including findings of relevance to this paper. In a chapter on rational number, ratio and proportion, Behr, Harel, Post and Lesh (1992) stated their belief that “the elementary school curriculum is definitely failing to include the basic concepts and principles relating to multiplicative structures necessary for later learning” (Behr et al., 1992, p. 300). Fifteen years later Lamon (2007), in her chapter on rational numbers and proportional reasoning, emphasised the need to look at the topic across the grade levels. She asserted that “Many scholars share the view that proportional reasoning is a long-term developmental process” (Lamon, 2007, p. 637). Both chapters, therefore, highlight – explicitly or tacitly – a need for school curricula to focus on the development of ratio-related concepts for all age levels from that at which multiplicative structures are first introduced.

Livy and Vale (2011) provide a recent and succinct summary of evidence that students in the middle years of schooling have poor understanding of ratio and proportional reasoning. Their paper is valuable also for teacher education. In their study of 297 prospective teachers, they found low levels of correct responses to relevant ratio and proportion test items. Apart from this, however, few studies seem to focus on teacher education and few have examined teachers’ knowledge of ratio for teaching. Hence, a project aiming to collect data from prospective and practising teachers – the ATEE (Association for Teacher Education in
Europe) Ratio Project – was started in 2011 and is ongoing (Berenson, Oldham, Price & Leite, 2013). It is described more fully in the fifth section of this paper.

INTERNATIONAL FOCUS: RATIO-RELATED CONCEPTS IN INTERNATIONAL MATHEMATICS CURRICULA

For the present discussion of ratio and proportional thinking in mathematics curricula, the most useful sources of data are those that reveal the emphasis given to curricular topics across different countries or education systems in some consistent form. Attention is therefore focused on the major cross-national studies that included some form of curriculum analysis: those conducted by the International Association for the Evaluation of Educational Achievement (IEA) and, for a short period, the International Assessment of Educational Progress (IAEP). Studies in which Ireland took part are obviously of particular relevance. In this section, a chronological account is presented, with a summary of general findings for each study being accompanied by references to the Irish data where relevant.

However, a word of caution is needed. Analysis of intended curricula – those typically decreed at national level and published in syllabus documents or their equivalents – is usually undertaken by ‘experts,’ who despite their best intentions may bring their own priorities and their own interpretations to the documents. The author has played the ‘expert’ role herself in this context, and she is well aware of her fallibility! Thus, judgments as to relative importance of topics within a curriculum are just that: judgments; they have a rather different status from the accumulated reports from teachers as to the amount of emphasis or time they give to particular topics in the classroom (hence, in implementing the intended curriculum). These too reflect judgments and also – in the case of time – estimates by individual teachers, but their collective responses may provide somewhat ‘harder’ data.

The first study involving Ireland was the IEA’s Second International Mathematics Study (SIMS), carried out in the early 1980s. Ireland participated in the curriculum analysis, though not in the student achievement component. To collect data from participating education systems, a ‘content × behaviours grid’ was devised, listing topics likely to figure in intended curricula. For Population A (the grade containing the greatest number of 13-year-old students), the category ‘Ratio, proportion, percentage’ was one of those listed in the Number section of the grid. Internationally, this category was rated as ‘Very important’ (reflecting that raters in many education systems gave it that status); the Irish raters deemed it to be only ‘Important’ (Travers & Westbury, 1989). However, the Irish raters were unusually sparing with their use of the ‘Very important’ rating (Oldham, 1989), so little can be deduced from the disparity.

Ireland participated also in the two IAEP studies, one carried out in 1988 and the other in 1990-91 (Martin, Hickey & Murchan, 1992). Data in the international report on the first of these (Lapointe, Mead & Phillips, 1989) are insufficiently detailed to be useful here, but the second gives relevant curricular information for the 13-year-old population (Lapointe, Mead & Askew, 1992). In the section of the report focusing on Number, so-called Ratio and Proportions is one of five topics, the others being Whole Number Operations, Common Fractions, Decimal Fractions and Percentages. Responses from Ireland show that the
The percentage of schools emphasising Ratio and Proportions ‘a lot’ in the modal grade for 13-year-olds (First Year) was lower than that for the other four categories. However, this is not unusual; six of the fifteen systems for which data are presented have Ratio and Proportions as the least emphasised category, with a further five having it bottom equal or in the bottom two. In fact the Irish percentage for Ratio and Proportions is the sixth highest of the fifteen systems. For systems giving lower emphasis, however, it is not clear whether the topic is studied more intensively in other grades – possibly earlier ones, possibly later ones. Altogether, therefore, the work of SIMS and the IAEP highlights the difficulties encountered, as well as insights obtained, in looking at cross-sectional descriptions of national curricula. They leave the case for Irish under-emphasis on ratio-related concepts ‘not proven.’

The Third International Mathematics and Science Study – now called TIMSS 1995, and the first study in the TIMSS series – attempted a longitudinal curriculum analysis. This examined not only curriculum guides, specifying the intended curriculum, but also textbooks, as likely indicators of the topics and skills actually taught. If the earlier studies were sometimes short of curricular data, TIMSS suffers from the opposite complaint; so much information is collected that only a small part can be presented in international reports, and the wood can be lost amongst the trees. The chief focus, therefore, is often on the grades at which student testing is carried out. For TIMSS 1995, these were American third, fourth, seventh and eighth grade. The instrument used for data collection contained a section on Proportionality, with categories ranging from the meaning of ratio and proportion to topics involving trigonometry and the slope of straight-line graphs. However, the findings reported in the curriculum analysis volume for the study (Schmidt, McKnight, Valverde, Houang & Wiley, 1997) focus only on ‘Proportionality Problems.’

Ireland participated in TIMSS 1995 at both primary level (therefore, administering the student achievement tests to Third and Fourth class children) and second level (with testing for First and Second Year). No Irish coverage of Proportionality Problems was reported for the two key primary school grades, reflecting the fact that ratio made its first (formal) appearance in the primary curriculum of the day in the syllabus for Fifth and Sixth Class (Department of Education, 1971). Details are given in the next section of this paper. The situation was fairly typical for participating education systems; according to Schmidt et al. (1997), at least half of the systems first introduced Proportionality Concepts and Proportionality Problems at some stage in grades 4-6. The findings for second level may be more significant. While Proportionality Problems are reported as appearing in the Irish curriculum before, at and after Second Year – again typical of the situation for most participating systems – such problems were not found in the textbook for Second Year. The point is followed up in the next section.

Ireland did not take part in TIMSS again until 2011, and then only at primary level. Unfortunately, therefore, internationally comparable data for the Irish second-level curriculum were not collected. In any case, curricular data are sparingly reported for TIMSS 2011, so a better picture of what other systems were offering recently can be obtained by examining the curriculum analyses for TIMSS 2007 (Mullis, Martin & Foy, 2008). Of 37 education systems returning data at fourth grade, 18 reported that problems involving proportions are in the intended curriculum for most students at that level; all but nine indicated that the topic was
intended to be introduced at least by sixth grade. If Ireland had participated, the country would probably have been in the minority group, depending on the exact definition of proportionality problems; ratio and proportion are not mentioned in the current primary school curriculum (Department of Education and Science, 1999). This also is discussed below. Almost all of the 50 systems participating at eighth grade reported that ‘Ratios’ were in the intended curriculum for most students at that grade, with 33 indicating that the topic was to be introduced at sixth grade or before. Again, therefore, if Ireland had participated, the country would have been in the minority group of ‘late starters.’

For TIMSS 2007, describing the degree of emphasis intended to be given to ratio-related topics was outside the scope of the report. With regard to the curriculum implemented in class, however, teachers’ responses indicated that, for the participating systems, the percentage of eighth-grade students who had been taught ‘Ratios’ was typically in the nineties; the international average of 87% was pulled down by data from systems in which the topic was not in the eighth grade curriculum. Additionally, in the Geometry section of the curriculum analysis for eighth grade, similar triangles – involving ratio concepts – are in the intended curriculum for half of the participating systems, with most others introducing them in ninth or tenth grade. In the Irish curriculum in operation at the time (Department of Education and Science, 2000), theorems implicitly or explicitly mentioning ‘equiangular’ triangles appear only at the end of the Geometry section of the Junior Certificate syllabus and only for Higher course students, so can perhaps be assigned typically to Third Year – ninth grade – hence again placing Ireland among the later starters.

Altogether, therefore, the evidence points towards a tentative conclusion that Ireland – in comparison to many other countries – is now a late starter with regard to introducing ratio-related concepts, and perhaps gives somewhat perfunctory attention to them at lower secondary level. This is not necessarily a criticism. It may be a conscious choice, reflecting matters such as importance given to other topics and the time available for mathematics in Irish schools. However, if it is an unconscious choice – made somewhat incidentally without due regard for consequences – then it may need further consideration. The issue is discussed further in the next section.

NATIONAL FOCUS: RATIO-RELATED CONCEPTS IN IRISH MATHEMATICS CURRICA

In this section, the Irish curricula are considered through an examination of Irish curriculum documents and textbooks. Other natural sources for consideration, especially in view of the major effect they have on the implementation of intended curricula, are the State examination papers, together with the Chief Examiners’ Reports (where available). However, since the international studies pay little heed even to high-stakes examinations, which are also under-represented in research (Cai & Howson, 2013), it is hard to find systematic accounts of international data to provide comparisons in the area. In view of this fact, investigation of such documents is left – as mathematics textbooks say – as an exercise for the reader. The author’s own attempt at the exercise has in general supported the main contention of this paper: examination papers and reports, like syllabus and textbook materials (as illustrated below), have not been set out in such a way as to promote a view of ratio concepts as being
'golden threads’ throughout school mathematics; rather, the emphasis has been on isolated procedures. Describing the evidence is outside the scope of the paper.

Curriculum documents in Ireland over the years have been interpreted via custom and practice as well as via analysis of their text. The focus on custom and practice was particularly marked up to the 1960s, with curricula, or rather syllabuses, consisting only of short lists of topics to be covered. The expected scope and depth of coverage became familiar, for example through the way that topics were tested in State examination papers, or perhaps with reference to some well-known textbooks – books not designed specifically for the Irish markets [1]. However, since that time, the scope of curriculum documents has continually increased, with more detailed content lists being augmented by the specification of aims and objectives (or, latterly, learning outcomes) and by guidelines that are meant to support teachers in implementing curricular intentions appropriately. Sample examination papers now feature also. Moreover, ever since the adoption of ‘modern mathematics’ – to a greater extent than was the case in the neighbouring jurisdiction – textbooks were purpose-written for the Irish curricula, and so can provide additional evidence of how these were likely to be implemented.

The situation as regards ratio-related concepts in the primary curriculum has already been mentioned. The so-called ‘new’ primary curriculum was introduced in 1971 (Department of Education, 1971). While it provided brief lists of content to be covered in each of the two-year bands from Junior Infants to Sixth Class, it augmented these with many attractively presented suggestions for how children might be introduced to the concepts. However, although fractions “as ratios” were mentioned in the syllabus for Fifth and Sixth class (p. 209), the ratio aspect was not discussed in the guidelines. When the curriculum was revised in the 1990s (for implementation as regards mathematics in 2002), the content load in Fifth and Sixth class was reduced, and ratio was one of the concepts removed.

At second level, syllabuses were published in Department of Education’s Rules and Programme for Secondary Schools, issued annually. In what follows, the versions cited are among many that list the relevant syllabuses. The junior cycle syllabus in operation in 1970 presented ‘Ratio and proportion’ as the second item in the list of contents, suggesting early treatment in the three-year period (Department of Education, 1972). While ratio concepts were implicit in other topics (such as elementary trigonometry and co-ordinate geometry), there was no further explicit mention. The syllabus introduced in 1973 provided more detail for several topics (Department of Education, 1974). It listed ratio separately from proportion (e.g. p. 65) and stated one of the synthetic geometry theorems – for students taking the course at Higher level – in the form “Two sides of a triangle are divided proportionately by a line drawn parallel to the third side” (p. 77). Its successor, introduced in 1987, used a similar format (Department of Education, 1997). However, “Ratio and proportion” are again linked in brief entries for what were then called Syllabuses A and B (pp. 31 & 35), with Syllabus C – a less demanding course – omitting explicit mention of the concepts, though it introduced “Drawing to scale” (p. 40). The synthetic geometry theorem noted above is presented in the form “A line drawn parallel to one side of a triangle divides the other sides in the same ratio”; similar triangles are included via the theorem “If the angles of two triangles are, respectively, equal in measure, then the lengths of the corresponding sides are proportional” (p. 34). Again
it should be noted that ratio concepts are implicit in other entries, notably with regard to co-
ordinate geometry and trigonometry; but they are not emphasised by use of the terms. The
organization of the 1973 and 1987 syllabuses suggests that, typically, the basic work on ratio
and proportion would occur in First Year, with the geometry theorems for Higher level being
addressed in Third Year.

The next syllabus, introduced in 2000, had a mandate to make changes to the syllabus content
only where these were deemed really necessary: necessary because areas were viewed as
problematic, or because a response was needed to changes in the primary curriculum
(Department of Education and Science, 2000; Department of Education and Science /
National Council for Curriculum and Assessment [DES / NCCA], 2002). For ratio, no
significant changes were made. The area was not identified as problematic; it was already
addressed ab initio in second-level textbooks, and the Guidelines for Teachers accompanying
the syllabus (DES/NCCA, 2002) – a ‘first’ at this level – did not mention ratio as being
“among [the topics] excluded from the revised [primary] curriculum” (p. 18). The Guidelines
do include an interesting lesson idea on using the Golden Ratio to introduce the topic, and
ratio is highlighted in another lesson idea on “discovering pi” (p. 36). Analysis could be
continued, but perhaps the point has been made that the intended curricula have not been
written in such a way as to highlight the importance of ratio and proportional thinking as
major curricular themes. As the author was involved in work on the Irish curricula in the
1990s, perhaps she bears some of the blame.

With regard to textbooks – typically influential in determining curriculum implementation –
those purpose-written for the Irish syllabuses from the 1960s initially may have focused
intentionally on modern material that was new and unfamiliar to many teachers. In one series
of texts, for example, ‘Stage 1’ devotes one chapter late in the book to ratio and proportion,
the latter being introduced only via the concepts of direct and inverse proportion (Holland &
Madden, 1970). The focus is somewhat procedural, whereas ‘modern’ topics are given
somewhat more context and rationale. In ‘Stage 2,’ trigonometric functions are introduced as
the coordinates of points on a unit circle, rather than through emphasis on their properties as
ratios (Holland & Madden, 1971). It might have been expected that later series of texts would
have redressed the balance somewhat as teachers’ familiarity with pre-1960s texts weakened.
Moreover, the initial Irish textbooks were slender volumes (produced in haste to support new
curricula, and under budgetary constraints?); teachers felt the lack of more substantial sets of
exercises, such as appeared in the older textbooks (Oldham, 1980). Recent books are much
more substantial. However, the patterns set with regard to ratio-related concepts seem
generally to have persisted. Even in the 2000s, ratio tends to be treated explicitly only early
in the cycle, with tacit or sparse explicit reference to the concept later in topics such as
coordinate geometry, trigonometry (though now typically using ratio rather than unit-circle
definitions; see for example Humphrey (n.d.) and Humphrey, Reeves, Guildea and Boylan
(2011)), and (for Higher course students) synthetic geometry theorems.

Thus, did Irish students meet ratio concepts ‘head on’ only in First Year and in Third Year
(revising for the Junior Certificate, or in geometry theorems where relevant)? Have our
curricula – both intended and implemented – emphasised ratio-related concepts less than do
curricula in other countries? It seems that the answer may be “yes.” Neither the curriculum
documents nor the textbooks gave much support to teachers in helping students make those
conceptual connections that Hiebert and Grouws (2007) highlight as being so important for
developing understanding. Of course, such documents cannot cover every aspect of all
topics; there will continue to be reliance on custom and practice, and on the content
knowledge and pedagogical content knowledge of good teachers who draw attention to the
connections. Perhaps, however, there was room for development in the area.

As suggested above, some such development can be seen in the extensive documentation
produced around Project Maths. In the expanded syllabus documents, ratio and proportion do
appear as at least slender ‘golden threads,’ with references occurring in introductory
paragraphs and in specific content statements in different content strands. Moreover, there
are teaching and learning plans on ratio-related concepts, including for example one
highlighting the role of ratio in trigonometry and aimed at Second-Year students.
Documentation is evolving fast, so the reader is referred for up-to-date information to the
Project Maths website (http://www.projectmaths.ie).

RATIO-RELATED CONCEPTS IN TEACHER EDUCATION

The paucity of research on teachers’ knowledge for teaching ratio-related concepts was
mentioned above, and attention was drawn to the ATEE Ratio Project initiated recently in the
area (Berenson et al., 2013). In this section, the study is described and the findings from Irish
involvement are highlighted.

For the initial phase of the study, the following research questions were chosen:

a) What meanings do prospective teachers at primary and secondary levels in [specific
institutions in the participants’ home countries] give to the term ‘ratio’?
b) What multiple representations do these prospective teachers associate with the term
‘ratio’?
c) Do the prospective teachers’ descriptive meanings and representations indicate
different levels of understanding for teaching ratio?

The instrument for data collection was designed to be administered within ten to fifteen
minutes in appropriate classes: typically, of undergraduate or graduate students in teacher
education courses. Five items were included, the ones relevant to this paper being:

1. What does the term ‘ratio’ mean to you?
2. How do you represent a ratio using mathematical symbols?
3. Draw several representations of how ratios are used.

The items were arranged on a single sheet of A4 or – for American participants – letter-size
paper, with appropriate spaces left to allow the participants to write or draw their responses.
Localised versions of the instrument, chiefly reflecting differences in language, were prepared
to allow for data collection in three countries: Ireland, Portugal and the USA.

The study used opportunity samples in participating researchers’ institutions, not
representative samples, so analysis was focused on identifying similarities rather than
differences. A grounded theory approach was used. Data from 158 students were collected
and analysed, with the members of the research team sharing insights progressively. Based on the data from items 1 and 4 in particular, and using Skemp’s (1976) classification, conjectures were eventually made around whether or not the participants’ responses might indicate *relational understanding* of the concept of ratio, as shown in Table 1. It should be noted that the authors were not able to make conjectures around participants’ *instrumental understanding*, as none of the items asked them to carry out a calculation or other procedure.

<table>
<thead>
<tr>
<th>Displays relational understanding</th>
<th>Does not display relational understanding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meaning of ratio reflects two variables</td>
<td>Meaning of ratio does not reflect two variables</td>
</tr>
<tr>
<td>Provides many representations</td>
<td>Provides few representations</td>
</tr>
<tr>
<td>Uses multiple types of representation</td>
<td>Uses few types of representation</td>
</tr>
<tr>
<td>Cites / draws relevant applications</td>
<td>Provides symbolic representations only</td>
</tr>
</tbody>
</table>

**Table 1: Conjectured indicators of presence or absence of relational understanding**

Further points can be made. Respondents in general did not relate ratio to advanced curricular topics, such as trigonometry or rate of change. For item 4, some participants responded using words or symbols, rather than providing the drawings specified – though some did produce art work, of varying levels of sophistication; some made no response at all.

The researchers also focused on their own institutions’ data, looking for interesting patterns. The 16 Irish respondents were Professional Diploma in Education students – graduates studying to become secondary teachers – taking Mathematics Pedagogy as one of their two ‘methods’ modules. For some, mathematics was their main subject. Others, typically scientists and business studies majors, took Mathematics as their minor ‘methods’; such students’ degrees often contain little advanced mathematics. (Similar variety in mathematical backgrounds obtained for participants in the other countries; for instance, some were prospective mathematics teachers, some were prospective science teachers and some were undergraduates preparing for primary teaching.) A brief summary of Irish findings follows.

For their meaning of ratio (item 1), 11 of the 16 Irish students referred to either comparisons or relationships or both; this was taken as an indication that, explicitly or tacitly, they recognised the importance of two variables in understanding the ratio concept. Of the other five, three were among the seven students who mentioned proportion; the remaining two referred to notions of dividing or splitting. Also, four students referred to fractions, though none gave these as their only meaning for ratio. Fractions may be associated more with part-whole situations than with ‘true’ ratio (Berenson et al., 2013).

For the mathematical symbols that represent ratio (item 3), all students used the colon, with or without numerical or algebraic examples (as in “2 : 3” or “x : y”); one of them did so incorrectly. Three also used the word “to” or fraction notation.

With regard to showing how ratios are used (item 4), seven students, including those identified as mathematics specialists, provided drawings of some sort: geometric figures
(similar shapes, or drawings for example indicating length and width on a rectangle), bar charts, or artwork of various kinds. Of the other nine, four made no response. The remaining five offered only symbolic or verbal representations; two, presumably reflecting their business studies background, wrote down “assets : liabilities.”

Overall, therefore, it can be said that in terms of the characteristics set out in Table 1, the Irish students showed varying degrees of relational understanding. On the whole, they did not provide very many representations of ratio; some did offer multiple representations, but often of the same type. For many students, the responses were consistent with a focus on the work covered in the textbook chapters on ratio and proportion, mentioned above: hence, say, dividing a sum of money in a given ratio. The mathematics specialists were more inclined to offer geometric or statistical representations, but no-one mentioned trigonometry or slope. Moreover, the responses in general were rather terse or scrappy, especially by comparison with those from the other participating groups. This may have reflected some local conditions on the day (the data were collected during the last lecture of the module, and students may not have engaged fully); alternatively, it may have been evidence that, given limited time, the students did not easily call to mind fluent and connected knowledge in the area [2].

CONCLUSION

Enough evidence has been produced to suggest that, although Project Maths shows signs of improvement in this respect, there is a case to answer with regard to the emphasis – or lack of emphasis – on ratio in Irish mathematics education. As a minimal move, therefore, it is appropriate to air the issue. Additionally, a larger-scale study than that reported in this paper might profitably be undertaken in order to examine the focus on ratio concepts and proportional thinking in Irish curriculum documents, textbooks and examinations. Student performance could also be investigated. Baseline data might be collected from school students who have not yet experienced the full impact of Project Maths, and compared in a few years with data from those who have experienced a settled version of the course. Data might also be collected from prospective and practising teachers over a longer period, perhaps through ongoing participation in the ATEE Ratio Project, to monitor (hopefully) growing understanding in the system. Meanwhile or in any case, there is an obvious role for teacher education, both initial and in-service.

As noted initially, this paper has been written in an attempt to open up discussion. The MEI conference offers the ideal forum for instigating both discussion and research. If it were to transpire that the ‘ratio problem’ lies more with the author than with Irish mathematics education … that would be a satisfactory outcome!

NOTES

1. Thanks are due to Professor Tony O’Farrell for this observation, and to Elizabeth Caird for information on books of the time.

2. Further data have been collected from the corresponding class of students this year. An amended form of the instrument was used, aiming to encourage a wider range of response to items 3 and 4, and in particular explicitly to allow non-pictorial representations in item 4. Details will be reported elsewhere, but it can be said here that, despite satisfactory engagement by the students, the range of responses is similar to that for the original set.
REFERENCES


Textbooks are an important resource in Irish mathematics classrooms, which can have both a positive and negative impact on teaching and learning. The Project Maths initiative is prompting teachers and students to cross boundaries and interact with mathematics in ways that had not been considered previously. Publishers have produced new texts in response to the expectations of the revised curriculum and the changed needs of the classroom. This paper presents a framework to consider the degree of novelty presented in tasks found in mathematics textbooks. Novelty is something that has been referred to, yet not addressed directly, in existing frameworks for the analysis of mathematical tasks. A particular strength of our framework is that it takes into account the experience of the solver, as opposed to just focusing on how a task has been structured. Sections of textbooks currently being used in Irish classrooms at second level have been analysed using this framework and the results indicate that while all textbooks incorporate a significant level of novelty, there is still room for more novel tasks to be included.

INTRODUCTION

Textbooks are widely accepted as a commonly used resource in mathematics classrooms. According to Jones, Fujita, Clarke and Lu (2008) “on average, internationally, over 60 per cent of teachers report using a textbook as a supplementary resource” (p. 142). Little research has been conducted into the nature of post-primary mathematics textbooks in Ireland (Conway & Sloane, 2005). However, there is some evidence that textbooks play an important role in Irish classrooms. O’Keeffe and O’Donoghue (2009) suggest that “over 75 per cent of Irish second level teachers use a textbook on a daily basis” (p. 283). Even in early-childhood mathematics classrooms, Dunphy (2009) identifies teachers as using the textbook for “structuring the programme of work and for guidance” (p. 118). A lot of the time in the classroom appears to be related to textbook usage and very often it is the only resource which students have access to during the lesson aside from the teacher, while most of the problems assigned for classwork and homework come from the textbook (Project Maths, 2012).

Harbison (2009) points out that “textbooks have a role in suggesting a possible pathway for navigating through the strands and strand units of the Mathematics curriculum” (p. 131), while Moffet (2009) acknowledges that “different textbooks lead to different instruction, and different instruction leads to different learning results” (p. 265). The latter may explain why the use of textbooks can be problematic. O’Keeffe and O’Donoghue (2009) suggest that mathematics textbooks in the Irish system promote “retention and practice, with little focus on active learning” (p. 290). Classroom inspections in Ireland have shown that teaching is highly dependent on the class textbook which has a tendency to reinforce this drill and practice style (NCCA, 2005). The Project Maths Development Team (Project Maths, 2012) has cautioned teachers in their choice of textbook: it points out that there is no single textbook which can
suit the learning needs of all students and it has advised schools, when choosing a textbook, to take into account the abilities, needs and interests of their students, as well as the quality of the book. In this paper, we use our novelty framework to classify tasks on the topic of Sequences and Series from two of the textbook series written for use in Project Maths classrooms.

When considering such a choice, it would be useful to look at the differing levels of novelty offered by textbook tasks. This paper presents a framework to consider the degree of novelty presented in tasks found in mathematics textbooks. The framework has been applied to sections of textbooks currently being used in Irish classrooms at second level. A particular advantage of this framework is that it considers the previous mathematical experiences of the student within the textbook chapter and how this relates to the task posed, something that may have been neglected in existing frameworks. The reasons for considering novelty will be explored in more detail in the next section.

There has recently been a lot of change in the mathematics classroom at second level in Ireland. Project Maths is an initiative, led by the National Council for Curriculum and Assessment (NCCA), to bring about positive change in the teaching and learning of mathematics. This new curriculum introduced to schools has a greater emphasis on conceptual understanding and advocates the development of problem-solving skills. Rather than practicing routine procedures to solve predictable problems, students are encouraged to make connections between different mathematical ideas and to develop a more flexible way of thinking. One of the key objectives of the Project Maths curriculum (NCCA, 2012) is “the ability to apply their mathematical knowledge and skill to solve problems in familiar and unfamiliar contexts” (p. 6). As a result of these changes, mathematics classroom activities are now moving away from instrumental understanding ‘knowing how’ and more time is devoted to developing relational understanding ‘knowing why’.

The publishers of mathematics textbooks have gradually produced new texts in response to the changed needs of the classroom. O’Keeffe & O’Donoghue (2012) carried out a review of the Project Maths textbooks available for schools using a modified instrument from the 1995 Third International Mathematics and Science Study and concluded that “all textbooks included in the study fall short of the standard needed” to support the intended Project Maths curriculum effectively, while acknowledging that all textbooks examined “display a genuine attempt to match Project Maths expectations but no one textbook meets all the Project Maths expectations” (p.21). Given the importance of the textbook as a classroom resource and the need for more research on the nature of mathematics textbooks at second level in Ireland at present, it is evident that there is a need for further work to be completed on analysing textbooks.

**FRAMEWORKS AND NOVELTY**

Before analysing tasks in a textbook, it is necessary to be clear on what is exactly being examined. Mason and Johnston-Wilder (2006, p. 4) define a task to be “what learners are asked to do in the mathematics classroom”, while a concept is described as “a label for the flow of images, thoughts, sensitivities, connections, possible actions and so on associated with
an idea”. For this paper, a task is considered to be an activity where a student interacts with a
mathematical topic by attempting to solve a textbook exercise either as homework or within
the classroom. We have used various existing frameworks for the classification of textbook
tasks: these include the Levels of Cognitive Demand (LCD) framework of Smith and Stein
(1998) and Lithner’s (2000) reasoning framework, which are outlined below.

Stein, Grover and Henningsen (1996) examine mathematical tasks in terms of how they
influence the kinds of thinking processes in which students engage when solving tasks. The
framework for LCD used by Stein et al. (1998), describes four levels of cognitive demand for
tasks. The LCD framework has four levels: Lower level (memorisation) - these tasks involve
either reproducing previously learned facts, rules, formulas or definitions or committing these
to memory; Lower level (procedures without connection to meaning) - these tasks are
generally algorithmic and the use of a procedure is either specifically called for or is evident
from prior instruction; Higher level (procedures with connection to meaning) - these tasks
focus students’ attention on procedures for the purpose of developing deeper levels of
understanding; Higher level (doing mathematics) - these tasks require complex and non-
algorithmic thinking. Charalambous, Delaney, Hsu and Mesa (2010) use the LCD framework
in their analysis of textbooks but do not take the prior experience of students into account, on
the basis that the presence of a certain topic in textbooks used in earlier grades or years is not
a guarantee that students understand the material.

Lithner (2008) characterises key aspects of reasoning when engaging with mathematical
tasks, characterizing such reasoning as either imitative or creative reasoning. Imitative
reasoning involves the use of memorization or well-rehearsed procedures while creative
reasoning requires novel reasoning with arguments to back it up and attached to appropriate
mathematical foundations. Lithner (2008) asserts that a novel task requires students to engage
in complex thinking in order to come up with an approach or suitable method to find a
solution. The student either devises a new (in the sense that it has not been encountered or
utilised previously) sequence of reasoning to formulate a solution or a forgotten sequence is
recreated. If a task can be solved by imitating an answer or a solution procedure then it is not
seen as novel. The LCD framework also touches on the notion of the student’s familiarity by
making reference to prior experience with a task but does not provide explicit criteria for
judging how such experience can be gauged. Similarly, Lithner’s characterisation of
reasoning refers to the presence of novelty and tasks being seen as familiar by the student.
However, it does not provide a clear method for measuring familiarity in tasks. These existing
frameworks attach importance to novelty but, given the absence of any method for gauging
such novelty in tasks, we found it necessary to create a new framework focused specifically
on novelty and measuring the degree of its presence in tasks. This was informed through our
experience of classifying tasks using the LCD framework and Lithner’s reasoning framework.

Other authors have considered students’ prior experience when faced with a task. Berry,
Johnson, Maull and Monaghan (1998, p.15) encountered an interesting issue when attempting
to characterize routine questions. In their words, they see the implication that routineness is
“located in a question rather than being a psychological construct of the relation between an
individual, or group and a question” as problematic and conclude that the psychological
relation is likely to be a socio-psychological one. In other words what is routine for one individual is not necessarily so for another. Selden, Mason and Selden (1989, p. 45) go some way towards acknowledging this aspect of human experience by identifying two components in a problem, namely, task and solver. A problem is defined by Selden et al. as a non-routine or novel task, the solution of which consists of finding a method of solution and carrying out such a solution. Selden et al. (1989, p.45) describe a solver as “usually a person, but possibly a group of persons or a machine”. Selden, Selden and Mason (1994, p.67) take the view that while some problem-solving studies do not explicitly mention the solver, it is essential to consider the solver and the skills and information that are brought to a task when examining a mathematical task. They point out that “tasks cannot be classified as problems independent of knowledge of the solver’s background” (p. 67). Most importantly, Selden et al. (1989, p.45) highlight the fact that the solver “comes equipped with information and skills, perhaps misconceptions, for attempting the task”. A consequence is that novel problems cannot be solved twice by the same person without a loss of novelty, as the solver would possess a method for solution the second time.

Berry et al.’s and Selden et al.’s work suggests that a solver’s previous experience is worthy of attention. It is clear that this is something that should be considered when attempting to classify tasks within a framework. In terms of textbook tasks, a certain amount of repetition is to be expected as it assists students in becoming more comfortable with a particular concept; however, it is useful to see how much novel material is introduced by authors in order to challenge the student in terms of problem solving and to promote the use of non-routine thinking and reasoning. We are also interested in examining whether the recommendations advocated by the Project Maths initiative are being implemented in textbooks.

FRAMEWORK FOR MEASURING NOVELTY

It is the aim of this paper to provide a new framework for the identification of novelty within tasks through the provision of clear criteria for three different categories described as novel, somewhat novel and not novel. Figure 1 gives an outline of the criteria used in this classification. This framework requires the examination of the expository material and each example in a chapter and the identification of the skills or concepts that are being demonstrated in the exemplar. For this framework, skills are taken to refer to the methods and techniques used in the solutions to tasks. Please note, it is not necessary for all characteristics in the description of the categories to apply in order for a task to be classified under a particular label. However there should be sufficient evidence to distinguish between the different categories and as many characteristics as possible should be identified before settling on a particular classification.
Novel | Skills involved in finding the solution are not familiar from preceding exercises or from any previous point in the chapter being analysed. The mathematical concept involved is not familiar from previous exercises or examples. Significant adaption of the method outlined in examples and exercises must be made in order to get the required solution.

Somewhat Novel | The presentation of the task makes the question appear unfamiliar. However its solution requires the use of familiar skills. The context (perhaps the use of an unfamiliar real-world situation) makes the task appear unfamiliar but familiar skills are used in its solution. A new feature or aspect of a concept is encountered but the solution to the task only involves the use of familiar skills. A minor adaption of the method outlined in the examples has to be made in order to get the required solution. The skills required are familiar but the use or application of such skill is slightly modified.

Not Novel | The presentation, context and concepts of the task are familiar. The solution to the exercise or problem has been modelled in preceding exercises or has been encountered earlier in the same chapter. The skills required are very familiar to the user and the method of solution is clear due to the similarity between the exercise and preceding examples and exercises.

**Figure 1: Framework for classification of novelty in tasks encountered in textbooks.**

**DESCRIPTION OF FRAMEWORK**

**Novel**

Something is said to be novel when the solver requires skills necessary for finding the solution or mathematical concepts that have not been covered in preceding exercises or at any previous point in the chapter being analysed. There cannot be any substantial similarity between the given exercise and the previous examples in this case.

The novel exercise should be original and essentially demand a new form of thinking from the solver that has not been encountered before in that chapter. This includes the situation where a task requires different methods of solution when the solver is only familiar with one method from the contents of the chapter, or when the solver has to make a significant change or alteration to the method that is familiar in order to find the required solution.

Once a novel exercise has been encountered, it diminishes the novelty for any similar exercise that follows it. While an exercise may be novel when first encountered, any exercises that follow which are of a similar likeness or approach can no longer be classified as novel.
Somewhat Novel

If there is only a superficial difference between the examples and the exercise, then it is labelled as *somewhat novel*. The presentation of the question may be different to preceding examples but when the solver goes about solving the task, it is apparent that the skills required are quite familiar from the preceding examples or earlier in the chapter. Such differences can occur when a task is presented in a different context such as a real-world scenario, which can serve to render the task unfamiliar to the solver. There can be some difficulty for the solver when determining how to solve the task but this is diminished once the familiar material is identified. Such variation necessitates the creation of an intermediate category to acknowledge that a task can have unfamiliar aspects but relies on skills that are actually quite familiar at that point.

Not Novel

An exercise or task is *not novel* if its solution has been dealt with in preceding exercises or other examples in previous parts of the chapter. The task could be direct repetition of material covered in examples or very similar to it. If material is not covered in the examples immediately preceding a set of exercises but the solver has experience from an earlier part of the chapter then the task is not novel.

By using these criteria, we suggest that it is possible for the researcher to get a clearer picture of what a solver has been exposed to in terms of the textbook examples and previous exercises. This gives an insight into how much novelty is present when an exercise is attempted. Such information complements the goals of the other frameworks as it allows one to look not only at a task but also the background against which the task is set. This framework allows one to consider the previous experiences of the solver within a mathematics textbook chapter and how this impacts upon the solver when solving a task.

**METHODOLOGY**

**Creation of framework**

When creating the framework, it was necessary to consider a number of elements before finalising the criteria for classifying tasks. For this analysis, the examples and information preceding a set of exercises for each textbook were examined. These included definitions, explanation of key words, exemplars demonstrating key concepts and illustration of methods of solution for problems. Before classifying tasks, each feature of an exercise was identified clearly. These included using a procedure or formula, proving a mathematical statement, constructing or interpreting a diagram, using a definition, investigating mathematical properties and more. Having considered these features of the tasks and examined the skills necessary for solution, the experience gained by the solver from the preceding examples and previous exercises was considered. Previous chapters were not examined as it is not usual in Irish schools for the teacher to follow the order of the textbook in a linear fashion and thus it is not possible for us to say which chapters a student would have seen. In both textbook series studied here, the topic of Sequences and Series is covered in a single chapter. Of course, students could have other experiences but the classification is confined to the material
contained in the textbook chapter only. Each of the three authors classified the tasks independently. Then the individual classifications were compared and elements influencing the choice of classifications were discussed. This analysis was carried out repeatedly using various topics, different chapters and textbooks in order to consider all aspects that could be encountered when classifying tasks for novelty. The criteria for classification were revised as necessary until it was felt that all aspects of novelty in a task were accounted for. The three classifications of novel, somewhat novel and not novel were settled on and the table in figure 1 was eventually agreed upon. It was decided that questions which formed multiple parts of an exercise would be considered as stand-alone tasks while preceding questions were taken into account when considering their impact on the next exercise. Any questions containing ambiguity or unclear elements were not considered for classification.

EXAMPLES OF CLASSIFICATION OF TASKS IN FRAMEWORK

This paper looks at applying the framework to the topic of sequences and series in two textbook series recently introduced on the Irish market and now being widely used in schools. These textbook series have been given the pseudonyms Text A and Text B. The topic of sequences and series was chosen because it was present at both Higher and Ordinary levels and allowed for the comparison of the treatment of the topic between these levels. While the series of textbooks have each been introduced at both junior and senior cycle, we decided to focus on the senior cycle material particularly because of the high-stakes examination that accompanies it. One chapter from each textbook series at each of Higher level (HL) and Ordinary level (OL), has been analysed using this framework (that is, four textbook chapters in all) and the results are presented later.

To illustrate how the framework is used and the criteria are applied, three examples showing classifications of tasks as novel, somewhat novel and not novel are given. In order to preserve the anonymity of the textbooks, it was decided to illustrate the framework using an older book, which was widely used in Irish second level schools during the 1990s (Roantree, 1994). An extract from the explanatory material in this chapter is given below. A definition of an arithmetic sequence, a formula for finding the general term in an arithmetic sequence, and two worked examples arising from this are provided. This is followed by a formula for finding the sum of terms in an arithmetic series, accompanied by a third worked example. Three tasks are then classified in light of this material using the criteria outlined in figure 1. An explanation for how each task was classified is provided directly after the classification.

Examples of task classification using the novelty framework

Definition

General term
The general term in an arithmetic sequence is
Example 1
Prove that the sequence $T_n = 5n - 1$ is arithmetic. Find $a$ and $d$.
Examine $T_n - T_{n-1}$

\[
T_n - T_{n-1} = (5n - 1) - (5(n - 1) - 1) = 5n - 1 - 5n + 6 = 5
\]

As $T_n - T_{n-1}$ is constant, the sequence $T_n$ is arithmetic.
Hence $d = 5$ and $a = T_1 = 5 - 1 = 4$

Example 2

Formula
If $S_n = T_1 + T_2 + \ldots + T_n$ is an arithmetic series, then

\[
S_n = \frac{n}{2} [2a + (n - 1)d]
\]

Example 3
The sum of the first 21 terms of an arithmetic series is zero. Express, in terms of $a$, the sum of the next 21 terms.

\[
S_{21} = 0 \implies \frac{21}{2} [2a + 20d] = 0, \text{ by Formula 4}
\]

\[
\implies d = \frac{-a}{10}
\]

(We have to find the value of $T_{22} + T_{23} + \ldots + T_{42}$)

\[
S_{42} = \frac{42}{2} [2a + 41d]
\]

\[
= 21 \left[ 2a - \frac{41}{10}a \right]
\]

\[
= 21 \cdot \frac{-2a}{10} = \frac{-441a}{10}
\]

Task 1
Is the following sequence arithmetic? $T_n = 1 - 3n$

This task is classified as not novel as the exercise is based on Example 1. The solver finds $T_n - T_{n-1}$ to be a constant and thus proves that it is arithmetic.

Task 2
Find a formula for the sum of the first $n$ odd numbers.

This task is classified as somewhat novel. The presentation of the task, due to its wording, would make it appear unfamiliar. The student might be put off by the notion of $n$ odd terms.
However once students focus on finding $a$ and $d$ for the series $1+3+5+7\ldots$, similar skills to previous tasks would be used, based on the formula used in Example 3 except that it is expressed in terms of $n$.

**Task 3**

If $a^{-1}, b^{-1}, c^{-1}, d^{-1}$ are in arithmetic sequence, prove that $b = \frac{2ac}{a + c}$

This task is classified as *novel*. The skills involved in finding the solution here are not familiar from preceding exercises or from any previous point in the chapter being analysed.

**RESULTS**

Table 1 contains the results of our classification of the exercises on the topic of sequences and series in the two textbook series under consideration. It shows the number and percentage of exercises falling into each classification.

<table>
<thead>
<tr>
<th></th>
<th>Novel</th>
<th>Somewhat Novel</th>
<th>Not Novel</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Text A (H)</strong></td>
<td>17 (8.3%)</td>
<td>58 (28.4%)</td>
<td>129 (63.3%)</td>
</tr>
<tr>
<td><strong>Text B (H)</strong></td>
<td>30 (9.2%)</td>
<td>68 (20.9%)</td>
<td>227 (69.9%)</td>
</tr>
<tr>
<td><strong>Text A (O)</strong></td>
<td>0 (0%)</td>
<td>27 (11.5%)</td>
<td>208 (88.5%)</td>
</tr>
<tr>
<td><strong>Text B (O)</strong></td>
<td>27 (7.7%)</td>
<td>33 (9.4%)</td>
<td>291 (82.9%)</td>
</tr>
</tbody>
</table>

**Table 1: Results of classification of sequences and series exercises in two textbook series.**

Without exception, the *not novel* categorisation makes up the majority of every exercise set analysed in the textbooks within this study. The lowest occurrence of the *not novel* category within the topic of sequences and series was in Text A (Higher Level) with 63.3%, while there was one exercise set in Text B (Higher Level) where every task was found to be *not novel*. Overall *not novel* accounted for 82.9% of tasks in Text B (Ordinary Level), 69.9% in Text B (Higher Level), 88.4% in Text A (Ordinary Level). While there was a high incidence of tasks classified in the *not novel* category in all of the textbooks analysed, it appears to be more frequent in Ordinary Level textbooks than the Higher Level ones. The *somewhat novel* category has a significant difference between levels. 20.9% of tasks in Text B (Higher Level) and 28.4% in Text A (Higher Level) compared to 9.4% in Text B (Ordinary Level) and 11.5% in Text A (Ordinary Level). The *novel* tasks had a low level of occurrence within the exercises analysed. There appears to be a relatively big difference between the proportions of *novel* tasks at Higher and Ordinary Level in Text A with 8.3% of tasks in Text A (Higher Level) in the *novel* category with 0% of tasks categorised as *novel* in Text A (Ordinary Level). There is much less difference between levels in Text B where 7.7% of tasks in Ordinary Level were found to be *novel*, while 9.2% of tasks in Text B (Higher Level) were categorised as *novel*. The main difference between the distributions of the tasks in the Higher Level and Ordinary Level textbooks seems to be that there are more tasks in the *somewhat novel* category at Higher Level, with far fewer tasks at Higher Level categorised as *not novel*. The implication of these results is that students are not currently exposed to a high degree of
novel tasks as they complete classwork and homework exercises and there is scope for textbook authors to include more novelty when designing exercises.

DISCUSSION

We have seen that the majority of tasks in both textbook series and at both levels have been classified in the *not novel* category. These results would appear to support the view outlined in O’Keeffe and O’Donoghue (2009) that mathematics textbooks in the Irish system promote “retention and practice” (p. 290). Given the large percentage of tasks classified as *not novel* in the particular chapters analysed, it would appear that a lot of exercises still promote the practising of skills. If teachers are to remain dependent on such textbooks as a source of classroom tasks then it is likely that the much criticized ‘drill and practice’ style of teaching will continue to be a feature of Irish mathematics classrooms. One of the main objectives of Project Maths is that students should to be able to solve problems in unfamiliar contexts; it would appear from our results that textbooks are not currently facilitating this aim adequately.

O’Keeffe and O’Donoghue (2012) provide an analysis of specific junior cycle and senior cycle mathematics textbooks in terms of their structure, content and how they meet the expectations of the syllabus. Their research provides a variety of data such as the distribution of: narration and narration type, graphics and their purpose, exercises, worked examples, motivational factors, comprehension cues and technical aids. In relation to this paper, the analysis of the distribution of routine and non-routine problems throughout specific junior cycle and senior cycle mathematics textbooks is of particular interest. They define a routine problem to be “all problems which are ‘dressed up exercises’”, while a non-routine problem refers to “all problems which cannot be answered by a routine procedure or problems in which it not immediately obvious what one must do” (p.23). It can be seen from their analysis that routine problems outnumber the non-routine ones, something that is borne out in our results. However, there is no explanation of how problems were classified as being routine or non-routine. The novelty framework could be used to clarify this situation and to expose the influence of examples and previous exercises on new tasks. While their work provides a distribution of worked examples through the textbooks, the novelty framework presented here allows one to gauge the impact that explanatory material included in the textbook has on the student and gives an indication of the degree of experience that the solver brings to a task from the textbook chapter and its exemplars. When applied correctly, the novelty framework could provide a more informed categorization of a task in terms of its novelty, particularly with reference to the perspective of the solver.

Other authors have considered the characterizing of routine questions and encountered issues when attempting to complete the categorization. Berry et al. (1998) define routine questions as “those for which students may be expected to execute a rehearsed procedure consisting of a limited number of steps” (p. 105). They had difficulty when it came to the categorization of parts of examination questions as routine or not. They called for further investigation into refining what is meant by the term ‘routine question’. Their study relied on “debate by a group of people with considerable experience of the type of examination paper in question” (p. 108) in order to determine which parts were routine and which were non-routine. However, when a group of students who had studied the paper were asked to categorize the
same parts of questions as routine or non-routine, it was found that there was only partial agreement between the authors’ categorization and that of the students. It is hoped that use of the novelty framework would help avoid such problems as it addresses in more detail how a task can be categorized in terms of the experience that a solver brings to it.

Selden et al. (1994) point out that tasks cannot be classified independent of knowledge of the solver’s background. They view ‘cognitively non-trivial’ problems as those where “the solver does not begin knowing a method of solution” (p. 119). While they acknowledge that sample solutions and examples can be used to make problems routine, they do not provide a definition of what is meant by ‘routine’. The novelty framework can be used to give further clarification to the difference between routine and non-routine and might provide greater insight into the degree of novelty that is present in a task for any set of textbook exercises. It also takes into account some of the knowledge that a solver brings to a task, rather than just focusing on the task’s structure or the demands that a particular task places on the solver. Each student will have different mathematical experiences which are not dependent on the textbook material, but our framework is not able to take these into account. This work forms part of a larger project where we consider tasks from examination papers, and old and new textbooks. We hope to use the information from our classifications to design new tasks for use in the classroom.

REFERENCES


In a first year university linear algebra course, students were presented with five purported proofs of the statement that a linear transformation of $\mathbb{R}^2$ must fix the origin. The students were familiar with the definition of a linear transformation as an additive function that respects scalar multiplication, and with the matrix representation of a linear transformation of $\mathbb{R}^2$. They were asked to consider each argument and decide on its correctness. They were also asked to rank the five “proofs” in order of preference and to comment on their rankings. Some recent work of K. Pfeiffer proposes a theoretical frame for the consideration of proof evaluation and of students' responses to tasks of this kind. With this in mind, the students' assessments of the five presented arguments are discussed. The main proposition of this article is that exercises of this nature can give lecturers some useful understanding of students' thinking about the mechanisms of proof and (in this instance) about essential concepts of linear algebra. In keeping with the conference theme of Crossing Boundaries, the motivations, concepts and tasks involved in this exercise are related to key transitions in the journey of a novice mathematician.

INTRODUCTION AND BACKGROUND

This paper reports on a proof evaluation task that was used in an introductory course on linear algebra for first year university students of mathematics. Students were asked to comment on each of five purported proofs of a statement involving elementary concepts of linear algebra that were under discussion in their lectures, and they were asked to rank the “proofs” according to personal preference. This paper is written from the point of view of the lecturer, and it is a report on an episode of “ordinary” practice rather than an investigation of a research question. The main theme is a discussion of what a lecturer can learn from such an exercise, about students' understanding of the subject concepts involved in the task, and more significantly about students' thinking on the purposes and mechanisms of mathematical proof. Potential effects of proof evaluation exercises on the mathematical learning and mathematical behaviour of students are discussed here only briefly and speculatively. The motivation for including this task in the assigned coursework was the hope of prompting thoughtful consideration of what mathematical proof is for. This theme had been discussed in lectures, but since the exercise took place during the first few weeks of the students' university education, it is likely that any relevant background for them was dependent on their individual experiences at earlier stages of education. We hoped that the invitation to consider and compare different proofs of the same statement would prompt students to adopt the role of critical assessor as a mathematician would; to assume their own authority to make a determination on the adequacy of each proof, and then to identify features or styles that might
motivate or explain a preference for one proof over another, perhaps according to personal
taste.

The course in question is a specialist one for first year students of Science and Arts who have
declared an interest in mathematics, and many of the students in it will continue to degree
level in the subject. The course involved an introduction to linear algebra, and this proof
evaluation exercise was part of the students' first assignment, which was completed
approximately in their fourth week at the university. The students were familiar with the
concept of a linear transformation of $\mathbb{R}^2$ as a function that respects addition and multiplication
by scalars, and they were familiar with the matrix representation of a linear transformation
and with the procedure of using matrix-vector multiplication to evaluate a transformation at a
particular point.

We stress that this is a report on a coursework exercise and not on a research study.
Nevertheless our approach and observations are broadly consistent with those of a number of
authors who have conducted research on the approaches taken by experienced
mathematicians and by students to the task of proof validation. These include Alcock and
Weber (2005), who focus on first year students and an example from real analysis. Selden
and Selden (2003) analyse the behaviour of students when validating four proposed proofs of
a statement from elementary number theory, and discuss the implications of their findings for

The PhD thesis of Pfeiffer (2011) proposes a schema that facilitates discussion of the content
of particular evaluations of proofs (as well as the business of proof evaluation in general).
This theoretical tool is described briefly here, as it will be occasionally used in the discussion
in this article. In accordance with Hemmi (2008), Pfeiffer regards proof and proofs as
artefacts of mathematical practice. She adapts ideas of Hilpinen (2004) on evaluation of
artefacts in general and specializes them to the case of mathematical proofs. In this context an
artefact is a (physical or conceptual) object that is designed and made by an author (or
authors) in order to fulfil a specific purpose (or purposes). Thus the quality of an artefact can
only be judged in terms of its success at achieving its purpose(s). In the case of a proof of a
mathematical statement, a primary and non-negotiable purpose is that the argument
establishes the truth of the statement. Other purposes might include provision of a satisfying
explanation, enhancing understanding of the concepts involved, and so on. Motivated by
Hilpinen, Pfeiffer proposes that evaluating a proposed proof might involve three
considerations: that the author's intention or “proof design” is appropriately matched to the
purpose of the proof, that the author's intention is appropriately realized in the actual written
proof, and that the written proof appropriately achieves its purpose. A mismatch between the
author's intention and the purpose of the proof might be recognized in an argument that does
not contain internal errors but that is inadequate to prove the statement that it claims to prove.
A mismatch between the intended and actual characters of a proof might be manifested as a
faulty deduction, an error in calculation or a missing case in a supposedly exhaustive
analysis. An omission of this nature might lead also to a mismatch between the actual
character of the proof and the purpose, even in a case where the author's intention is well
matched to the purpose. The point of Pfeiffer's schema is certainly not to suggest that every

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instance of proof evaluation must involve deliberate or documented attention to each of these three relationships, but rather to provide some context and terminology for discussion of what proof evaluation entails and for discussion of specific evaluations of particular proofs. It is intended not as a rigid framework but as a helpful theoretical construction.

THE TASK

Students were presented with the following text.

Alison, Bob, Charlie, Deirdre and Ed are thinking about proving the following statement.

If the function $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation, then $T$ fixes the origin, i.e. $T(0,0) = (0,0)$.

Alison’s Proof

Suppose that $T(1,1) = (a,b)$. Then

$T[(1,1)+(0,0)] = T(1+0,1+0) = T(1,1) = (a,b)$.

But on the other hand since $T$ respects addition

$T[(1,1)+(0,0)] = T(1,1)+T(0,0) = (a,b)+T(0,0) = (a,b)$ from above.

So $T(0,0) = (a,b)-(a,b) = (0,0)$.

Bob's Proof

We know that for any element $u$ of $\mathbb{R}^2$ and for any real number $k$ we have

$T(ku) = kT(u)$.

Then applying $T$ to $(0,0)$ and multiplying the result by any real number $k$ must give the same result as multiplying $(0,0)$ by $k$ first and then applying $T$. But multiplying $(0,0)$ by $k$ always results in $(0,0)$ no matter what the value of $k$ is. So it must be that the image under $T$ of $(0,0)$ is a point in $\mathbb{R}^2$ which does not change when it is multiplied by a scalar. The only such point is $(0,0)$. So it must be that $T(0,0) = (0,0)$.

Charlie's Proof

Think of $T$ as the function that moves every point one unit to the right. So $T$ moves the point $(0,0)$ to the point $(1,0)$. Then $T$ is a linear transformation but $T$ does not fix the origin. This example proves that the statement is not true.

Deirdre's Proof

Suppose that $(a,b)$ is a point in $\mathbb{R}^2$ for which $T(a,b) = (0,0)$. Then

$T[2(a,b)] = T(2a,2b) = 2T(a,b) = 2(0,0) = (0,0)$.

Thus $T(2a,2b) = T(a,b)$, so $(2a,2b) = (a,b)$, so $2a = a$ and $2b = b$. Thus $a = 0$, $b = 0$ and $T(0,0) = (0,0)$. 

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Ed’s Proof

Since \( T \) is a linear transformation it can be represented by a matrix. Suppose that the matrix of \( T \) is \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then the image of \((0,0)\) under \( T \) can be calculated as follows:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a0 + b0 \\ c0 + d0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

So \( T(0,0) = (0,0) \).

The students were asked the following questions about the text above.

(a) Does Alison's answer prove that the statement is true? If not, why not?

(b) Does Bob's answer prove that the statement is true? If not, why not?

(c) Does Charlie's answer prove that the statement is not true? If not, why not?

(d) Does Deirdre's answer prove that the statement is true? If not, why not?

(e) Does Ed's answer prove that the statement is true? If not, why not?

(f) Please rank the five answers in order of your preference (according to your own opinion). Include some comments to explain your ranking.

The following remark was included in the assignment that was given to students.

This is not a trick question and there is not necessarily a “right answer”, certainly not to part (f). What you are expected to do is read each of the proposed proofs and think about the content. Is the argument correct? Does it do what it claims? There may be more than one correct proof here, and part of the point of this question is the fact that there can be many different approaches to proving a mathematical statement. You might prefer one argument to another for example if you find it easier to follow or if you find it particularly convincing. If you are in doubt, discuss the problem with classmates or post a message on the Discussion Board.

THE RESPONSES

The 28 students whose responses are discussed here are those who fully answered the questions in the task (many other students provided only yes/no answers without explanation).

Alison’s Answer: Of the 28 respondents, 17 stated (or indicated in their comments) the view that Alison proves that the statement is true. One was non-committal, and the other 10 stated that Alison's answer does not prove that the statement is true.

Bob’s Answer: 21 students accepted Bob's proof as correct. One student described it as partially correct, and 6 considered it to be incorrect. It was the second most popular of all the proposed proofs, being ranked first (or joint first) by 8 students.

Charlie's Answer: Charlie's proof was considered incorrect by 25 students, and correct by three. It was ranked 5th by 21 students.
**Deirdre's Answer:** Deirdre's proof was considered to be correct by 19 students, and ranked 1st or 2nd by 11 of these. It was considered to be incorrect by 8 students, with one student reporting no decision on its correctness.

**Ed's Answer:** Ed's proof was accepted as correct by all 28 students, and was by far the most popular of the five proofs, being ranked 1st by 18 students and 2nd by 5 students.

**DISCUSSION OF THE RESPONSES**

An account of the 28 student responses might be sensibly organised in a number of different ways. It is organised here under five headings that correspond to the five proofs, in an effort to highlight features of students' thinking about proof that emerged from their responses to the specific arguments. There is no suggestion that these observations constitute in any sense an exhaustive analysis of the data. They are selected on the basis of their potential interest to a lecturer hoping to facilitate a meaningful learning experience involving proof, and in some cases on the basis that they were unexpected.

**Alison's proof – showing is telling**

On the whole we found the responses of students who considered Alison's proof to be incorrect to be more informative than those who (correctly in our opinion) approved it. This is probably an inevitable feature of an exercise of this nature – a statement of acquiescence with a proposed argument does not tell us much about what process of thinking led to this position. An articulated objection (whether correctly reasoned or not) tells us much more.

Alison's argument establishes that every additive function from $\mathbb{R}^2$ to $\mathbb{R}^2$ fixes the origin, so it applies to a broader class of functions than linear transformations. In order to exploit the additivity of $T$ she chooses the point $(1,1)$ and looks at the image under $T$ of the sum $(1,1)+(0,0)$. Five students objected to the introduction of the point $(1,1)$, apparently believing that focusing attention on this chosen point constituted a restriction of the statement to a particular example. Of these five, three concluded that Alison's entire argument amounted only to the statement that $(1,1)-(1,1) = (0,0)$. Another accepted Alison's proof as correct, but commented that

I would prefer if she used $(c,d)$ where $(c,d)\in\mathbb{R}^2$ instead of $(1,1)$.

Another feature of the students' comments on Alison's answer, and one that may interest lecturers, is that they revealed problematic aspects of the use of the word “show” in mathematical discourse. Mathematicians use the words “show” and “prove” interchangeably and consider them to be basically synonymous, the only difference perhaps being that the term “show” may have less formal connotations. Logically, for a mathematician, what is required to “show” that a statement is true is the same as what is required to “prove” it to be true. However, these words are not synonymous in ordinary usage, and it is valuable for lecturers to be aware that novice members of the mathematical community may not automatically share its special understanding of the word “show”. This is noticeable in some of the students' comments on Alison's argument, which variously assert that she “uses the fact” that $T$ respects addition (to reach her conclusion) and that she “showed” that $T$ respects addition. The phrase “uses the fact” is unambiguous here, but the word “showed” is open to
two quite different interpretations. If “show” is interpreted as synonymous with “prove”, then a reasonable conclusion is that the student is confusing what needs to be proved with what is included in the hypotheses and entirely misunderstanding the mechanism and purpose of the proof. However, if “show” is interpreted in the everyday sense, as “point out” or “draw attention to”, then it is perfectly reasonable to describe a step in Alison's proof as showing that $T$ respects addition. The following comment from one student possibly uses both senses of “show”.

She proves it by showing that points of linear transformations are additive and she manipulated this to show that the origin will be fixed.

Strikingly, two students seemed to realize in the process of writing their comments that use of the word “show” to describe Alison's emphasis on the additivity of $T$ is problematic. These two students revised their wording as indicated below.

Alison's proof shows uses the fact that linear transformations respect addition . . .

She shows knows that it respects addition because it is a linear transformation.

**Bob's proof – half the story**

The six students who rejected Bob's proof stated two reasons for doing so. Two students objected to the assertion in Bob's proof that $(0,0)$ is the only element of $\mathbb{R}^2$ that “does not change when multiplied by a scalar”, pointing out that (for example) $(2,3)$ does not change when multiplied by the scalar $1$. The students who approved this proof presumably interpreted from the context that Bob's intended meaning was “never changes when multiplied by any scalar”. Perhaps this slightly imprecise form of words in Bob's proof can be legitimately criticized; in the context of Pfeiffer's schema, such a criticism would identify a minor and easily rectifiable mismatch between the intended and actual characters of the proof.

The other four students who rejected Bob's proof (as well as three who accepted it) complained that it only used part of the definition of a linear transformation, namely the property of respecting scalar multiplication. All seven of these students criticized Alison's proof on the same grounds; the following is a representative comment.

Bob supplies the other half of Alison's proof, he proves the statement for scalar multiplication. He is also half right.

The students who reject or criticize Bob's and Alison's proofs on these grounds appear to be attending insufficiently to the logical structure of their arguments, and appear to be focussing on internal features of these arguments almost independently of the statement that is to be proved. These students recognize the strategy of reasoning from a definition towards a desired conclusion, but their conclusion that the argument cannot be complete if it uses only part of the definition seems to be automatic. Their written comments do not indicate attempts to assess the significance to the argument of the “missing” part of the definition; their verdicts appear to be founded purely on an inspection of features of the proof without reference to its purpose.
No student cited as a reason to favour the proof of either Bob or Alison that both of these arguments prove a more general statement that they are directly concerned with. Alison's argument proves that every additive function fixes the origin, and Bob's proves that every function that respects scalar multiplication fixes the origin. One student did note that Bob's proof uses “one” property of linear transformations as a reason to rank it first.

**Charlie's proof – true and false**

Of the 25 students who rejected Charlie's proof, only 11 did so on the grounds that the proposed counterexample is not or may not be a linear transformation. Of these 11 students, 7 clearly stated or demonstrated that Charlie's function is not a linear transformation. Another objected that Charlie would have to show that his function is a linear transformation in order for his proof to be conclusive, and two complained that Charlie's proof introduces a particular function without any consideration of the properties of linear transformations. One student asserted that

Charlie's proof only applies for functions that are not linear transformations.

This response is surprising given that Charlie's argument involves only one specific function; nevertheless the student appears to be aware of the essential error in Charlie's reasoning.

Not all of the remaining 14 students who rejected Charlie's proof gave clear reasons. The possibility is open that the conflict between Charlie's conclusion and those of the other authors prompted many to object, but only two students cited this as a reason for rejecting the proof. Six students objected to the restriction of attention to a particular function. There is no sign in the work of these six students of acknowledgement that Charlie's goal is exceptional amongst these five, that he is trying to disprove the statement by exhibiting a counterexample. In the context of Pfeiffer's schema, their evaluations of Charlie's proof do not appear to include consideration of the relationship between the content of the proof and its objective.

A possibly surprising feature of the responses to Charlie's argument is that of the three students who considered it to be correct, each also accepted at least one of other four proofs as correct. For example, one commented as follows on Charlie's proof

As he notes, the linear transformation could possibly move every point one unit to the right. Therefore $T$ does not fix the origin.

The same student accepted (for example) Bob's proof, and seemed to recognize the conflict between Bob's and Charlie's positions, commenting on ranking Charlie's proof 3rd that

Even though Charlie disproves the statement, it's a very valid reason to disprove it.

The three students who approved Charlie's proof did not appear to be troubled by the inconsistency of their own positions, and apparently believed that the statement could simultaneously be validated by a correct proof and contradicted by a counterexample. The intriguing phenomenon of such beliefs is investigated and thoughtfully discussed by Stylianides and Al-Murani (2010). Their study involved a written exercise that was followed by interviews with participants who appeared to simultaneously accept proofs and
counterexamples for the same statement. These authors caution against summarily concluding that such a position indicates serious misconceptions about the nature of proof. Their interviews allowed a nuanced approach and revealed features of their participants' thinking that were not apparent from the written exercise.

**Deirdre's proof – “probably ok”**

Deirdre's argument begins with a linear transformation $T$ and a hypothesized point $(a,b)$ whose image under $T$ is the origin. (There is no *a priori* guarantee that such a point exists.) From the fact that $T$ respects multiplication by scalars, it is established that $(a,b)$ and $(2a,2b)$ have the same image under $T$. It is then erroneously deduced that these two points must be the same and hence that $a = b = 0$. This is a specific error in the line of reasoning documented in Deirdre's argument. The argument also suffers from a structural error in its logic: what it attempts to establish is not that $T(0,0) = (0,0)$ for every linear transformation $T$, but that if a point is mapped by a linear transformation to $(0,0)$, then that point must be $(0,0)$. Since this statement is not true there is no hope of proving it, and even if it were true some further work would be required to deduce from it that every linear transformation fixes the origin. The structural error in Deirdre's proof is akin to replacing the statement *all bicycles have wheels* with the statement *every object that has wheels is a bicycle*. In the context of the schema of Pfeiffer, this error corresponds to an opposition between the author's proof strategy (the intended character of her proof) and her purpose (establishing that $T(0,0) = (0,0)$ for a linear transformation $T$). The “internal” error in Deirdre's proof, (that $T(a,b) = (0,0) = T(2a,2b)$ means $(a,b) = (2a,2b))$ corresponds to a failure in the author's implementation of her strategy, a mismatch between the intended character and actual character of her proof. That such an error must exist is inevitable in this instance, since the author's intention is to prove an untrue statement.

Obviously notable is the fact that two-thirds of the students accepted as correct an argument that has (at least) two serious flaws, one in its overall logical structure and one in its internal deductions. Many of these students identified similarities between Deirdre's proof and Bob's, and commented that Deirdre's proof uses scalar multiplication. It is possible that these students' vigilance in assessing Deirdre's proof was compromised by its resemblances to Bob's proof, which was accepted by all but two of them. Having already seen the “scalar multiplication” property of linear transformations appropriately used to prove the statement, students may have concluded correctness from a superficial inspection of Deirdre's argument. This may partly explain the students' readiness to accept this fundamentally flawed proof. However, a possibly surprising feature of this general pattern of approval was that at least three students accepted Deirdre's proof as correct despite stating that they did not understand it. One of these three wrote

> The way Deirdre's answer is laid out is confusing. I think that it probably does prove it but she needs to write it better.

This comment prompts a comparison between the behaviour of these novice students and that of experienced academic mathematicians, whose work often requires them to make judgements on the correctness and quality of proposed proofs, for example when refereeing...
journal submissions. In this context it is not unusual to request expansion, clarification or entire rewriting of a proof on the grounds of finding it impossible to understand and being therefore unable to decide on its correctness. It would be irresponsible of a referee to recommend publication of an article whose proofs she is unable to validate. Authors have a responsibility to provide proofs that are intelligible to a willing and knowledgeable reader. However, there is tension between what is reasonable to demand of the author and of the reader, as well as vagueness about the meaning of “knowledgeable”. For a referee to reject or criticize a proof on the grounds that she has not succeeded in understanding it requires her to believe that the difficulty is not only due to shortcomings in her own subject knowledge, that she has made every reasonable effort to engage with and understand the proof, and that the author has not made his proof sufficiently intelligible. Making a judgement like this means taking a definite and critical position on the balance of the responsibilities of author and reader. To do this requires a measure of confidence that can only come with experience. Thus it is perhaps not so surprising that novice students may be inclined to accept a proof that they do not fully understand, if they consider that it looks plausible and if they do not find a specific error in it. Schoenfeld (1994) discusses the matter of mathematical authority in classrooms, and reports that students rarely see themselves as having any authority to make mathematical judgements.

They have little idea, much less confidence, that they can serve as arbiters of mathematical correctness, either individually or collectively.

We are interested in the idea that proof evaluation exercises can stimulate students to develop their own sense of responsible mathematical authority and to progressively test and extend the scope of that authority.

The eight students who rejected Deirdre's proof did so for a variety of reasons. Two of these eight (as well as two students who accepted the proof) criticized the use of the scalar 2 instead of a general $k$. Two of these students suggested that this specialization amounted to restriction to a special case and constituted a reason to reject the proof, the other two only that it compromised the quality of the argument (as opposed to its correctness).

Two students rejected Deirdre's proof on the basis of the erroneous deduction that $T(a,b) = T(2a,2b)$ means $(a,b) = (2a,2b)$. One wrote

Her second line contains a mistake, when she states $T(2a,2b) = T(a,b) \Rightarrow (2a,2b) = (a,b)$. This is not necessarily true. She is incorrect.

For us, the most remarkable feature of the data on our students' responses to Deirdre's proof is that not one student noted its significant structural flaw. The only possible reference to the unexpected structure of Deirdre's argument is an oblique one from a student who accepted the proof and commented that

she works backwards to reach her conclusion.

From their comments it is not evident that any of the students gave careful critical attention to the question of “fitness-for-purpose” of Deirdre's proof. In the context of Pfeiffer's schema, none of the students made written comments that could be construed as consideration of the
relationship between the intended or actual characters of this argument and its purpose. As mentioned above this may be attributable in part to the apparent similarities between Deirdre's proof and Bob's. Nevertheless, the mathematician's habit of constantly testing the connection between the text of an argument and the statement that it purports to prove does not seem to be automatically adopted by students. That a passage of text seems sensible and error-free is not enough to confirm that it establishes the truth that it claims to establish. To validate it as a proof a reader must attend not only to its internal content but also to its logical structure in relation to what it claims (implicitly or explicitly) to achieve. Constant alertness to the possibility of logical failures in an argument is a habit of mind whose development may need explicit attention from both teachers and students. It is an essential feature of mathematical practice, and one that might plausibly be encouraged by critical study of proofs some of which are incorrect or inadequate in different ways. We propose that understanding of the role and power of proof in mathematics can be both tested and strengthened by critical study of proofs that have subtle but serious problems, and recall the following words from *Proofs and Refutations* (Lakatos, 1976, p.23) [1].

I think that if we want to learn about anything really deep, we have to study it not in its 'normal’, regular, usual form, but in its critical state, in fever, in passion. If you want to know the normal healthy body, study it when it is abnormal, when it is ill. If you want to know functions, study their singularities.

**Ed's proof – clear and simple**

Few students commented in detail on Ed's proof. Typical remarks included that it was clear, simple, short and easy to understand. Overall the group demonstrated a clear preference for Ed’s translation of the problem into an easy matrix calculation over Alison and Bob's processes of reasoning from the defining properties of a linear transformation. The matrix representation of a linear transformation had been discussed in detail in lectures, and manipulations with matrices featured in several other questions on the assignment that included this proof evaluation exercise.

Much has been written about the challenges involved in the transition from the procedural focus that tends to dominate mathematical learning at second level to the conceptual approach that students of mathematics are expected to adopt at third level. Tall (1991) describes the process through which students develop the mental dispositions of advanced mathematicians in terms of

>a stony path that will require a fundamental transition in their thinking processes.

The students involved in this account are in their first weeks of university education, taking their first steps along Tall's stony path. In view of this it seems unsurprising that amongst the five proofs in the evaluation exercise, their overwhelming preference was for the one that demonstrates the truth of the statement by formulating it in terms of elementary arithmetic. A more experienced mathematician might feel that Ed's proof establishes the truth of the statement without offering much insight into why it is true, and might prefer the more general and arguably more informative explanations of Alison and Bob.
CONCLUDING REMARKS – CROSSING BOUNDARIES

The themes of this paper are closely connected to a number of boundaries that may be crossed and recrossed in the experiences of students and mathematicians and in the extensive business of mathematical education. Some of these boundaries are fine and often imperceptible lines like the one that separates the two meanings of the word “show” that are discussed in connection with the responses to Alison's proof. Such distinctions involving words that have related but distinct meanings in mathematical and everyday usage are worthy of the attention of educators, as they have the potential to corrupt our best efforts to communicate. Other boundaries are not fine lines that can be crossed unwittingly, but more resemble deep inhospitable rifts whose traverse requires courage and persistence, and of which there can be no second crossing. The image of the transition to advanced mathematical thinking as an arduous wilderness expedition is a bit hackneyed, but as educators of future mathematicians (and future teachers of mathematics), our approach cannot be to try to pick the stones out of the path. Mathematics at the advanced level is conceptually difficult. To engage with it requires patience and resolve, to unravel the strands of sometimes convoluted arguments and attend to the overall structure of a network of inferences, while dealing with fussy details of objects defined in terms of their abstract properties. It requires a simultaneous appreciation of proof as the objective and reliable standard by which mathematical knowledge is certified, and of proofs as communications by humans, whose transmission is subject to the general fallibility of human communication and reliant on allowances by author and reader. These two views of proof are not in opposition, and working mathematicians operate with both, as discussed by Bass (1999). Through tasks that require the use of knowledge and judgement to critique proposed arguments, students at an early stage of education can be invited to take small steps across the boundary that separates spectators from participants in the human mathematical enterprise. Tasks that require students to construct proofs of course also invite such a step, but at a point where the boundary seems to have the form of a high and impermeable wall. Exercises in such activities as proof validation and proof evaluation might be located at a point where the wall has a few gaps or low points, and importantly also allow students to communicate their mathematical preferences to their lecturers.

Another boundary that has become more porous in recent years is the one that separates the research communities in mathematics and in mathematics education. We suggest that interaction between these communities is essential if our goal is to improve the state of mathematical capability amongst our students and graduates, and the state of the subject itself. If advances in pedagogical and educational thinking are to make their way into third level classrooms and benefit students, mathematicians need to engage with them. On the other hand research on mathematical education at third level must surely be informed by the experiences of the mathematicians and students who are involved in that endeavour. “Practice reports” like this one have hopefully a role to play in the conversation between the two communities. The MEI conferences have done a great deal to provide opportunities for mutually beneficial interactions between researchers in mathematics and mathematics education in Ireland.
NOTES

1. These sentiments are attributed by Lakatos to Denjoy

REFERENCES


THE FIGURATIVE METHOD: A BRIDGE FROM NUMERICAL TO QUANTITATIVE REASONING

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We designed and implemented a constructivist teaching experiment with the purpose of helping students’ transition from numerical reasoning to quantitative reasoning. We report our work using one student in 5th grade. We used the Figurative Method, which allowed the student to comprehend the quantitative situations of the tasks we proposed. The Figurative Method consists of representing a quantitative situation using magnitudes of lengths of line segments in lieu of magnitudes of qualities of objects involved in the situation. At the same time, the Figurative Method promotes symbolic representations of the quantitative relationships. At the end of our teaching experiment, we concluded that the student understood the Figurative Method and started to reason quantitatively.

RATIONALE

Many researchers describe students’ difficulty to reason algebraically in the early grades (Kaput, Carraher, & Blanton, 2008; Kieran & Chalouh, 1993). This difficulty has persisted despite an increase in remediation efforts nationwide (Greenes & Rubenstein, 2008; Kilpatrick, Mesa, & Sloane, 2007). Multiple approaches to Early Algebra have been proposed and implemented, among them a curriculum that gives plenty of space to the development of quantitative reasoning (Smith & Thompson, 2007). It is the focus on relationships – the key feature of the quantitative reasoning – that makes this approach so suitable for Early Algebra. One such approach, Measure Up, inspired by Davydov’s work (1975), was conducted at the University of Hawaii (Dougherty, 2008). The teaching experiment presented here illustrates our efforts to reform our mathematics curriculum by adding a bridge from numerical reasoning to quantitative reasoning. In such a curriculum, the algebraic structures are ‘built-in’ naturally, developed through quantitative relationships as they arise in meaningful tasks carefully chosen to connect with students’ experiences. Having meaningful relationships at its core, quantitative reasoning is important for Algebra and beyond; for modelling, analysing and interpreting situations from context-based mathematics and science (Mayes, Hatfield, & Belbase, 2012; Mayes, Peterson, & Bonilla, 2013).

QUANTITATIVE REASONING AND THE FIGURATIVE METHOD

We will now introduce our criteria of judgment for quantitative reasoning and understanding of the Figurative Method. The constructs presented here follow Thompson’s (1990) model of quantitative reasoning, augmented by our Figurative Method. To illustrate, we present an example of student reasoning while solving the following problem:

Lily tells her mother: “Now I’m 19 years younger than you are, but in 10 years your age will be double my age.” How old is each of them now?
To solve the problem, the student needs to conceive two quantities, Lily’s age and mother’s age, in two different instances in time: now, and ten years later. A quantity is a quality of an object together with its magnitude (Freudenthal, 1983; Thompson, 1990). Assigning magnitudes to objects implies conceiving of a unit of measurement, an origin from where the measurement starts, and a sense of measurement (Freudenthal, 1983). When referring to a quantity, we use the notation (quality of an object, magnitude).

If the student considers the instance of time “now” - then the quantities conceived are: (Lily’s age, L) and (Mother’s age, M), where L and M are unknown and represent Lily’s age and mother’s age measured in years. If the student considers the instance of time “ten years later” - then the quantities conceived are: (Lily’s age ten years later, L) and (Mother’s age ten years later, M), where L and M are unknown, and represent Lily’s age and mother’s age measured in years. To conceive the new quantities (Lily’s age ten years later, L) and (Mother’s age ten years later, M), the student needs to perform quantitative operations using the old quantities, (Lily’s age, L) and (Mother’s age, M). In this case, we have the new magnitudes L=L+10 and M=M+10 (since ten years will pass by), and therefore the new quantities are: (Lily’s age ten years later, L+10) and (Mother’s age ten years later, M+10). Furthermore, the student needs to conceive the relationships between quantities by performing quantitative operations. For example, to establish relationships between the quantities (Lily’s age, L) and (Mother’s age, M), the student may conceive any of the relationships (Lily’s age, L=M-19), (Mother’s age, M=L+19), (Lily’s age, L<M), or (Mother’s age, M>L). These quantitative relationships (and others that may be conceived from the given problem), together with the quantitative operations, constitute a quantitative structure. We say that the student reasons quantitatively if the student can analyse the quantitative structure.

The Figurative Method consists of representing a quantitative situation using magnitudes of lengths of line segments in lieu of the magnitudes of qualities of objects involved in the situation. Those representations allow the student to “see” the relationships between quantities and translate them into symbolic representations (algebraic relations). We use the method as a way to communicate about a quantitative situation. Below is an example of student use of the Figurative Method, a way to communicate her understanding of the quantitative situation in our problem (see Figure 1):

![Figure 1: Example of student use of the Figurative Method](image)

As can be seen in Figure 1a), representing the present moment (‘now’), the student correctly compares the quantities (Lily’s age, L) and (Mother’s age, M). Indeed, the line segment
representing (Mother’s age, M), is composed of two smaller line segments or parts - one with a magnitude congruent with L, and the other one with a magnitude of 19. Therefore, we may infer that the student has conceived the quantity (Mother’s age, M=L+19). When we examine Figure 1b), representing the situation “ten years later”, we notice that the student adds some parts of magnitude 10 to the line segments representing the old quantities. We may infer that the student has conceived the quantities (Lily’s age ten years later, L+10) and (Mother’s age ten years later, M +10). However, the student fails to represent correctly the relationship from the statement “in 10 years your age will be double my age.” We can see that, in the absence of the right relationship, she tries various arithmetic calculations (e.g., adding 10 and 19, doubling 29).

The Figurative Method is considered understood if the student is able to:

- Start with a part (preferably but not necessarily the shortest part, representing the quantity with the smallest magnitude); represent that part (as a line segment), and name that part.
- Represent the other quantities from the problem, function of that part, figuratively (as line segments), and ‘name’ those quantities.
- Write appropriate algebraic equations or inequalities, as suggested by the relationships between quantities and their figurative representations.
- Use appropriate arithmetic, as suggested by the algebraic relations and/or figurative representations, to solve the problem.

CONCEPTUAL ANALYSIS

The following is a description of the theoretical framework we used to analyse our data. A conceptual analysis (Thompson, 2000) was performed with the goal of proposing answers to the following questions:

1. What was the student’s understanding of the Figurative Method?
2. Did the student begin to reason quantitatively?

To perform a conceptual analysis (Dewey, 1991; Thompson, 2000; von Glasersfeld, 1995), we examined the mathematical constructions elicited by the student. Thinking is dependent on mental operations. Although we could not directly observe mental operations, we tried to infer what they might be by analysing the student’s constructions and actions. To understand somebody else’s understanding is a matter of fitting coherently with our own interpretation of somebody’s meanings rather than matching our meanings in sameness. A direct comparison with what the student had in mind was not possible, and therefore the compatibility refers strictly to fitting our interpretation. The fit took place if the student did nothing to contradict our interpretation. The data we collected, sometimes with discrepant fragments, went through a linkage process. We supplied the connecting links and subjected the data to repeated and systematic inferences, both inductive and deductive. We started with the data, the facts (things said or written by the student, actions, as observed). A perturbation, a lack of coherence, a discrepancy with our own constructs or a difficulty felt by the student stimulated our thought in new directions. We suggested one possible meaning to fit the entire data, thus performing an act of inductive reasoning. This possible meaning acted as a working, ‘viable’ hypothesis.
and was tested for coherence against *all* the data we had, thus performing an act of deductive reasoning. The process was repeated until we were satisfied with the meaning proposed, in the sense that it fitted coherently our subjective understanding of the student’s understanding. As such, our last viable hypothesis should be seen as a ‘suspended’ conclusion, a model that does not match the meanings of the student in sameness, but which fits our interpretation of the student’s meanings.

**METHODOLOGY**

**About the teaching experiment**

This was an experiment in the same constructivist spirit as those described by Steffe and Thompson (2000). We abided by the following principles:

- This was an attempt of a *technical constructivism* experiment, with students constructing their own knowledge while being ‘directed’ on the right path. *The teacher had the role of “orienting function”* - as described by von Glasersfeld (1995, p.184): “The teacher cannot tell students what concepts to construct, or how to construct them, but by a judicious use of language they can be prevented from constructing in directions which the teacher considers futile but which, as he knows from experience, are likely to be tried.”
- *Communication was central* to our experiment. Not only did we employ the Figurative Method as a public way of communication, but we also asked the students to communicate about their solutions through letters.
- The students reasoned about *relationships between quantities in various situations* from the students’ experiential world. We were optimistic and relied on “children’s powers” (Mason, 2008) to overcome epistemological obstacles.
- We used as often as possible *open problems, open situations*. This way, the students had the feeling of ‘owing’ the problems.
- We considered *complexity* a key component for success. The students reasoned about more and more complex situations.
- While conducting the experiment we accounted for our *students’ ways of thinking*, while looking for ways to *transfer* our experience to a whole class of students.

Both authors participated in all the phases of the teaching experiment:

- We started by *establishing* some *working hypotheses* regarding our students’ ways of thinking.
- We *designed appropriate tasks that perturbed our students*, and hypothesized about students’ responses to such perturbations.
- After observing *where the students really ended* after perturbations, we *reflected* back on our first hypotheses.

Valentina conducted the experiment, and Florin was the witness whose role was to provide “constraints” to keep the experiment on the planned learning trajectory (Simon, 1995).
The implementation of the Figurative Method

With respect to our students’ prior knowledge, we assumed that they knew how to calculate with numbers (arithmetic operations), and had some geometric conceptual knowledge (line segment, length of the line segment, rectangle, length and width of a rectangle). Worth mentioning is the initial disagreement between the two authors. For Florin, the Figurative Method was a tool to empower the student, while for Valentina it was a way of knowing how the students think. As such, Florin proposed to begin by showing the students how the tool worked, i.e., modelling how the Figurative Method was used to solve tasks, while Valentina insisted on letting the students construct their own method. The agreement, when reached, shifted the attention to the type of tasks we proposed, tasks that could ‘direct’ the students on the right path. As such, from now we focus on the tasks we designed, and present the tasks together with our working hypotheses about the students’ ways of thinking, the intended perturbations, and hypotheses about the students’ responses.

We started with two simple tasks, Tasks 1a and 1b, where the need for representations of quantities as line segments was obvious:

**Task 1a.** Draw a rectangle with the length twice as long as the width.

**Task 1b.** Draw a rectangle with the length 1256 times as long as the width.

In Task 1a we wanted the students to be able to represent the relationship between length and width, first figuratively and then algebraically. If the students failed to describe the relationship (length is twice as long as the width, etc.) and/or failed to represent this relationship on their drawing, then we would intervene with didactic objects (Thompson, 2002) (e.g., sticks of various lengths). Task 1b was designed to produce a perturbation that would prompt the students to use an algebraic relation (symbolic representation).

We continued with the open Tasks 2 and 3, where the students could choose the names of the children referred to in the task, the magnitudes of the quantities, or the question to be answered in the situation described in the task. We wanted to give the students a sense of ownership over the tasks, and to enforce a conversation about quantitative relationships rather than about numerical operations. Also, we hypothesized that the use of the Figurative Method would help the students make the comparison between the quantitative situations from Task 1a and Task 2.

Task 2 was similar to Task 1a, but this time the quantities were amounts of money, and we increased the complexity by asking about the total amount:

**Task 2.** Two children from the first grade, …, and …, and two children from the fifth grade, … and …, counted their money from their piggy-banks. The children from the first grade each have the same amount of money. The other two children from the fifth grade each have the same amount of money, too. Each of the children from fifth grade has … times as much as any of the children from the first grade. If all four children put their money together, the total amount would be ….
Task 3, also an open task, was designed to be solved with several sets of numbers, in order to allow the students to generalize, and describe the quantitative situation by using quantitative relationships with symbolic representations:

**Task 3.** Two sisters, …and …, counted their money from their piggy banks. Together they have…. One of the sisters, …, has… more than the other sister, ….

In Task 4, we wanted the students to start from the algebraic relation toward the figurative representation with line segments, and operations with numbers: We hypothesized that the use of the Figurative Method would act like a catalyst between the symbolic representation of the linear equation and the correct sequence of calculations necessary to find solutions:

**Task 4.** While solving a problem, your friend wrote this relation: \( b = 3a + 12 \)

i) Can you represent this relation with line segments?

ii) Let’s say that \( b=165 \). Can you find \( a \)? What operations would you use?

**DATA ANALYSIS**

The participating students, Becky and Tom, were two fifth-grade students from a public school in Arizona. Both were in the ‘advanced group’ in mathematics, the top third of their class. We met twice with each of them, one hour each time. The sessions were audio taped. We focus here only on Becky, Tom’s role being to ensure a certain type of communication that otherwise appears naturally in a class of students.

Task 1a was successfully completed by Becky. She could draw the rectangle (see Figure 2, below) and state the relationship between length and width. When asked, Becky drew the width and the length as line segments (see Figure 2, below).

Becky: A rectangular shape. There are two sides that are shorter and the other two sides are twice as long as the other sides.

![Rectangle from Task 1a](image)

When prompted to comment on the relationships between the opposite sides of the rectangle, Becky characterized the rectangle as being with two pairs of congruent sides:

Becky: To make a rectangle I think you have to have all four sides, two sides being congruent, two pairs of congruent sides.

VP: What does “congruent” mean in this situation?

Becky: Two sides are the same length.

At the beginning, Task 1b posed difficulty for Becky, when she tried, unsuccessfully, to represent the relationship between the length and the width. She pictured the rectangle in her mind this way: “The length is very long and the width is very short”. We asked Becky to draw the rectangle from Task 1b in such a way that Tom would be able to figure out the relationship between the length and the width of the rectangle without reading Task 1b. Becky
proposed to show Tom, using tick marks, that “the width would have to fit into length 1256 times” (see Figure 3, below), but she had to give up.

Becky: It’s kind of small. I don’t think I can …
VP: How many could you fit in there?
Becky: Fifty-seven.

Still looking for a way to communicate to Tom the relationship between the length and width of the rectangle in Task 1b, Becky wrote “width $= \frac{1}{1256}$” and “Length $= 1256$ times as long as the width.” In the first relation, Becky considered the length as having one unit (of measurement), and she partitioned it into 1256 widths. In the second relation, although correctly formulated, the arithmetic operation was absent. After we gave Becky several magnitudes for the length of the line segment representing the width of the rectangle (e.g., 2 ft.), and asked for the magnitude of the length of the line segment representing the length of the rectangle (e.g., 2512 ft.), Becky was able to generalise: “$L = 1256 \times W$”. When we gave her the magnitude for the length of the line segment representing the length of the rectangle (5024 ft.) and asked for the magnitude of the line segment representing the width, Becky could not perform the division (see Figure 4, below). Instead, she attended to the numerical operation through multiplication.

Becky: At first I was going to divide, but I thought it would be easier to multiply. I decided that because 1000, … 1256 is supposed to go into 5000, it should be less than 5, so I started with 4.

When we asked Becky to write the relation between width and length, given the length, Becky wrote “$W = L \div 1256.$” We could say that Becky successfully attended to the quantitative relationships between the length and the width, and the width and the length, respectively. She attended to those quantitative relationships through quantitative operations although she had difficulty performing the division. She could write the algebraic relations because she understood the quantitative situation.

In Task 2, Becky filled in without difficulty the children’s names, and decided that each of the 5th graders had “two times” as much money as any 1st grader because “two” was her favourite number. Because she hesitated to assign a number to the total amount of money the children
had, we proposed $120. About our contributions to the open tasks, we decided to help Becky with the magnitude for the total amount of money because we wanted to avoid possible difficulties with computations, and/or with rational numbers representing amounts of money. Also, we needed to be sure that the student did not start with a ‘pre-made’ solution. Becky understood the situation as being “alike” the one with the rectangle in Task 1a because of the two pairs of congruent sides and two pairs of children with the same amount of money, and she recognized the similarity between “twice as long” and “twice as much,” as well as the difference “now it’s about money” (which implied it was not about the lengths anymore). The perturbation consisted in the addition of the total amount of money (equivalent with the perimeter of the rectangle in Task 1a). Being an open task, we let Becky figure out what solving Task 2 meant for her: “How much money each person had.” At her first attempt to solve Task 2, Becky was unsuccessful. She divided the total amount of money ($120) by the number of children (4) and considered $30 as being the amount of money each person would have if they had equal shares. To find the amount of money each 1st grader had, she divided $30 by 2 and got $15. To find the amount of money each 5th grader had, she added $15 to $30 and got $45. Prompted to check her answer, Becky realized that $45 was not “twice as much” as $15. We directed Becky to use the Figurative Method throughout her second attempt to solve Task 2 (see Figure 5, below):

VP: Who has the least amount of money?
Becky: 1st graders.
VP: Can you represent the amount of money one 1st grader has?
Becky: Yes (She draws a line segment).
VP: Who has the most amount of money?
Becky: 5th graders (She draws a longer line segment).
VP: Can you represent the amount of money one 5th grader has?
Becky: Yes (she draws a longer line segment).
VP: Compare this line segment with the one you drew before. What do you notice?
Becky: 5th graders have twice as much as 1st graders (She places a tick mark on the line segment representing one of the 5th graders’ money).
VP: What can you say about the other two children? Draw line segments for their amounts of money, too.
Becky: … (She draws two line segments).
VP: Would you put all their money together, please? What do you notice?
Becky: They would have together $120.
VP: Can you solve this problem now?
Becky: Yes, I need to divide by 6 because there are 6 amounts of money, or two extra amounts of money because the 5th graders have twice as much as the 1st graders (she divides 120 by 6 and writes the amounts of money each child has).
Since Becky solved Task 2 only after employing the Figurative Method, we believed we offered her an intellectual need for the use of the method in subsequent tasks.

In Task 3, Becky started with the sisters’ names: Ann and Belle. We agreed to name with A and B - the amounts of money each sister had, S - the amount of money they had together, and N - the excess of money Belle had. We asked Becky to compare A and B and write a relationship, like in Task 1b, with L and W. After some time and some serious thinking she wrote “B – A = N.” Becky correctly solved the problem for two sets of numbers, one chosen by her (S=$240 and N=$160, see Figure 6 below), and the other one chosen by us (S=$500, N=$32, not presented here).

To check if Becky could generalize her scheme/algorithm of solving Task 3, we asked her to write a letter to Tom (see Figure 7, below), explaining how she solved Task 3. We emphasized that Tom might choose different numbers when solving Task 3. Notwithstanding that in her letter A > B (it should have been the other way around, see Figure 6, above), and her intentions when writing $S - N \div 2 = B$ were to execute the operations in the order they appeared, from left to right, Becky proved that she understood the Figurative Method. Indeed, she started with one of the amounts of money, represented it as a line segment, and named it A. She also referred to “sister A” as a pointer to the amount of money sister A had, and to “sister B” as a pointer to the amount of money sister B had, respectively. She represented and named the other amount of money, B. The first line segment was longer that the second one, hence the amount A was greater than B. The equation $N + B =A$ accounted for the fact that A exceeded B by N. After Becky removed the excess N from S (she performed S – N), she
obtained two equal amounts, B. She figured B out by dividing \((S - N)\) by 2, then found A by adding \(N\) to B. She wrote the correct quantitative relationship (represented both figuratively as line segments and symbolically as equations). She carried out the appropriate arithmetic when working with numbers and explained her solution: “Take the total amount (S) and subtract \(N\) from it, then divide by 2, that is how much sister B has. If you would add sister \(B+N\) that is how much sister A has.”

**Figure 7: Task 3 general solution**

Becky solved Task 4 without difficulty. Given the symbolic representation of the quantitative relationship, \(b = 3 \times a + 12\), she successfully represented it figuratively. First, she represented \(a\) as the smaller line segment (see Figure 8, below), then she represented \(b\) as a line segment with the length three times the length of \(a\), plus an excess of 12. When prompted, she could name the line segments \(a\) and \(b\), respectively.

**Figure 8: Figurative representation in Task 4**

Becky understood the quantitative relationship, since she could carry out the correct sequence of arithmetic calculations. Given \(b = 165\), to find \(a\) she removed the excess by subtracting 12 from 165 and obtained 153, then she divided 153 by 3 (see Figure 9, below). She checked her division by doing the inverse operation (multiplication).

**Figure 9: Arithmetic calculations in Task 4**
DISCUSSION AND CONCLUDING REMARKS

From the data presented, given our criteria for understanding the Figurative Method, we may conclude that Becky understood the method. We can also make the case that the Figurative Method was a helpful tool in comprehending the quantitative situations in Tasks 1-4. For example, the representations by line segments of the quantities in Task 2 helped Becky “see” why her old way of thinking about the quantitative situation did not work (she needed to divide the total amount of money by the number of equal shares represented by congruent line segments, and not by the number of children with perceived equal shares of money).

Moreover, the symbolic representations were introduced organically by the Figurative Method. In Task 3, after Becky named the amounts of money $A$ and $B$, she expressed the fact that one of the amounts was greater than the other with ease by writing, “$B - A = N$” (when she considered $B > A$) or “$N + B = A$” (when she considered $A > B$).

We can also say that Becky started to reason quantitatively, since she comprehended the quantitative situations in Tasks 1-4. We are aware that quantitative reasoning takes a long time to develop, and a key feature of the quantitative reasoning is the complexity of the situations. In Tasks 1-4, the quantitative situations are simple. Also, we used only line segments and their lengths, and amounts of money. The result might have been different if we used quantities that are difficult to construct, such as time with different units of measurement. With respect to Becky, she was one of the advanced students in her class. Should we have conducted our teaching experiment with other students, their learning trajectories might have been different. To conclude, given the short period of time we met with the student, we found the Figurative Method to be a very useful bridge between numerical and quantitative reasoning.

NOTES

1. After this teaching experiment, Florin successfully introduced the Figurative Method to his Algebra I students. For 15 weeks, once per week for 20 minutes, his students reasoned quantitatively about more and more complex situations.

REFERENCES


SPOON-FEEDING TO TONGUE-BITING: AN EVOLVING INSTRUCTIONAL FRAMEWORK FOR PRIMARY SCHOOL MATHEMATICS

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In this paper, I examine the evolution of an instructional framework for primary mathematics during a sustained, on-site professional development project with one case study school. The project attempts to bridge the gap between the perpetually-reported issues with pedagogical practices in Irish mathematics lessons and those espoused as best practice in international literature; and, to a lesser extent those highlighted in the Primary School Mathematics Curriculum. This instructional framework is used in an attempt to support teachers in addressing these pedagogical shortcomings, in addition to enriching the quality of mathematics lessons through a heightened emphasis on mathematical thinking. The paper draws on one aspect of my doctoral thesis and so this partial analysis is limited to lesson observations and teacher interviews. Findings were that the instructional framework changed considerably during the project. Teachers reported finding this instructional framework to be very useful in their classroom teaching, particularly as a planning and a reflection tool. Findings suggest that the transition from the traditional role of the teacher to a more facilitative role where teachers and pupils have an equal voice was particularly challenging for teachers.

INTRODUCTION

Since the introduction of the 1999 Primary School Curriculum in Ireland three reports have been published by the Inspectorate regarding teaching and learning in mathematics. Similar concerns in mathematics lessons are highlighted across all three reports:

- an over-reliance on whole-class teaching in a majority of classrooms including teacher-dominated discussion;
- classroom environments where pupils are not provided with adequate opportunities to work collaboratively;
- an over-reliance on textbooks as the chief teaching aid;
- insufficient provision and use of resources, in particular, concrete materials; and
- insufficient differentiation to meet the needs of children with varying learning abilities and needs.

Furthermore, the Review of English, Mathematics and Visual Arts (DES, 2005) reported that in more than two-thirds of mathematics classrooms, teacher talk dominated where pupils worked individually and silently for excessive periods. Unsurprisingly, the recommendations from these reports include that the over-reliance on textbooks as the primary teaching aid should be discontinued; pupils should be encouraged to use a range of reasoning and problem-solving strategies; teachers’ awareness of the potential of co-operative or collaborative
learning should be heightened; talk and discussion should feature more prominently in mathematics lessons; and that pupils should ‘have access to the objects, equipment and materials necessary for them to discover, learn and consolidate their learning’ (DES, 2010, p.17).

These findings regarding the over-reliance on textbooks in Irish primary classrooms are mirrored in national and international assessments (e.g. TIMSS, 1995; NAMA, 1999; NAMA, 2004; NAMA, 2009; NAMIS, 2010). Furthermore, the Inspectors’ findings regarding the insufficient use of concrete materials are also corroborated in national assessments (e.g. NAMA, 2004; NAMA, 2009). Finally, the findings regarding the insufficient opportunities for collaborative learning are mirrored by the NCCA (2005) findings that whole-class teaching was the organisational setting most used, followed closely by individual work; whilst there was only limited use of pair or group work.

However, it would be naïve to assume that pair or group work should be championed at the expense of whole-class discussion or indeed that whole-class teaching is undesirable. Dooley’s (2010, p. 229) research in Irish mathematics classrooms highlights the potential of whole-class discussion suggesting that “extended whole-class conversation can be a vehicle for the construction of mathematical insight.” The need for various organisation settings is further illuminated by her finding that the effect of group work increased pupils’ contribution in whole-class discussion in that “pupils often consolidated or generated ideas at this stage…” (p. 236). However, the quality of whole-class discussion is important and Dooley (p. 253) contends that “the construction of insight by pupils in whole-class settings is a complex interaction of task, classroom discourse style and pupil engagement” where the discourse embraces a conjectural atmosphere. Similarly, analysis of a Fourth class mathematics lesson in an Irish primary school leads NicMhuirí (2011) to assert that the discourse was not truly mathematical and that opportunities for high level mathematical thinking were limited. Both studies illuminate the pivotal role played by discourse styles, in particular, teacher follow-up moves during mathematical conversations.

So, general agreement exists that the pedagogical approaches employed in Irish primary mathematics lessons are misaligned with the constructivist principles which are advocated in the Primary School Mathematics Curriculum (PSMC) and so need to be reformed and enhanced. Achieving this in individual classrooms is challenging; attempting to do this at a whole-school level is even more ambitious. Thus, teachers require guidance and support in attempting to address these perpetually-reported pedagogical shortcomings. In this paper, I analyse one particular aspect of my doctoral thesis – the evolution of an instructional framework for teaching and learning mathematics. This instructional framework is used in an attempt to support teachers in addressing the pedagogical shortcomings outlined above, in addition to enriching the quality of mathematics lessons through a heightened emphasis on mathematical thinking.

OUTLINE OF RESEARCH PROJECT

The research project aimed to explore the experiences and perspectives of primary school pupils and teachers during the implementation of a reform approach to mathematics. The case study school is a mixed gender, vertical school in a small rural town. At the beginning of the
study, the school was in existence for three years resulting from an amalgamation of an all-girls’ and an all-boys’ school. There were 205 pupils enrolled in the school during the project: 103 girls and 102 boys. The participants in the study included all of the teaching staff of the case study school which comprised of thirteen teachers: an administrative principal, eight class teachers and four support teachers.

METHODOLOGY

The methodology involved collaborative on-site professional development (PD) whereby the teachers were firstly up-skilled in the use of an instructional framework and secondly, the teachers collaboratively devised mathematics lessons which they subsequently taught. After analysis of their standardised test results, the school chose the strand of Measures as the focus for the project. The research project took place over a year. The PD aspect of the research took place over two school terms or seven months and was rolled out in two phases. Phase One focused on an instructional framework devised by Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier & Human (1997) and was implemented through the topic of Length. Phase Two focused on a revised version of this instructional framework and was implemented through the topic of Weight. The PD was mainly collaborative, for example, sharing of expertise and the generation of collaborative lesson plans; however, for the first few weeks of each phase, the PD was front-loaded with input from the researcher in order to up-skill teachers in:

1. an evidence-based instructional framework for teaching and learning mathematics with understanding;
2. recommended approaches in the school-selected strand of Measures, in particular, the strand units of Length/Weight; and
3. pupil conceptions/misconceptions in the school-selected strand units of Length/Weight.

Both the PD and the data collection took place simultaneously. The following data collection instruments were used: observation, review of documents, individual interviews, focus groups and field notes. Although all teachers were invited to participate in the PD, early in the site visit, teachers were asked to express an interest in being ‘tracked’ throughout the study. All teachers volunteered to be ‘tracked’; however, only four teachers were chosen. Data from these teachers and classrooms form the bulk of the research. Two mathematics lessons were observed in each of the ‘tracker’ classes – one in each phase. Although multiple data sources were collected, the focus of this paper – the evolution of the instructional framework – relies chiefly on lesson observation and teacher interviews.

ANALYSIS

Rationale for using the instructional framework

Although a significant amount of research has been conducted regarding the teaching and learning of mathematics in primary schools, very little tangible guidance for teachers appears to be available in one place. In order to garner information about good practice, it is often
necessary for teachers to draw from many varied, often far flung sources. Using this instructional framework was an attempt to highlight good practice for teachers in a meaningful, user-friendly format. Furthermore, the Instructional Framework for Teaching and Learning Mathematics with Understanding (Hiebert et al, 1997) was chosen as a tool for guiding teachers “toward designing classrooms that encourage understanding” (p. xv) because its genesis lies in multiple research projects. Four mathematics teaching and learning projects contributed to the development of this framework: Cognitively Guided Instruction at the University of Wisconsin-Madison; Conceptually Based Instruction at the University of Delaware; Problem Centred Learning at the University of Stellenbosch; and Supporting Ten-Structured Thinking at Northwestern University. The framework “within which teachers can reflect on their own practice, and think again about what it means to teach for understanding” (p.xix) arose out of five years of collaboration between the researchers in all four projects. Although the projects were quite different a “rather striking consensus about the features of classrooms that are essential for supporting students’ understanding” (p.xix) grew. These core features form the basis of the instructional framework.

**Evolution of the instructional framework**

**Phase One**

Table 1 below illustrates Hiebert et al’s (1997) instructional framework. This is a summary of the book in which each dimension has a corresponding chapter. The teachers analysed this framework in detail, for example, groups of teachers each studied, discussed and analysed different chapters of the book before summarising the main points and sharing with other groups.

**Table 1: Summary of dimensions and core features of classrooms that promote understanding (Hiebert et al, 1997)**

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>Core Features</th>
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</thead>
<tbody>
<tr>
<td>Nature of Classroom Tasks</td>
<td>Make mathematics problematic</td>
</tr>
<tr>
<td></td>
<td>Connect with where students are</td>
</tr>
<tr>
<td></td>
<td>Leave something behind of mathematical value</td>
</tr>
<tr>
<td>Role of the Teacher</td>
<td>Select tasks with goals in mind</td>
</tr>
<tr>
<td></td>
<td>Share essential information</td>
</tr>
<tr>
<td></td>
<td>Establish classroom culture</td>
</tr>
<tr>
<td>Social Culture of the Classroom</td>
<td>Ideas and methods are valued</td>
</tr>
<tr>
<td></td>
<td>Students choose and share their methods</td>
</tr>
<tr>
<td></td>
<td>Mistakes are learning sites for everyone</td>
</tr>
<tr>
<td></td>
<td>Correctness resides in mathematical argument</td>
</tr>
<tr>
<td>Mathematical Tools as Learning Supports</td>
<td>Meaning for tools must be constructed by each user</td>
</tr>
<tr>
<td></td>
<td>Used with purpose – to solve problems</td>
</tr>
<tr>
<td></td>
<td>Used for recording, communicating and thinking</td>
</tr>
<tr>
<td>Equity and Accessibility</td>
<td>Tasks are accessible to all students</td>
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<tr>
<td></td>
<td>Every student is heard</td>
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<tr>
<td></td>
<td>Every student contributes</td>
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</tbody>
</table>
Similarly, Table 2 outlines the same instructional framework but with one addendum – an additional information column. This additional information was extrapolated from various chapters of the book and was included as a more detailed reference guide for teachers. Both versions were used with teachers during the PD in Phase One; however, teachers reported using Table 2 more often, both as a reference guide and when planning mathematics lessons.

**Table 2: Dimensions and core features and additional information of classrooms that promote understanding**

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>Core Features</th>
<th>Additional Information</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nature of Classroom Tasks</strong></td>
<td>Make mathematics problematic</td>
<td>Tasks should encourage reflection and communication</td>
</tr>
<tr>
<td></td>
<td>Connect with where students are</td>
<td>Tasks should allow students to use tools</td>
</tr>
<tr>
<td></td>
<td>Leave something behind of mathematical value</td>
<td>Tasks should leave behind important residue - there are 2 types of residue: a) insights into the structure of mathematics (mathematical relationships) and b) strategies or methods for solving problems</td>
</tr>
<tr>
<td><strong>Role of the Teacher</strong></td>
<td>Select tasks with goals in mind</td>
<td>Explanations and demonstrations by the students become more important than those by the teacher</td>
</tr>
<tr>
<td></td>
<td>Share essential information</td>
<td>Teachers need to select sequences of tasks not just individual tasks</td>
</tr>
<tr>
<td></td>
<td>Establish classroom culture</td>
<td>Teachers should remove themselves from a position of authority (deciding whether answers are correct) in order to promote the autonomy of students’ intellectual activity</td>
</tr>
<tr>
<td><strong>Social Culture of the Classroom</strong></td>
<td>Ideas and methods are valued</td>
<td>Students work together to solve problems and interact intensively about solution methods</td>
</tr>
<tr>
<td></td>
<td>Students choose and share their methods</td>
<td>Collaboration depends on communication and social interaction (individual work can be followed by small group or whole class discussion of methods and ideas)</td>
</tr>
<tr>
<td></td>
<td>Mistakes are learning sites for everyone</td>
<td>Students must learn to live with a certain amount of uncertainty</td>
</tr>
<tr>
<td></td>
<td>Correctness resides in mathematical argument</td>
<td></td>
</tr>
<tr>
<td><strong>Mathematical Tools as Learning Supports</strong></td>
<td>Meaning for tools must be constructed by each user</td>
<td>Tools include oral language, physical materials and symbols</td>
</tr>
<tr>
<td></td>
<td>Used with purpose – to solve problems</td>
<td>Students develop meaning for tools by actively using them in a variety of situations, to solve a variety of problems</td>
</tr>
<tr>
<td></td>
<td>Used for recording, communicating and thinking</td>
<td>Using tools can free our thinking for more creative activities</td>
</tr>
<tr>
<td><strong>Equity and Accessibility</strong></td>
<td>Tasks are accessible to all students</td>
<td>All ideas and methods are potential learning sites</td>
</tr>
<tr>
<td></td>
<td>Every student is heard</td>
<td>A variety of ideas are essential for fuelling rich discussions</td>
</tr>
<tr>
<td></td>
<td>Every student contributes</td>
<td>Each person learns to respect and value each other’s thinking</td>
</tr>
</tbody>
</table>
During the initial stages of PD, concerns regarding this type of approach were raised by several teachers. Examples of these concerns included a) the very different role of the teacher to that of the traditional role, in particular, when teachers should ‘tell’; b) the ability of less-able pupils to be involved in problematic tasks and any subsequent self-esteem issues; and c) the suitability of this approach for certain types of mathematics, for example, the possibility of this approach being more suitable for traditionally hands-on content such as Length and less suitable to more “abstract content such as Number”. Finally, questions were also posed regarding the possible reactions of parents to this type of approach, in particular, parents of more-able pupils considering the collective responsibility for understanding which is espoused in this approach. Teachers expressed concerns regarding negative reactions from parents if their children have to regularly help others, clarify ideas for others and explain concepts to others. However, despite these concerns, teachers experimented with the implementation of this approach.

Following several weeks of implementation, the tracker teachers were very complimentary about the instructional framework, in particular, its use as a teaching aid. The reported benefits of using the instructional framework included a) using it as a reflection and planning tool for teaching; b) ensuring a focus on problem solving; c) encouraging pupil explanations; and d) giving pupils control of their own learning. Interestingly, all of these teachers reported the same challenge in using the framework – the role of the teacher. In particular, teachers reported a difficulty in changing from a more traditional teacher role to one where teachers and pupils have an equal voice:

Well the biggest challenge would be the teacher stepping back and trying to give that control…rather than the teacher voice it is the pupil voice that needs to be heard. That was the biggest challenge.

Teacher has removed him/herself from a position of authority – that was hard…

I suppose the biggest challenge was that change where you weren’t the teacher anymore. That is a big thing for a teacher. That was the biggest thing.

Another teacher mirrored this latter view that the teacher role was being relinquished by “handing over the role of the teacher to the children.” Although teachers found this role-change challenging, they also highlighted some benefits to it, for example, teachers viewed it as giving more agency to pupils by “holding back and letting them (the pupils) come around to solutions’ and ‘the teacher was purely observational really while they (the pupils) were working rather than the teacher spoon-feeding which we tended to do. …That is our nature to spoon-feed them everything.” Another teacher illuminated the dichotomy that exists between the teacher telling and the pupils telling:

So the teacher trying to talk less and giving control over to the children was great, it took that little bit of biting your tongue but once they got into it like they were great…and rather than you telling them, they were telling you. It was great.

Based on this teacher feedback, the suggestion that the role of the teacher should be further clarified in any revised instructional framework is unsurprising. In particular, teachers requested clarification regarding the amount of guidance to give pupils, for example, “It is
just to know have you said too much or too little”. In other words, by stepping back are teachers stepping out or stepping aside? Teachers referred to this change in many ways from “stepping back” to “handing over” to “letting them off” to “biting your tongue” to “observation” to “facilitation”. Derived from this, teachers suggested the need for additional guidance by including “the language of questioning and affirming comments” in any revision of the framework. The uncertainty regarding questioning and guidance was a very real one for teachers and is highlighted by one of the tracker teachers:

Should you ask any question or by asking a certain question are you giving them (the pupils) too much…are you like helping them to discover? Do you want to ask the bare essentials and let them completely come up with everything or do you want to steer the question?...so you want questions that are going to get them completely thinking kind of openly or do you want a kind of steering question? It is important to know what type of question to ask because you weren’t sure like, should I even have said that, or is that still me being in control…

This teacher role uncertainty was also evident in the observed lessons in Phase One. For the majority of these, teachers had completely ‘stepped back’ and were providing little or no guidance to pupils. Teachers appeared to have taken on an observational role rather than a proactive, facilitative role. This is despite the fact that some teachers had reported taking on a facilitative role. So facilitation, its meanings and practical application in mathematics lessons needed to feature in the Phase Two PD and also in any revised instructional framework.

Observations of the mathematics lessons also revealed that although ample collaborative opportunities were evident through pair and group work, there was very little cross-fertilisation between the groups, in that learning seemed to remain within each small group and was not shared with other groups or with the whole class. Arising from this, sharing and building on mathematical thinking was not obvious at a whole-class level because whole-class mathematical discussion did not take place. This is regrettable considering Dooley’s (2010) finding that group work can increase pupils’ contributions to whole-class discussion. Moreover, the problems did not appear to be particularly rich or challenging. In conclusion, based on the teacher interviews and lesson observations, the following chief additions were necessary in an effort to refine and enhance the instructional framework for Phase Two:

- an emphasis on the teacher’s role as an active, skilled facilitator;
- inclusion of teacher talk which encourages reflection and communication, in particular, the language needed for facilitation and the questioning needed for discovery learning;
- an emphasis on whole-class discussion in developing mathematical thinking including reasoning, and revision of conjectures and solutions;
- an emphasis on students using language to refine, revise, clarify, build on and communicate mathematical thinking;
- inclusion of revoicing to deepen mathematical understanding and to enrich mathematical thinking; and
• an emphasis on students choosing to record verbally, concretely, pictorially/graphically, symbolically or in written form.

Two dimensions of Hiebert et al’s instructional framework appeared to be misplaced – Social Culture of the Classroom and Equity and Accessibility appeared to be pre-requisites for a certain type of classroom. Therefore, in the revised instructional framework, these dimensions were combined and moved to a foreword focusing on pre-requisites for a classroom environment. Three dimensions remained so combined with the new dimension regarding teacher talk, the revised instructional framework had four dimensions all beginning with T – Tasks, Teachers, Tools, Talk; hence, the revised instructional framework evolved into the 4Ts Instructional Framework for Maths (see Table 3).

Table 3: The 4Ts instructional framework for maths

<table>
<thead>
<tr>
<th>4Ts</th>
<th>Core Features</th>
<th>Additional Information</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tasks</strong></td>
<td>Make mathematics problematic</td>
<td>Tasks should encourage reflection and communication</td>
</tr>
<tr>
<td></td>
<td>Connect with where students are</td>
<td>Tasks should allow students to use tools</td>
</tr>
<tr>
<td></td>
<td>Select tasks with goals in mind</td>
<td>Teachers need to select sequences of tasks not just individual tasks</td>
</tr>
<tr>
<td></td>
<td>Leave something behind of mathematical value</td>
<td>Tasks should leave behind important residue - there are 2 types of residue: a) insights into the structure of mathematics (mathematical relationships) and b) strategies or methods for solving problems</td>
</tr>
<tr>
<td><strong>Teacher</strong></td>
<td>Take on an active, skilled facilitator role</td>
<td>Teachers facilitate discussion and value silence (Pratt, 2002) in the course of mathematical discussions</td>
</tr>
<tr>
<td></td>
<td>Share essential information</td>
<td>Explanations and demonstrations by the students become more important than those by the teacher</td>
</tr>
<tr>
<td></td>
<td>Establish classroom culture</td>
<td>Teachers remove themselves from a position of authority (deciding whether answers are correct) in order to promote the autonomy of students’ mathematical thinking</td>
</tr>
<tr>
<td></td>
<td>Encourage revision of conjectures</td>
<td>Revising conjectures (Lampert, 2001) and solutions is encouraged</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Reasoning is used to judge whether a conjecture/solution is mathematically sound</td>
</tr>
<tr>
<td><strong>Tools</strong></td>
<td>Meaning for tools must be constructed by each user</td>
<td>Tools include oral language, physical materials, pictures/diagrams (Askew, 2012) and symbols</td>
</tr>
<tr>
<td></td>
<td>Used with purpose – to solve problems</td>
<td>Students develop meaning for tools by actively using them in a variety of situations, to solve a variety of problems</td>
</tr>
<tr>
<td></td>
<td>Used for recording, communicating and thinking</td>
<td>Using tools can free our thinking for more creative activities</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Students choose to record verbally, concretely, pictorially/graphically, symbolically or in written form</td>
</tr>
<tr>
<td><strong>Talk</strong></td>
<td>Teacher talk encourages reflection and communication</td>
<td>Teacher questioning is open-ended and probing</td>
</tr>
<tr>
<td></td>
<td>Students use language to refine, revise, clarify and communicate mathematical thinking</td>
<td>Teacher responses are neutral (Pratt, 2002) and encourage further discussion</td>
</tr>
<tr>
<td></td>
<td>Talk is used to encourage and communicate reasoning</td>
<td>Revoicing is used to deepen mathematical understanding and to share mathematical thinking</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Students build on the mathematical ideas of others</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Talk is used to encourage and communicate reasoning – both in verbal and in written form (either in a journal or a notebook)</td>
</tr>
</tbody>
</table>
Phase Two

The 4Ts Instructional Framework for Maths was used to teach the strand unit Weight. Due to space restrictions, the Pre-requisites foreword, in addition to the suggestions for teacher language and questioning, are not included in this paper; however, these aspects featured as part of the revised instructional framework.

The observed differences in maths lessons between Phases One and Two were noteworthy. In particular, the sharing and promotion of mathematical thinking appeared to be more prevalent in the lessons in Phase Two. This finding is consistent with the teacher feedback from the Phase Two interviews. It is important to note that these observations are reflective of the observed lessons only. They cannot be used to generalise about other maths lessons, either in these tracker classes during these phases or indeed in other classes in the school. A number of similarities and differences are outlined in Table 4. The similarities are denoted by italics.

Table 4: Observed similarities and differences in maths lessons between phases one and two

<table>
<thead>
<tr>
<th></th>
<th>Phase One Lessons</th>
<th>Phase Two Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Hands-on tasks</strong></td>
<td>Use of hands-on tasks in collaborative group settings</td>
<td>Use of hands-on tasks in collaborative group settings</td>
</tr>
<tr>
<td><strong>Concrete materials</strong></td>
<td>Use of concrete materials</td>
<td>Use of concrete materials</td>
</tr>
<tr>
<td><strong>Discovery learning</strong></td>
<td>Use of discovery learning</td>
<td>Use of discovery learning</td>
</tr>
<tr>
<td><strong>Collaboration</strong></td>
<td>Use of extensive pair and group work</td>
<td>Use of extensive pair and group work</td>
</tr>
<tr>
<td><strong>Groupings</strong></td>
<td>Use of mostly ability groupings</td>
<td>Use of mostly mixed-ability groupings</td>
</tr>
<tr>
<td><strong>Whole-class discussion</strong></td>
<td>Little if any whole-class discussion was evident</td>
<td>Whole-class discussion was used extensively to share ideas and concepts following small-group work</td>
</tr>
<tr>
<td><strong>Problem solving</strong></td>
<td>Some problem solving evident but mainly related to a practical task</td>
<td>Problem solving was evident in all maths lessons and ranged from the practical through to abstract</td>
</tr>
<tr>
<td><strong>Types of problems</strong></td>
<td>Many of the problems were simple, routine, or lower-order in nature</td>
<td>Most of the problems were rich and higher-order in nature</td>
</tr>
<tr>
<td><strong>Sharing solution methods</strong></td>
<td>Little evidence of how pupils solved a problem or task</td>
<td>Evidence of pupils sharing and explaining how they arrived at solution methods, for example, using representative materials, drawing a table, trial and error, discussion, etc.</td>
</tr>
<tr>
<td><strong>Role of teacher</strong></td>
<td>Teachers appeared unsure of their role and often stepped back completely from the lesson</td>
<td>Teachers appeared more sure of their role and took on a facilitative role in asking probing, open-ended questions, promoting the sharing</td>
</tr>
</tbody>
</table>
of ideas, suggesting revoicing, etc.

<table>
<thead>
<tr>
<th>Mathematical authority</th>
<th>Mathematical authority appeared to reside with pairs and groups of pupils rather than with the teacher</th>
</tr>
</thead>
<tbody>
<tr>
<td>Communicating mathematical ideas</td>
<td>Little evidence of pupils communicating their mathematical ideas</td>
</tr>
<tr>
<td>Refining mathematical ideas</td>
<td>Little evidence of pupils refining their mathematical ideas</td>
</tr>
<tr>
<td>Learning logs</td>
<td>Learning logs were often descriptive in nature – often describing the task rather than reflecting on the learning</td>
</tr>
<tr>
<td></td>
<td>Learning logs appeared to be more reflective of the learning that took place. This may have been aided by teacher-suggested prompts, for example, What I learned…; What helped me to learn…; etc.</td>
</tr>
</tbody>
</table>

In summation, rich problem solving in addition to communicating and expressing mathematical thinking appeared to be notably more prevalent in the lessons in Phase Two compared to those in Phase One. Furthermore, teachers appeared to be more confident and relaxed with the approach and importantly, with their role as facilitator within this approach. It is important to note that the mathematics lessons in Phase One satisfy most of the recommendations from the Inspectorate reports. These include the judicious use of textbooks; pupils using a range of problem solving strategies; collaborative learning; focussed use of materials and resources; and talk and discussion. However, as outlined in Table 4, although this is a marked improvement from traditional mathematics practice, a dearth in communicating and expressing still exists in these lessons. It is only in the Phase Two lessons that these absent elements became interwoven with the other pedagogical elements resulting in classrooms where pupils were thinking deeply and communicating frequently about mathematics.

**DISCUSSION**

The evolution of the instructional framework during the project was chiefly influenced by the experiences of teachers coupled with lesson observation and professional reading. The corresponding changes in the mathematics lessons were striking; however, a word of caution is necessary. This instructional framework was not developed in a vacuum; neither was it implemented by teachers in a vacuum. It was one segment of a sustained, on-site professional development programme that included mathematical content knowledge, mathematical pedagogical knowledge, reformed pedagogical approaches, pupil feedback from mathematics lessons, professional dialogue, collaborative planning, professional reading, video footage and podcasts. The changes to classroom practice cannot therefore be solely attributed to the 4Ts Instructional Framework for Maths. The process of this collective endeavour is important.
CONCLUDING REMARKS

Teachers’ experiences of using the 4Ts Instructional Framework for Maths were extremely positive in this case study school. Classroom practice appears to have changed considerably in the tracker classes for the strand units of Length and Weight. Communicating ideas, reasoning, refining mathematical thinking, rich problem solving and democratic collaboration appear to be to the forefront of these classrooms. However, challenges have also been apparent throughout this evolutionary process including challenges to teacher beliefs, in addition to time constraints. The teacher’s role as an active, skilled facilitator appeared to pose the utmost challenge for teachers, particularly when facilitating mathematical discussion. However, the importance of interpretive flexibility and professional discretion cannot be over emphasised. Successful mathematics classrooms do not mean conforming to a highly prescribed method of teaching (Stigler & Hiebert, 1999; Hiebert et al, 1997). Instead, it means “taking ownership of a system of instruction, and then fleshing out its core features in a way that makes sense for a particular teacher in a particular setting” (Hiebert et al, p. 14). In this way, the 4Ts Instructional Framework for Maths can act as a temporary bridge or scaffold between recommended pedagogical features and pupil learning where planning, implementation and reflection feature in an iterative feedback loop. Equally, it has the potential to guide teachers in progressing from (in the words of one of the teachers) “spoon-feeding to biting your tongue” … and beyond!

REFERENCES


EXPLORING THE ROLE OF PATTERNS: EARLY ALGEBRA IN THE IRISH PRIMARY MATHEMATICS CURRICULUM

Aisling Twohill
St Patrick’s College, Drumcondra

Studying arithmetic in primary school to the exclusion of an algebraic perspective hinders children in developing skills in algebraic reasoning and analytical thinking (Kaput, 2008; Shmittau, 2011; Cai, Fong Ng and Moyer, 2011). An alternative to studying algebra after years of studying arithmetic is the early algebra approach. Early algebra does not imply introducing formal manipulation of symbols and expressions in primary school, but rather the development of skills which young children already possess in order to nurture algebraic habits of mind (Mason, 2008). In the Irish primary school curriculum instruction in algebra commences in Junior Infants but the challenges presented by formal algebra in secondary school remain a reality (OECD, 2009). In this paper I aim to examine the design of the algebra strand of the Irish primary school curriculum, and in particular to compare this design to the findings of research in the learning and teaching of early algebra. Specifically this paper will focus on patterning as an effective step into algebra. Lannin (2009) suggests that when algebra begins early patterns must play an intrinsic role in the development of children’s thinking (also Threlfall, 1999; Owen, 1995; Dooley, 2012; Warren, 2005; and others).

INTRODUCTION

In 1999, the Department of Education and Skills launched the revised Primary School Curriculum for Irish schools. This curriculum reflected the research available at the time regarding teaching approaches and methodologies (Government of Ireland, 1999). Since the publication of the mathematics curriculum in 1999 there has been considerable discussion in research circles regarding the teaching and learning of algebra. Algebra was identified as a ‘gatekeeper’ in education (Mason, 2008; Kaput, 2000). When students encountered formal algebra in early secondary school, the difficulties they encountered curtailed their progress through the education system. Much discussion centred round the introduction of algebra early in a child’s education to avoid such challenges. In Irish schools, the algebra strand of the Primary School Mathematics Curriculum (PSMC) is introduced in Junior Infants, and yet in the Programme for International Student Assessment 2003 Irish students found the items assessing algebra to be of relative difficulty in comparison to other areas of mathematics (OECD, 2009).

There exists also a large body of research literature concerning the content and design of instruction in early algebra (Radford, 2011; Lannin, Barker and Townsend, 2006; Threlfall, 1999; among others). The purpose of this paper is to consider the design of the algebra strand of the PSMC and to interrogate in what way it reflects the findings of research into how children develop skills in algebraic reasoning. To begin, I will outline a rationale for why it is pertinent to investigate the teaching and learning of algebra in Irish primary schools. I will
proceed to describe the structure of the Irish primary school curriculum algebra strand and discuss the progression of thinking therein. In the discussion section of this paper I will focus on two elements of the development of algebraic reasoning, namely the patterning route to generalisation and the explicit approach to pattern solving.

RATIONALE

On the basis of research conducted recently, one may find some positive indicators of how algebra is taught in Irish schools. The PSMC is organised into five content strands, one of which is algebra (Government of Ireland, 1999). The algebra strand begins when children commence school, in Junior Infants. In 2003 the Inspectorate of the Department of Education and Skills investigated how the revised primary school curriculum was being implemented by teachers. The implementation of the algebra strand was ‘good’ in most classrooms (DES, 2005). However, the findings of national and international assessments of attainment paint a different picture.

In 2009, the National Assessment of Mathematics and English Reading assessed the attainment of nearly 4000 Sixth class pupils. Most of the children in the Sixth class cohort (90%) could solve routine word problems and 65% of children could “demonstrate understanding of a letter as a placeholder in algebraic expressions, and complete two-step number sentences involving addition and subtraction” (Eivers, Close, Shiel, Millar, Clerkin, Gilleece and Kiniry, 2010, p.42). Only 35% succeeded in translating word problems into number sentences and vice versa and only 10% achieved a mastery of algebra which enabled them to evaluate “linear expressions and one-step equations” (ibid., p.42). It is a cause for concern that 90% of the children who are within months of entering secondary school failed to evaluate a one-step algebraic equation, and that 65% could not translate between word and number sentences. As a result, problem-solving strategies which involve deriving a number sentence from a word problem are not available to them (Polya, 1973; Carpenter & Levi, 2000). This situation may have contributed to the poor success of children in this study in the process skill of ‘Apply & Problem-Solve’. Of questions posed to Sixth class children requiring an application of problem-solving skills, only 44% were answered correctly, which is the lowest percent correct score of the five process skills (the other four being Understand & Recall, Implement, Integrate & Connect, and Reason (Eivers et al., 2010)).

In 2011, the Trends in International Mathematics and Science Study (hereafter TIMSS 2011) measured the mathematics attainment of 4560 children in Fourth class in Ireland. In the design of the mathematical assessment presented to participating students, the researchers targeted the cognitive processes of Knowing, Applying and Reasoning (Eivers and Clerkin, 2012). Mean scores were published as a subscale under each cognitive process. Reasoning was identified as including “intuitive and inductive reasoning based on patterns and regularities that can be used to arrive at solutions to non-routine problems” (Mullis, Martin, Ruddock, O'Sullivan and Preuschoff, 2009, p.45). The Irish cohort achieved a mean score on this subscale which was significantly lower than the corresponding overall mean for mathematics (Eivers and Clerkin, 2012).
There seems to be a mismatch between the perception of how algebra is being taught and the findings from empirical research in terms of how children are learning. Corcoran (2005) discusses the disparity between what is perceived to be taught and what is in actuality being learned in Irish primary classrooms. She suggests that a variety of contributing factors may be resulting in a situation where the curriculum as designed is not being implemented in classrooms. Further research is needed to explore how children’s algebraic reasoning skills are developing in primary school in Ireland.

IRISH PRIMARY CURRICULUM

As one of the five strands of the PSMC there are learning objectives within the algebra strand from Junior Infants (when children commence school) and for every subsequent year of primary school. In this section I will examine what constitutes the algebra curriculum in primary school in Ireland, focusing on the area of patterning.

The broad objectives within the Algebra strand state that, allowing for varying abilities and circumstances, children should be enabled to:

- explore, perceive, use and appreciate patterns and relationships in numbers; identify positive and negative integers on the number line; understand the concept of a variable, and substitute values for variables in simple formulae, expressions, and equations; translate verbal problems into algebraic expressions; acquire an understanding of properties and rules concerning algebraic expressions; solve simple linear equations; use acquired concepts, skills and processes in problem-solving (Government of Ireland, 1999, p.13).

Along with the broad objectives are detailed content objectives. The content of algebra study, as intended under this curriculum, includes the identification and extension of pattern and sequences and the application of patterns as an aid to computation. The concept of an unknown quantity within a mathematical expression is explored and this exploration is extended to include variables in sixth class. It is prescribed that children learn to translate number sentences into word sentences and vice versa, including sentences which contain an unknown quantity. Children should solve such sentences from third class. In Fifth and Sixth class, negative numbers are introduced when children should learn to identify them on the number line and add them. Finally, in Fifth and Sixth class, children are required to know the basic rules of mathematical operations, their priorities and the use of brackets.

An exploration of the entire content of the algebra strand is beyond the scope of this paper and I will focus therefore on the content objectives which relate to patterning. The patterning content objectives included under the Algebra strand of the PSMC are outlined in Table 1.
<table>
<thead>
<tr>
<th>Class</th>
<th>Content Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>Junior Infants</td>
<td>Identify copy and extend patterns in colour, shape and size.</td>
</tr>
</tbody>
</table>
| Senior Infants | Identify copy and extend patterns in colour, shape, size and number (3-4 elements);  
                  Discover different arrays of the same number;  
                  Recognise patterns and predict subsequent numbers.                                                                                                                                                    |
| First class   | Recognise [numeric] patterns, including odd and even numbers;  
                  Explore and use patterns in addition facts.                                                                                                                                                           |
| Second class  | As for 1st class but progressing to the prediction of subsequent numbers in the pattern recognition objective.                                                                                               |
| Third class   | Explore, recognise and record patterns in numbers, 0-999;  
                  Explore, extend and describe (explain rule for) sequences;  
                  Use patterns as an aid in the memorisation of number facts.                                                                                                                                             |
| Fourth class  | As for 3rd class but progressing to include numbers from 1000-9999.                                                                                                                                           |
| Fifth class   | Explore and discuss simple properties and rules about brackets and priority of operations;  
                  Identify relationships and record verbal and simple symbolic rules for number patterns.                                                                                                              |
| Sixth class   | Know simple properties and rules about brackets and priority of operations;  
                  Identify relationships and record symbolic rules for number patterns;  
                  Explore the concept of a variable in the context of simple patterns, tables, and simple formulae and substitute values for variables.                                                          |

Table 1: Algebra strand content objectives relating to patterning, in the Irish primary school curriculum (Government of Ireland, 1999).

Within the patterning content objectives outlined in Table 1, there are objectives concerned with generalisation of arithmetic in the identification and application of the properties of operations. Of more interest in this paper are the patterning objectives which could facilitate children in developing their ability to explore, identify, express and apply generalisations, in a broader sense and not limited to the area of arithmetic.
Examining the patterning content objectives there is a developmental sequence from Junior Infants to Sixth class as summarised in Figure 1.

![Diagram of developmental progression in patterning]

It is worth mentioning that within the PSMC, algebra is not referred to under any of the other strands either among the content objectives or the teaching guidelines (Government of Ireland, 1999). There is an attempt within the mathematics curriculum to guide teachers towards the possibility for both integration of mathematics with other subject areas and also linkage of strands and strand units. Specifically, each strand unit contains suggestions for subject area(s) which would be relevant for integration and many content objectives contain suggestions for linkage with other strand units. Within the algebra strand there is no suggestion for linkage or integration and there is no mention of linkage with algebra under any other strand unit of the curriculum. The Irish curriculum seems to endeavour to cover fundamental content which will enable students to engage with algebra in the secondary school, and also to introduce the concept of pattern and the underlying structure of number. These are vital elements of any instruction in algebra, but teaching them in isolation can be
seen as more arduous and less efficient than teaching algebra through an integrated curriculum (Carpenter, Levi, Franke & Zeringue, 2005; Blanton & Kaput, 2000).

PATTERN AS A ROUTE TO GENERALISATION

Mathematics is the science of patterns. The mathematician seeks patterns in number, in space, in science, in computers, and in imagination. Mathematical theories explain the relations among patterns; functions and maps, operators and morphisms bind one type of pattern to another to yield lasting mathematical structures. Applications of mathematics use these patterns to "explain" and predict natural phenomena that fit the pattern (Steen, 1988, p.616).

Much discussion regarding extending algebra to the early years of schooling involves patterns and patterning. Lannin (2009) suggests that the emphasis on patterning as an introduction to algebra stems from the role of patterns as “dynamic representation of variables” (p.233). The ability to generalise from and work with generalisations is fundamental to children’s developing thinking in mathematics (Kaput, 2008). Also Lannin (2009) suggests that generalizing through patterning activities may create a bridge between students’ knowledge of arithmetic and their understanding of symbolic representations.

Radford (2011) aims to specify where algebra is involved in activities which involve extending sequences. He researched the actions and underlying thought processes undertaken by Grade 2 students involved in extending non-numeric patterns. The findings of the study demonstrated that in order to extend a sequence, the students were compelled to “grasp a commonality”, and in order to do so, they coordinated spatial and numerical structures. Radford (2011) did not consider the process involved in this sequence extension to be algebraic but suggested that algebraic thinking is introduced when the child is expected to consider “remote figures [from] beyond the perceptual field” (p.318).

In comparison to Radford’s view, Threlfall (1999) asserts that patterning has been established as a vital preliminary component for children to progress to the study of algebra. He contends that involvement in patterning supports the children in their developing abilities in mathematics “including recognition and prediction, and the generalization and communication of rules” (p.21). Owen (1995) concurs with Threlfall’s thinking and highlights areas of mathematical thinking which cannot be developed in the pre-number child through activities other than patterning, namely “recognition of an event, prediction of a future event, generalization of a rule and communication of that rule” (Owen, 1995, p.126). In the next section I will examine the Irish curriculum with regard to patterning and consider what the opposing positions of Radford (2011) and Threlfall (1999) might suggest regarding its content.

The patterning content objectives of the Irish curriculum in Infant classes are supported therefore by the position of Threlfall and Owen. Radford’s viewpoint may highlight a deficiency in the exclusion of far terms under the suggested activities accompanying content objectives (Government of Ireland, 1999). Interrogating the suggested activities within the
curriculum, none of the non-numeric patterns are growing and no reference is made to near or far terms. The numeric pattern work suggested for Infant classes is limited to identifying missing numbers in forward and backward counting sequences. In First and Second class the patterning activities suggested are comprised entirely of group counting and there is no mention of non-numeric patterns. Within the recommended activities, there is no mention of near or far terms but rather the focus is on the prediction of subsequent numbers in a pattern (ibid.).

In terms of teaching approaches and the underlying knowledge base of teachers, it is imperative that non-algebraic activity not be presented as algebraic as to do so could result in a failure to nurture genuine algebraic thinking in young children (Radford, 2012). Clements and Sarama (2009) caution that teachers need to be aware of the role of repeating sequential patterns and of where they “fit into (but certainly do not alone, constitute) the large role of patterning and structure” (p.190). Papic (2007) also contends that when teaching patterning to young children, it is vital that teachers remain cognisant of focusing children’s attention on the unit of repeat. The algebra strand from Junior Infants to Second class may seem inconsistent in terms of the algebraic thinking required of children, as defined by Radford (2011). There are indications from research literature, that children at this class level could progress to activities which are algebraic in nature, and that the curriculum need not be limited to low order patterning activities (Britt and Irwin, 2011; Schifter, Bastable, Russell, Seyferth, and Riddle, 2008; Papic, 2007). However, there is a necessity for patterning activities wherein children progress from an ability to copy simple patterns to extending and creating patterns. Before children can proceed to the algebraic process of identifying or evaluating a remote figure, they must have established an understanding of what a pattern is and how a commonality may be identified (Threlfall, 1999; Owen, 1995).

Fox (2006) explored the patterning activities of children in Australian preschool settings, where the children are of similar age to children in Junior and Senior Infants in Irish primary schools. She concludes that “experiences with identifying, creating, extending and generalising patterns, recognising relationships, making predictions, and abstracting rules provide foundations for future algebraic development” (p.6). Disappointingly, in her research Fox (2006) found that such patterning activities which are supportive of development in algebraic reasoning were not evident in the preschools involved. In terms of the pre-algebraic activities discussed above, the children neither discussed repetition within the patterns they explored, nor did they identify a commonality. To summarise, there is much research which supports the view of patterning as supporting children’s prerequisite skills in the area of algebraic thinking. Much of the activity being carried out is not strictly algebraic in itself and there is the potential for children to engage with patterning in a routine, non-explorative way which may not support their developing thinking. There is a necessity for teachers to be aware of the role that patterning plays and how it contributes to children’s developing thinking.
RECURSIVE AND EXPLICIT APPROACHES TO PATTERNING

When children begin to examine patterns, their natural reaction is to reason recursively, meaning that they examine the mathematical relationship between consecutive terms in a sequence (Lannin, 2004). As children develop sophistication in their approach to patterning, they need to move beyond the recursive method of exploring a sequence. To gain an insight into a greater range of patterning and also a structural understanding beyond the most basic repeating pattern, children need to develop an understanding that rules underlie patterns and that to expand a pattern efficiently, the rule must be identified. Owen (1995) suggests building up a sequence element by element to facilitate the children’s realisation that recursive methods are not always accurate.

In the development of children’s reasoning, it may be necessary for a teacher to encourage the children to consider an explicit approach, as children naturally approach patterning activities with a tendency to reason recursively (Lannin, 2004). It is advisable therefore that teaching activities and materials avoid an immersion in sequences which favour a recursive approach. Students need recourse to both explicit and recursive methods of solving patterns, and their thinking should be developed to include an ability to determine which method is appropriate in a particular situation (ibid.). In exploring students’ approaches to patterning in Taiwan, Ma (2007) found that many patterns evident in textbooks involve a constant difference between terms and may in this way reinforce reliance on a recursive method.

In this context, it may be said that some of the number pattern activity suggested in the Irish curriculum, neither requires nor promotes algebraic reasoning. In First and Second class, children are drilled in extending and repeating linear patterns and the curriculum advises that they should be enabled to explore pattern as a support in their growing computational skills, e.g. 2+5=7; 2+15=17; etc. Children are engaged in observing patterns in odd and even numbers and in multiples of 2, 3, 4, 5 and 10. Children are asked to explain a rule for patterns but there is no mention in the curriculum of applying this rule to a general term or a far term. Importantly, there is also no justification in the curriculum for this patterning work and I would voice a concern that teachers see such activities as a support to arithmetic which becomes almost trivial in nature without a discussion regarding generalisation. Within the broad objectives of the curriculum there is the requirement that children should be facilitated in developing the ability to “reason, investigate and hypothesize with patterns and relationships in mathematics” (Government of Ireland, 1999, p.12). This objective does not seem present in the breakdown of content objectives and recommended activities.

Within the Irish primary school mathematics curriculum, children in Fifth class are expected to identify and record rules for sequences which should incorporate explicit rather than recursive thinking (Government of Ireland, 1999). However, the two exemplars given within the curriculum are of sequences with constant differences where the rules are to increase or decrease by a constant. This tendency towards sequences which are exclusively based upon a constant difference is reproduced in Irish textbooks. In the Fifth class textbook in the Mathemagic series (Barry, Manning, O’Neill & Roche, 2003), not only are all sequences
either arithmetic or geometric in nature but the accompanying teaching notes instruct the children to compare each element to its predecessor and successor, and never suggest that there may be a relationship between the term and its position in the sequence. Eivers et al. (2010) found, in an examination of teaching approaches in Irish classrooms, that textbook use was pervasive particularly in senior classes and suggested that teachers were possibly overly reliant on textbooks. If teachers are teaching sequences and patterning solely from textbooks such as Mathemagic 5, it is probable that many children in Fifth class will never have been encouraged to solve sequences explicitly.

CONCLUSION
Focusing on the algebraic skill of generalising, Mason (2009) states that “young children are able to generalise, because without it they could not function in the world and certainly could not grasp language” (p.159). When children enter Irish primary schools they bring with them nascent skills in generalisation and it is the role of the school to nurture and develop such skills so that by the time children leave primary school they possess the competences necessary for engagement in abstract symbol manipulation.

The algebraic strand of the Irish primary mathematics curriculum is very rich in patterning activity but the algebraization of this activity is very much the responsibility of the classroom teacher, and thus possibly inconsistent. Corcoran (2005) researched the mathematical knowledge and skills of a cohort of 71 preservice teachers and found that only 42% of the participants could correctly answer an item which involved algebra. While the study sample and number of items were too small to be generalisable, Corcoran expressed concern about the mathematical literacy of these preservice teachers as being overly procedural. Limited mathematical literacy and a procedural approach to mathematics are unlikely to lend themselves well to an independently algebraic approach to patterning activities, and I would conjecture that in many classrooms the patterning activity of children involves little algebraic thinking.

Many recommendations of research literature in the area of early algebra are fulfilled within the Irish curriculum. Instruction commences early, and patterning is considered as a support to computation throughout primary school. However, there is no mention made of generalisation or identifying a far term in a pattern. While mention is made of formulating rules for patterns, all the pattern examples provided lend themselves most readily to recursive solutions. Including a focus on generalisation, far terms and explicit pattern solving would benefit children in their development of skills in algebraic reasoning and in overcoming the challenges of formal algebra.

REFERENCES


