Numerical approximation of solution derivatives of singularly perturbed parabolic problems

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Abstract

Numerical approximations to the solution of a linear singularly perturbed parabolic problem are generated using a backward Euler method in time and an upwinded finite difference operator in space on a piecewise-uniform Shishkin mesh for a convection-diffusion problem. A proof is given to show first order convergence of these numerical approximations in appropriately weighted $C^1$-norm. Numerical results are given to support the theoretical error bounds. The analysis is also applied to singularly perturbed problems of reaction-diffusion type.

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1 Introduction

The solutions of singularly perturbed problems typically contain steep gradients in narrow regions of the domain, often referred to as layer regions. Layer adapted meshes, such as piecewise-uniform Shishkin meshes [6] or Bakhvalov meshes [4], have been designed to concentrate a significant proportion of the mesh points into these layer regions and thereby generate pointwise globally accurate piecewise-polynomial approximations to the continuous solution, irrespective of the size of the singular perturbation parameter. An additional feature of these layer-adapted meshes is that accurate approximations to the first derivative of the solution can be easily generated. For ease of reference, we shall refer to this additional feature of layer-adapted meshes as flux-capturing. In this paper, we present a proof of this flux-capturing property of Shishkin meshes in the case of a singularly perturbed parabolic problem.

When estimating the error in a numerical approximation, relative errors are more relevant than absolute errors. In many cases, the continuous solution is initially normalized to have a maximum value of $O(1)$ and then a pointwise bound in the maximum norm on the absolute error is equivalent to a bound on the relative error, measured in the maximum norm. In the context of singularly perturbed problems, these comments are pertinent, as

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there are different scales involved in the problem. In particular, the magnitude of the derivative can vary significantly within the layer regions as compared to its behaviour outside the layer regions. For this reason, the appropriate norm to measure the error in approximating the flux needs to be examined closely.

Given that the singularities appearing in the solution of singularly perturbed problems are pointwise singularities, it is natural \[6\] to employ pointwise norms to measure accuracy. Below we will discuss the following discrete versions of \(C^0, C^1\) and weighted–\(C^1\) norms, defined over a finite set of mesh points \(R^N := \{x_i\}_0^N\), by

\[
\|u\|_{R^N} := \max_{x_i \in R^N} |u(x_i)|, \\
\|u\|_{1,R^N} := \|D^-u\|_{R^N \setminus \{x_0\}} + \|u\|_{R^N}, \\
\|u\|_{1,w,R^N} := \|wD^-u\|_{R^N \setminus \{x_0\}} + \|u\|_{R^N},
\]

where \(D^-u\) is the discrete derivative defined by

\[
D^-u(x_i) := \frac{u(x_i) - u(x_{i-1})}{x_i - x_{i-1}}.
\]

However, the value of a nodal error estimate depends on the choice of mesh points. Global accuracy over the entire domain is a more neutral measure. Hence, we will consider the merits of various weighted–\(C^1\) norms defined over a measurable region \(R\) as follows

\[
\|u\|_{1,w,R} := \|wu_x\|_R + \|u\|_R, \quad \|u\|_R := \operatorname{ess sup}_{x \in R} |u(x)|.
\]

For a singularly perturbed boundary value problem of the form

\[
\varepsilon u'' + au' = f, \; x \in \Omega := (0,1); \; u(0) = A, u(1) = B; \; a(x) \geq \alpha > 0; \quad (1)
\]

it was established in \[5, 6\] that for a numerical solution \(U^N\) generated using a standard upwind finite difference operator and an appropriate piecewise-uniform Shishkin mesh, one has a global error bound of the form

\[
\|u - \bar{U}\|_{1,\varepsilon,\Omega} := \varepsilon \|u' - (\bar{U})'\|_\Omega + \|u - \bar{U}\|_\Omega \leq CN^{-1} \ln N, \quad (2)
\]

where \(\bar{U}\) denotes a piecewise linear interpolant over the domain \(\bar{\Omega}\) of the discrete solution \(U^N\). Throughout this paper, \(C\) denotes a generic constant that is independent of the singular perturbation parameter \(\varepsilon\) and of all discretization parameters. That is, the bound in \(2\) states that the numerical method is parameter-uniform \[6\] in the \(\varepsilon\)-weighted norm \(\cdot\|\cdot\|_{1,\varepsilon,\Omega}\).

Gartland \[8\] measured the errors from an exponentially-fitted compact finite difference operator on a locally quasi-uniform exponentially graded mesh in a discrete version of this \(\varepsilon\)-weighted norm \(\cdot\|\cdot\|_{1,\varepsilon,\Omega}\). However, the number of mesh points required in the Gartland mesh depends (albeit logarithmically) on \(1/\varepsilon\). Moreover, in the context of parameter-uniform numerical methods \[9\], exponentially-fitted finite difference schemes (which are designed to be nodally exact in the case of constant coefficients) are limited to certain classes of singularly perturbed problems. Andreev \[11\] presented sharp bounds on the continuous solution measured in \(\cdot\|\cdot\|_{1,\varepsilon,\Omega}\) and the discrete solution of a monotone three point difference
scheme on arbitrary non-uniform grids in a discrete version of the norm $\| \cdot \|_{1, \varepsilon, \Omega}$ [2]. These results can be used to derive parameter-uniform global error bounds in $\| \cdot \|_{1, \varepsilon, \Omega}$ in the case of problem [1].

Note that the error in the estimate of the derivative term in [2] has been normalized by the factor $\varepsilon$, as $\varepsilon \| u' \|_\Omega = O(1)$.

The results can be used to derive parameter-uniform global error bounds in $\| \cdot \|_{1, \varepsilon, \Omega}$ in the case of problem [1]. For example, in the case of problem [1], we note that

$$|u'(x)| \leq C, \quad x \geq C \varepsilon \ln(1/\varepsilon).$$

Hence the scaling by the factor $\varepsilon$ in the error bound [2] is not appropriate, if this error bound is restricted to points outside the layer region $[0, C \varepsilon \ln(1/\varepsilon)]$.

In [14] Kopteva and Stynes established an error estimate of the form

$$\varepsilon |u'(x_{i-0.5}) - DU(x_i)| \leq CN^{-1} \ln N, \quad x_i \leq C \varepsilon \ln N,$$

$$|u'(x_{i+0.5}) - DU(x_i)| \leq CN^{-1} \ln N, \quad x_i \geq C \varepsilon \ln N,$$

(where $DU$ denotes a discrete derivative of $U$) for Shishkin and (corresponding bounds) for Bakhvalov meshes. The bound outside the computational layer region $[0, C \varepsilon \ln N]$ is now an unweighted $C^1$ error bound. In the context of singularly perturbed elliptic problems in two dimensions, we note that Kopteva [13] used a second order asymptotic expansion to establish first order nodal convergence (appropriately scaled in the fine mesh areas) of the discrete first order partial derivatives, under the restriction $\varepsilon \leq CN^{-1}$. In passing, we observe that the bounds established below (for a parabolic problem) cover both cases of $\varepsilon \leq CN^{-1}$ and $\varepsilon \geq CN^{-1}$.

In the context of nodal accuracy on a certain mesh $\Omega^N$, the following discrete weighted norm

$$\|u\|_{1, \varepsilon, \Omega^N} := \|zD^{-1} u\|_{\Omega^N} + \|u\|_{\Omega^N}, \quad z(x_i) := \begin{cases} \varepsilon, & \text{if } \alpha x_i \leq \varepsilon \ln N, \\ 1, & \text{if } \alpha x_i > \varepsilon \ln N, \end{cases}$$

appears to be a reasonable discrete norm to use to measure accuracy in the approximating solutions of singularly perturbed problems. However, observe that in the classical case of $N^{-1} \leq \varepsilon$, the scaling factor of $\varepsilon$ for mesh points within the region $(\varepsilon \ln(1/\varepsilon), \varepsilon \ln N)$ is not appropriate.

It is also worth remarking that, in the case of singularly perturbed ordinary differential equations, if a scheme is nodally second order (ignoring logarithmic factors) in $\| \cdot \|_{\Omega_N}$ on a Shishkin mesh, then it is nodally first order in the $\varepsilon$-weighted $C^1$-norm $\| \cdot \|_{1, \varepsilon, \Omega^N}$. In particular, Andreev and Savin [3] analysed a modification of Samarskii’s monotone finite difference operator on a piecewise-uniform mesh to establish an error bound in $\| \cdot \|_{\Omega^N}$ of the form $C(N^{-1} \ln N)^{2}$ and thereby one has an error bound of the form $CN^{-1} \ln N$ in the discrete norm $\| \cdot \|_{1, \varepsilon, \Omega^N}$ for the scheme presented in [3].

In this paper, we confine our attention to a simple finite difference scheme on a standard piecewise-uniform Shishkin mesh as it applies to a singularly perturbed partial differential equation defined over a region $G := \Omega \times (0, T]$, which is a time-dependent version of problem [1]. More sophisticated finite difference operators on various layer-adapted meshes, which are second order in space and first order in time, exist in the literature. However, in contrast
to the case of an ordinary differential equation, one cannot directly deduce a first order error bound in the discrete norm

\[ \|u\|_{1,\varepsilon,G,N,M} := \varepsilon\|D_x u\|_{G^N,M \setminus \{(x_0, t_j)\}_{j=0}^M} + \|D_t u\|_{G^N,M \setminus \{(x_i, t_0)\}_{i=0}^N} + \|u\|_{G^N,M} \]

from such nodal error bounds. Kopteva [12] analysed a non-monotone finite difference scheme on a Shishkin mesh, which is second order in both space and time; thereby, this scheme is first order in the discrete norm \( \|\cdot\|_{1,\varepsilon,G,N} \) (assuming \( M = CN \)). In this paper, we choose to establish convergence in a global norm (specified below) for a monotone finite difference scheme, which is only first order in both space and time.

In the case of singularly perturbed parabolic problems, Shishkin [18] introduced a sophisticated global metric, which is designed to measure the pointwise relative error in estimating the first derivative both within and outside the layer region. In the case of the time dependent version of problem (1), this new weighted metric is, in essence, of the form

\[ \|v\|_{1,s,G} := \|s(x)v_x\|_G + \|v_t\|_G + \|v\|_G, \quad s(x) := \frac{\varepsilon}{\varepsilon + \varepsilon^{-a(x_1)x/\varepsilon}}. \]

In [18], Shishkin shows that the error bound (2) applies in the case of upwinding on a piecewise-uniform mesh, but the same numerical scheme is not \( \varepsilon \)-uniformly convergent in this new metric. Conditions can be imposed on the parameters in a generalized piecewise-uniform Shishkin mesh (see [18, §6] for details) so that the numerical approximations converge almost \( \varepsilon \)-uniformly in this metric \( \|\cdot\|_{1,s,G} \). To be precise, at a rate of \( O(\varepsilon^{-\nu}N^{-1}) \), where \( \nu > 0 \) is arbitrarily small. We refer the reader to [18] for further details. Shishkin extended these ideas on suitable metrics to the case of singularly perturbed elliptic partial differential equations in [17].

In this paper, we choose the simpler (but cruder) global metric of simply scaling the first derivative by the constant \( \varepsilon \) within the layer and using no scaling factor outside the layer. Hence, instead of \( \|\cdot\|_{1,s,G} \) we will measure the errors in the following weighted \( C^1 \)-norm:

\[ \|v\|_{1,\chi,G} := \|\chi(x)v_x\|_G + \|v_t\|_G + \|v\|_G, \quad \chi(x) := \begin{cases} \varepsilon, & \text{if } \alpha|x - p| \leq 2\varepsilon \ln(1/\varepsilon), \\ 1, & \text{if } \alpha|x - p| > 2\varepsilon \ln(1/\varepsilon), \end{cases} \]

where \( p = 0 \) or \( p = 1 \) depending on the location of the boundary layer. In this paper, we examine a problem with the boundary layer located on the right (where \( p = 1 \)). Note that the weighting function \( \chi(x) \) is excessive in the region where \( C\varepsilon \ll \alpha|x - p| \leq 2\varepsilon \ln(1/\varepsilon) \).

In §2 the continuous problem is stated and parameter-explicit bounds on the derivatives of the solution are established by decomposing the solution into a regular and singular components. In §3, the numerical method is described and appropriate bounds on the nodal errors are given. These estimates are used in §4 and §5 to establish scaled nodal approximations of the space and time derivatives, respectively. The main result of the paper, which establishes an error estimate in the norm \( \|\cdot\|_{1,\chi,G} \), is given in §6. Some numerical results are given in the final section of the paper. In the appendix, the corresponding weighted-\( C^1 \) bounds on the numerical approximations of the reaction-diffusion problem are derived.
2 Continuous problem

Consider the following class of singularly perturbed parabolic problems

\[ L_\varepsilon u := -\varepsilon u_{xx} + a(x,t)u_x + b(x,t)u + u_t = f(x,t), \quad \text{in } G := \Omega \times (0,T], \quad (4a) \]
\[ u = 0, \quad \text{on } \Gamma_B \cup \Gamma_L \cup \Gamma_R, \quad 0 < \varepsilon \leq 1; \quad a(x,t) > \alpha > 0, \quad (4b) \]
\[ b(x,t) \geq \max\{\|a_x\|_G, \|a_t\|_G\} + \beta, \quad \beta > 0, \quad (4c) \]

where \( \Omega := (0,1) \), \( \Gamma_B := \{(x,0) \mid 0 \leq x \leq 1\} \), \( \Gamma_L := \{(0,t) \mid 0 \leq t \leq T\} \), \( \Gamma_R := \{(1,t) \mid 0 \leq t \leq T\} \) and \( \Gamma := \overline{G} \setminus G \). Since the problem is linear, there is no loss in generality in assuming zero boundary/initial conditions. The constraint (4c) on the coefficient \( b(x,t) \) can be transferred to the time variable by using the change of variable \( u = ve^{\gamma t} \), where \( \gamma > 0 \) is sufficiently large. We assume that the data of the problem satisfy regularity and compatibility conditions so that the solution of problem (4) is such that

\[ u \in C^{6+\gamma}(G) \]

for the analysis (see [7] and [15]) presented below to be applicable.

It is well–known that the differential operator associated with (4) satisfies a comparison principle. From this, one can establish the stability estimate

\[ |u(x,t)| \leq \min\{\frac{x}{\alpha}, t\} \|f\|_G. \]

Motivated by the bounds given in [10] and [18] we present the following bounds on the derivatives of the regular and singular components of \( u \). For the sake of completeness, we outline a proof of these bounds here.

**Theorem 1.** The solution of (4) can be written in the form \( u = v + w \), where the regular component \( v \in C^{6+\gamma}(G) \) satisfies

\[ L_\varepsilon v = f, \quad \text{in } G, \quad v = u, \quad \text{on } \Gamma_B \cup \Gamma_L, \quad (5a) \]

and \( v = v^* \) can be specified on the boundary \( \Gamma_R \) so that

\[ \left\| \frac{\partial^{k+m}v}{\partial x^k \partial t^m} \right\|_G \leq C(1 + \varepsilon^{2-2k-2m}), \quad 0 \leq k + 2m \leq 6. \quad (5b) \]

The singular component \( w \) satisfies the homogeneous differential equation

\[ L_\varepsilon w = 0, \quad \text{in } G, \quad w = 0, \quad \text{on } \Gamma_B \cup \Gamma_L, \quad w = u - v \text{ on } \Gamma_R, \quad (6a) \]

and for all points \((x,t) \in G\) its derivatives satisfy the pointwise bounds

\[ \left| \frac{\partial^{k+m}w}{\partial x^k \partial t^m}(x,t) \right| \leq C\varepsilon^{-k}(1 + \varepsilon^{2-2k-2m})e^{-\alpha(1-x)/\varepsilon}, \quad 0 \leq k + 2m \leq 6. \quad (6b) \]

\[ ^1 \text{The space } C^{n+\gamma}(D) \text{ is the set of all functions, whose derivatives of order } n \text{ are Hölder continuous of degree } \gamma > 0. \text{ That is,} \]
\[ C^{n+\gamma}(D) := \{z : \frac{\partial^{i+j}z}{\partial x^i \partial t^j} \in C^\gamma(D), 0 \leq i + 2j \leq n\}. \]
Proof. Using the stretched variables $\zeta := (1 - x)/\varepsilon, \eta := t/\varepsilon$ and the a priori bounds \cite{15} pg. 320, Theorem 5.2, one deduces the bounds

$$\left\| \frac{\partial^{k+m} u}{\partial x^k \partial t^m} \right\|_G \leq C \varepsilon^{-k-m}, \quad 0 \leq k + 2m \leq 6.$$  

Consider the extended domain $G^* := \{(0, A) \times (0, B); A > 1, B > T\}$, with associated extended boundaries $\Gamma_L^*, \Gamma_B^*$ and $a^*, b^*, f^*$ are smooth extensions of $a, b, f$ to the extended domain $G^*$. The first order reduced operator $L_0^*$ is defined by

$$L_0^* z := a^* z_x + b^* z + z_t \quad \text{in} \quad \bar{G}^* \setminus (\Gamma_B^* \cup \Gamma_L^*), \quad z = \varepsilon R^*, \quad \text{on} \quad \Gamma_B^* \cup \Gamma_L^*.$$  

The regular component $v$ is composed of the reduced solution $v_0$, higher order terms $v_1, v_2$ in an asymptotic expansion and a remainder term $R$, given by

$$v^* = v_0^* + \varepsilon v_1^* + \varepsilon^2 v_2^* + \varepsilon^3 R^*,$$

$$L_0^* v_0^* = f^*, \quad \text{in} \ G^*, \quad v_0^* = u, \quad \text{on} \ \Gamma_B^* \cup \Gamma_L^*;$$

$$L_0^* v_i^* = (v_{i-1}^*)_{xx}, \quad \text{in} \ G^*, \quad v_i^* = 0, \quad \text{on} \ \Gamma_B^* \cup \Gamma_L^*, \quad i = 1, 2;$$

$$L_0^* R^* = (v_2^*)_{xx}, \quad \text{in} \ G^*, \quad R^* = 0, \quad \text{on} \ \bar{G}^* \setminus G^*.$$  

The bounds on the derivatives of $v^*$ (and hence $v$) are then easily deduced.

The singular component $w$ can be decomposed as follows

$$w(x, t) = (u - v)(1, t) \Psi(x, t) + \varepsilon R(x, t),$$  

where, for each value of $t$, the unit boundary layer function $\Psi$ satisfies

$$-\varepsilon \Psi_{xx} + a(x, t) \Psi_x = 0, \quad \Psi(0, t) = 0, \quad \Psi(1, t) = 1.$$  

Note that

$$\Psi(x, t) = \int_0^x e^{-\int_{s=r}^x \frac{a(x, t)}{\varepsilon} ds} \, dr - \int_1^0 e^{-\int_{s=r}^1 \frac{a(x, t)}{\varepsilon} ds} \, dr.$$  

Using the strict inequality $a > \alpha$ and $((1 - \theta)t)^m e^{-t} \leq m! e^{-\theta t}, \quad 0 < \theta < 1, \ t \geq 0$, we have that

$$\left| \frac{\partial^m \Psi(x, t)}{\partial t^m} \right| \leq C e^{-\alpha(1-x)/\varepsilon}, \quad 0 \leq m \leq 3.$$  

For the remainder term, $R(x, t) = 0, (x, t) \in \Gamma$ and for all $(x, t) \in G$

$$\varepsilon L_0^* R = -((u - v)(1, t)b(x, t) + (u - v)_t(1, t)) \Psi(x, t) - (u - v)(1, t) \Psi_t(x, t).$$  

Hence

$$|R(x, t)| \leq C e^{-\alpha(1-x)/\varepsilon}.$$  

Using the stretched variables and the localized bounds on the derivatives \cite{15} pg. 352, (10.5) one can deduce the bounds

$$\left| \frac{\partial^{k+m} R(x, t)}{\partial x^k \partial t^m} \right| \leq C \varepsilon^{-k-m} e^{-\alpha(1-x)/\varepsilon}, \quad 0 \leq k + 2m \leq 6.$$  

6
Hence,
\[ \left| \frac{\partial^{k+m}w(x,t)}{\partial x^k \partial t^m} \right| \leq C(1 + \varepsilon^{1-m})\varepsilon^{-k}\varepsilon^{-\alpha(1-x)/\varepsilon}, \quad 0 \leq k + 2m \leq 6. \]  
\hspace{7cm} (7a)

We improve the above bounds on the time derivatives by noting that for \( m = 1, 2, 3 \)
\[ \left| L_\varepsilon \frac{\partial^m w(x,t)}{\partial t^m} \right| \leq C \sum_{j=0}^{m-1} \left| \frac{\partial^j w(x,t)}{\partial \tau^j} \right| + C \sum_{j=0}^{m-1} \left| \frac{\partial^{j+1} w(x,t)}{\partial \tau \partial x} \right|, \quad (x,t) \in G, \]
\[ \frac{\partial^m w(x,t)}{\partial t^m} = 0, (x,t) \in \Gamma_B \cup \Gamma_L, \quad \left| \frac{\partial^m w(1,t)}{\partial t^m} \right| \leq C, \]
which implies that
\[ \left| \frac{\partial^m w(x,t)}{\partial t^m} \right| \leq C(1 + \varepsilon^{2-m})\varepsilon^{-\alpha(1-x)/\varepsilon}, \quad 1 \leq m \leq 3. \]  
\hspace{7cm} (7b)

From the equation \((L_\varepsilon w)_{tt} = 0\) we have that, for all \( t \geq 0, \)
\[ -\varepsilon \frac{\partial^2}{\partial x^2} w_{tt} + a(x,t) \frac{\partial}{\partial x} w_{tt} = g(x,t), x \in (0,1), \]
\[ w_{tt}(0,t) = 0, |w_{tt}(1,t)| \leq C, \]
\[ |g(x,t)| \leq C \sum_{j=0}^{3} \left| \frac{\partial^j w(x,t)}{\partial \tau^j} \right| + C \sum_{j=0}^{1} \left| \frac{\partial^{j+1} w(x,t)}{\partial \tau \partial x} \right| \leq C(1 + \varepsilon^{-1})\varepsilon^{-\alpha(1-x)/\varepsilon}. \]

For each time \( t \), we use this boundary value problem for \( w_{tt} \) to deduce (use argument from [6, pp 46-47] with \( x \rightarrow 1 - x \)) that
\[ \left| \frac{\partial^{i+2}}{\partial x^i \partial t^2} w(x,t) \right| \leq C(1 + \varepsilon^{-i})\varepsilon^{-\alpha(1-x)/\varepsilon}, \quad i = 1, 2. \]  
\hspace{7cm} (7c)

Collecting all these bounds together completes the proof. \( \square \)

**Remark 1.** The proof of the bounds in Theorem 1 simplifies significantly in the special case of \( a(x) \) being independent of time. In fact, in this particular case, all the time derivatives of the solution \( u \) of (4) are \( \varepsilon \)-uniformly bounded.

### 3 Numerical scheme

Consider a uniform mesh in time \( \tau^M = \{k\tau, \ 0 \leq k \leq M, \ \tau = T/M\} \) and a piecewise–uniform Shishkin mesh \( \Omega^N \) in space on which numerical approximations of the solution of problem (4) are generated. The subintervals \([0, 1 - \sigma] \) and \([1 - \sigma, 1] \) are each uniformly subdivided into \( N/2 \) mesh intervals, where the transition parameter \( \sigma \) is defined by

\[ \sigma := \min \left\{ \frac{1}{2}, 2 \varepsilon \frac{\varepsilon}{\alpha} \ln N \right\}. \]

Then, the grid in the space variable \( \Omega^N = \{x_i\} \) is given by

\[ x_i = \begin{cases} 
  iH, & \text{if } 0 \leq i \leq N/2, \\
  (1 - \sigma) + (i - N/2)h, & \text{if } N/2 \leq i \leq N,
\end{cases} \]  
\hspace{7cm} (8)
where the step sizes are \( h := 2\sigma/N \) and \( H := (1 - \sigma)/N \). We denote the local step sizes by \( h_j := x_j - x_{j-1} \) for \( j = 1, \ldots, N \), and we define the following sets of mesh points
\[
\bar{G}^{N,M} := \bar{\Omega}^N \times \bar{\omega}^M, \quad G^{N,M} := \bar{G}^{N,M} \cap G, \quad \Gamma^{N,M} := \bar{G}^{N,M} \setminus G^{N,M}.
\]

We combine this mesh with a simple fully implicit finite difference operator, which uses the classical upwind approximation for the space derivatives, to produce the finite difference method:
\[
\begin{align*}
L^{N,M} U(x_i, t_j) &= f(x_i, t_j), \quad (x_i, t_j) \in G^{N,M}, \quad U(x_i, t_j) = 0, \quad (x_i, t_j) \in \Gamma^{N,M}, \\
L^{N,M} V(x_i, t_j) &= (-\varepsilon \delta^2_x + a D^-_x + b I + D^-_t) U(x_i, t_j),
\end{align*}
\]
where \( U_{i,j} := U(x_i, t_j) \) and
\[
\begin{align*}
D^+_x U_{i,j} &:= \frac{U_{i,j} - U_{i,j-1}}{h}, \quad L^x U_{i,j} := -\varepsilon \delta^2_x U_{i,j} + a(x_i, t_j) D^-_x U_{i,j} + b(x_i, t_j) U_{i,j}, \\
D^+_x U_{i,j} &:= \frac{U_{i,j} - U_{i,j+1}}{h_i} - h_i, \quad D^-_x U_{i,j} := \frac{U_{i,j} - U_{i,j-1}}{h_i}, \\
\delta^2_x U_{i,j} &:= \frac{1}{h_i} (D^+_x U_{i,j} - D^-_x U_{i,j}), \quad h_i := \frac{h_i + h_{i+1}}{2}.
\end{align*}
\]

Throughout the analysis in this paper we assume that
\[
\sigma = \frac{2\varepsilon}{\alpha} \ln N, \quad C_1 N \leq M \leq C_2 N.
\] (10)

The classical case of \( \sigma = 0.5 \) is dealt with in Remark 2 at the end of §6.

It is well-known that the finite difference operator associated with problem \( (9) \) satisfies a discrete comparison principle. To obtain appropriate bounds of the error in the maximum norm, consider the following decomposition of the numerical solution \( U = V + W \), where the discrete regular \( V \) and singular \( W \) components satisfy the problems
\[
\begin{align*}
L^{N,M} V &= f, \quad V|_{\Gamma^{N,M}} = v|_{\Gamma^{N,M}}, \quad (11a) \\
L^{N,M} W &= 0, \quad W|_{\Gamma^{N,M}} = w|_{\Gamma^{N,M}}. \quad (11b)
\end{align*}
\]

In the next theorem we establish bounds on the error associated with the regular and singular components, which are used later in the error analysis in the weighted \( C^1 \)-norm \( \| \cdot \|_{1, \chi, G} \).

**Theorem 2.** Assume \( (11) \). For all \( t_j \geq 0 \), we have the following bounds
\[
\begin{align*}
|\langle V - v \rangle(x_i, t_j)\rangle &\leq C t_j N^{-1}, \quad \text{if } x_i \in [0, 1], \\
|\langle W - w \rangle(x_i, t_j)\rangle &\leq C N^{-2}, \quad \text{if } x_i \in [0, 1 - \sigma], \\
|\langle W - w \rangle(x_i, t_j)\rangle &\leq C N^{-2} + C t_j N^{-1} \ln N, \quad \text{if } x_i \in [1 - \sigma, 1],
\end{align*}
\]
where \( v, w \) are the solutions of problems \( (11a), (11b) \) and \( V, W \) are defined in \( (11) \).

**Proof.** For the regular component we deduce the truncation error bound
\[
\| L^{N,M} (V - v) \|_{G^{N,M}} \leq C N^{-1}.
\]

8
By the discrete maximum principle one then has the nodal error bound

$$|(V - v)(x_i, t_j)| \leq Ct_jN^{-1}.$$  

For the singular component, we distinguish two cases depending on the location of the grid point. First, outside the layer, we have that

$$|(W - w)(x_i, t_j)| \leq |W(x_i, t_j)| + |w(x_i, t_j)| \leq CN^{-2}, \quad x_i \leq 1 - \sigma.$$  

If $x_i \in (1 - \sigma, 1)$, then the truncation error satisfies

$$|L^{N,M} (w - W)(x_i, t_j)| \leq C(\tau + h|w_{xx}(x_i^*, t_j)| + \varepsilon h|w_{xxx}(x_i^{**}, t_j)|),$$

with $x_i^* \in (x_i - h, x_i)$, and $x_i^{**} \in (x_i - h, x_i + h)$. Using the inequalities

$$|w_{xx}(x_i^*, t_j)| = \left| \int_0^{t_j} w_{xxt}(x_i^*, s) ds \right| \leq Ct_j\varepsilon^{-2}e^{-\alpha(1 - x_i)/\varepsilon},$$

$$|w_{xxx}(x_i^{**}, t_j)| \leq Ct_je^{-3}e^{-\alpha(1 - x_{i+1})/\varepsilon},$$

within the layer region, we obtain the truncation error bound

$$|L^{N,M} (w - W)(x_i, t_j)| \leq C(\tau + \frac{N^{-1} \ln N}{\varepsilon}t_j e^{-\alpha(1 - x_{i+1})/\varepsilon}), \quad 1 - \sigma < x_i < 1.$$  

Use the discrete barrier function

$$C(t_jN^{-1} \ln N(1 + \alpha(h/\varepsilon)^i+2-N + \tau t_j + CN^{-2}),$$

and $|(w - W)(1 - \sigma, t_j)| \leq CN^{-2}$ to complete the proof. \(\square\)

## 4 Nodal approximation of space derivatives

Consistency and stability is a classical argument in numerical analysis, which is typically employed to deduce a nodal error bound. To bound the quantity $D_x(U - u)$ at the mesh points, we use an argument of this type, by employing a bound on a quantity of the form $\|\hat{L}_{N,M}(D_x(U - u))\|$, where the finite difference operator $\hat{L}_{N,M}$ is monotone and is defined below in (13).

We denote the nodal error by $e(x_i, t_j) := U(x_i, t_j) - u(x_i, t_j)$, and the associated truncation error by $\mathcal{T}(x_i, t_j) := L^{N,M}e(x_i, t_j)$. We define the discrete error flux to be

$$\mathcal{H}^{-}_{i,j} := D_x^{-}e(x_i, t_j), \quad \text{if } 0 < x_i \leq 1.$$  

The main purpose of this section is to deduce suitable bounds on $\mathcal{H}^{-}$.

We identify a discrete problem associated with the error flux defined over the region

$$G^N_{H} := G^N \cap \{(H, 1) \times (0, T]\}; \quad \hat{G}^N_{H} := \hat{G}^N \cap \{[H, 1] \times [0, T]\}.$$  

We define a new finite difference operator $\hat{\delta}_x^2$ by

$$\hat{\delta}_x^2 Z_{i,j} := \frac{1}{h_i} \left( \frac{h_{i+1}}{h_i} D_x^+ - \frac{h_i}{h_{i-1}} D_x^- \right) Z_{i,j},$$
which has the property that
\[ \hat{\delta}_x^2 D_x^- Z_{i,j} = D_x^- \delta_x^2 Z_{i,j}. \]

Note the following identity
\[ D_x^- (P_{i,j} Q_{i,j}) = P_{i,j} D_x^- Q_{i,j} + Q_{i-1,j} D_x^- P_{i,j}. \] (12)

Using these identities and \( D_x^- (L^{N,M} e(x_i, t_j)) = D_x^- \mathcal{T}(x_i, t_j) \), we see that for all mesh points within the region \( G_H^{N,M} \), the quantity \( \mathcal{W}_{i,j}^- \) satisfies
\[ \hat{L}^{N,M} \mathcal{W}_{i,j}^- = D_x^- \mathcal{T}(x_i, t_j) - e(x_{i-1}, t_j) D_x^- b(x_i, t_j), (x_i, t_j) \in G_H^{N,M}, \] (13)

where for the internal points \( (x_i, t_j) \in G_H^{N,M} \),
\[ \hat{L}^{N,M} Z_{i,j} := (-\varepsilon \delta_x^2 + a(x_{i-1}, t_j) D_x^- + (b + D_x^- a)(x_i, t_j) I + D_t^-) Z_{i,j}, \]
and \( \hat{L}^{N,M} Z_{i,j} := Z_{i,j} \) for \( (x_i, t_j) \in G_H^{N,M} \setminus G_H^{N,M} \).

**Remark 2.** Note that on the piecewise uniform mesh \( G^{N,M} \)
\[ \delta_x^2 Z_{i,j} = \begin{cases} \delta_x^2 Z_{i,j}, & \text{if } x_i \neq 1 - \sigma, 1 - \sigma + h, \\ \frac{1}{H}(2\sigma D_x^+ - D_x^-) Z_{i,j}, & \text{if } x_i = 1 - \sigma, \\ \frac{1}{h}(D_x^+ - 2\sigma D_x^-) Z_{i,j}, & \text{if } x_i = 1 - \sigma + h. \end{cases} \]

When bounding the term \( D_x^- \mathcal{T}(x_i, t_j) \) we will make use of the following truncation error bounds
\[ |D_x^- (u_t - D_t^- u)(x_i, t_j)| = \frac{1}{\tau} \left| \int_{t_{j-1}}^{t_j} \int_{s=r}^{t_j} D_x^- u_t(x_i, s) ds dr \right| \leq C \tau \| u_{tx} \|_{(x_{i-1}, x_i) \times (t_{j-1}, t_j)}. \] (14a)

If \( h_{i-1} = h_i \), then
\[ |D_x^- (u_x - D_x^- u)(x_i, t_j)| \leq Ch_i \| u_{xxx}(x, t_j) \|_{x \in (x_{i-2}, x_i)}, \] (14b)

and if \( h_{i-1} = h_i = h_{i+1} \), then
\[ |D_x^- (u_{xx} - \delta_x^2 u)(x_i, t_j)| \leq Ch_i^2 \| u_{xxxx}(x, t_j) \|_{x \in (x_{i-2}, x_{i+1})}. \] (14c)

From the assumption that \( \beta > \| a_x \|_G \), the discrete operator \( \hat{L}^{N,M} \) satisfies a discrete comparison principle.

Now we deduce bounds on the regular \( \mathcal{V}^- := D_x^- (V - v) \) and the singular \( \mathcal{W}^- := D_x^- (W - w) \) components of the discrete error flux \( \mathcal{W}^- \). We begin with the singular component. For the mesh points along the right hand boundary \( x = 1 \), we will need an appropriate bound on the outgoing error flux \( |D_x^- (W - w)| \). We achieve this by sharpening the nodal error bound given in Theorem 2, within the layer region, to reflect the fact that \( (W - w)(1, t_j) = 0. \)
Lemma 1. For the solutions \( w, W \) of the problems (9a) and (11b), and for sufficiently large \( N \),

\[
\varepsilon |D_x^-(W - w)(1, t_j)| \leq CN^{-1}(\ln N)^2, \quad t_j \geq 0.
\]  

(15)

Proof. For each \( t_j \geq 0 \), consider the discrete function \( \psi(x_i, t_j) \) as the solution of the discrete problem

\[
-\varepsilon \delta_x^2 \psi + (a(x_i, t_j) + \beta t_j) D_x^- \psi = 0, \quad x_i \in (1 - \sigma, 1), \quad \psi(1 - \sigma, t_j) = 1, \quad \psi(1, t_j) = 0.
\]

Note that the mesh is uniform in the layer region \((1 - \sigma, 1) \times (0, T]\). Define the discrete flux to be

\[
F_i^j := D_x^- \psi(x_i, t_j) < 0,
\]

which satisfies the inequalities

\[ h \sum_{i=N/2+1}^N F_i^j = -1, \quad F_i^j = F_i^j \prod_{k=i}^{N-1} \left(1 + \frac{(a(x_k, t_j) + \beta t_j)h}{\varepsilon}\right)^{-1}, \quad i < N. \]

Hence, for sufficiently large \( N \),

\[ |D_x^- \psi(1, t_j)| \leq \frac{1 - \left(1 + (\|a\|_G + \beta T)h \varepsilon\right)^{-1}}{h\left(1 - (1 + (\|a\|_G + \beta T)h \varepsilon)^{-N/2}\right)} \leq \frac{C}{\varepsilon}. \]

Here we have used the inequality

\[ (1 + K \frac{\ln N}{N})^{N/2} > e^K \frac{\ln N}{N} \]

and \( N \) is sufficiently large. Note also that for \( x_i \in (1 - \sigma, 1), \)

\[ (-\varepsilon \delta_x^2 + (a(x_i, t_j) + \beta t_j)D_x^-) D_i^- \psi(x_i, t_j) = -(D_i^- a(x_i, t_j) + \beta)D_x^- (\psi(x_i, t_{j-1})) \geq 0, \]

where we have used the identity [12] and \( D_i^- \psi(1 - \sigma, t_j) = D_i^- \psi(1, t_j) = 0. \) Also

\[ |(D_i^- a(x_i, t_j) + \beta)D_x^- (\psi(x_i, t_{j-1}))| \leq C \varepsilon^{-1}(1 + \frac{\alpha h}{\varepsilon})^{i-N}. \]

Using a discrete comparison principle, we deduce that

\[ D_i^- \psi(x_i, t_j) \geq 0, \quad |D_i^- \psi(x_i, t_j)| \leq C \left(1 + \frac{\alpha h}{\varepsilon}\right)^{i+1-N}. \]

Now we define a barrier function to deduce appropriate bounds for \( \mathcal{W}_N^-. \) First, we note that, at each time level \( t_j \), the grid function \( x_i - 1 + \sigma \psi(x_i, t_j) \) is the solution of the following problem

\[
-\varepsilon \delta_x^2 + a(x_i, t_j)D_x^- (x_i - 1 + \sigma \psi(x_i, t_j)) = a(x_i, t_j) - \sigma \beta t_j D_x^- (\psi(x_i, t_j)), \quad x_i \in (1 - \sigma, 1),
\]

\[ (x_i - 1 + \sigma \psi(x_i, t_j))_{x_i=1-\sigma} = (x_i - 1 + \sigma \psi(x_i, t_j))_{x_i=1} = 0. \]
So, by the discrete maximum principle \( x_i - 1 + \sigma \psi(x_i, t_j) \geq 0 \). Note that,
\[
L^{N,M}(x_i - 1 + \sigma \psi(x_i, t_j)) \geq a(x_i, t_j) - \sigma \beta t_j D_x^- \psi(x_i, t_j) + \sigma D_t^- \psi(x_i, t_j) \geq a(x_i, t_j).
\]
Define the following discrete barrier function
\[
B_1(x_i, t_j) := C\|L^{N,M}(W - w)\|_{(1-\sigma,1)}(x_i - 1 + \sigma \psi(x_i, t_j)) + CN^{-2};
\]
where \( L^{N,M}(W - w) \) is the truncation error associated with the singular component. Recall that in the boundary layer region
\[
\|L^{N,M}(W - w)\|_{(1-\sigma,1)} \leq C\varepsilon^{-1}N^{-1} \ln N.
\]
For \( x_i \in (1 - \sigma, 1) \) we then have that
\[
|(W - w)(x_i, t_j)| \leq B_1(x_i, t_j), \quad x_i \in [1 - \sigma, 1].
\]
Therefore,
\[
\varepsilon|\mathcal{W}_{i,j}^-| = \frac{\varepsilon}{h} |(W - w)(1 - h, t_j)| \\
\leq C\varepsilon\|L^{N,M}(W - w)\|_{(1-\sigma,1)}(1 + \sigma|D_x^- \psi(1, t_j)|) + CN^{-1} \\
\leq C N^{-1}(\ln N)^2,
\]
which is the required result. 

**Theorem 3.** Assume (10). Then, for all \( t_j \geq 0 \),
\[
|D_x^- (W - w)(x_i, t_j)| \leq CN^{-1}, \quad \text{if } x_i \leq 1 - \sigma,
\]
\[
\varepsilon|D_x^- (W - w)(x_i, t_j)| \leq CN^{-1}(\ln N)^2, \quad \text{if } x_i > 1 - \sigma,
\]
\[
(16)
\]
where \( W \) is the solution of (11b) and \( w \) is the solution of (9).

**Proof.** Note that outside the layer region, Theorem 2 implies that
\[
|\mathcal{W}_{i,j}^-| \leq CN^{-1}, \quad \text{if } x_i \in [0, 1 - \sigma], \ t_j \geq 0.
\]
Also, for \( x_i = 1 - \sigma + h, 1 - \sigma + 2h \)
\[
|w(x_i, t_j)| \leq C e^{\alpha h/\varepsilon} e^{-\alpha/\varepsilon} \leq CN^{-2}, \\
|W(x_i, t_j)| \leq C(1 + \alpha h/\varepsilon)^2(1 + \alpha h/\varepsilon)^{-N/2} \leq CN^{-2}.
\]
Hence,
\[
\varepsilon|\mathcal{W}_{i,j}^-| \leq CN^{-1}, \ x_i = 1 - \sigma + h, 1 - \sigma + 2h.
\]
In the layer region \((1 - \sigma + 2h, 1) \times (0, T)\) we will obtain the bounds by using (13). Initially, \( \mathcal{W}_{i,0}^- = 0, \ x_i \in (1 - \sigma, 1) \) and we established the required bound on the right boundary in (15). For \( x_i \in (1 - \sigma + 2h, 1), t_j > 0 \) and \( G^j_i := (x_{i-1}, x_i) \times (t_{j-1}, t_j) \), using (14) we get that
\[
|\hat{L}^{N,M}\mathcal{W}_{i,j}^-| \leq C \tau \|w_{u,x}\|_{G^j_i} + Ch(\varepsilon h)\|w_{xxx}\|_{G^j_i \cup G^j_{i+1} \cup G^j_{i+1}} \\
+ \|a_x\| G_{x} w_{xxx} \|_{G^j_i} + \|a\| G_{x} w_{xxx} \|_{G^j_{i-1} \cup G^j_{i}} + CN^{-1} \ln N \\
\leq \frac{C N^{-1} \ln N}{\varepsilon \varepsilon} e^{-a(1-x_i)/\varepsilon} + CN^{-1} \ln N.
\]
Use the discrete barrier function (and the strict inequality $a(x, t) > \alpha$)
\[
N^{-1} \ln^2 N (1 + (1 + \alpha h \varepsilon^{-1})^{i+1-N}),
\]
with the stability properties of $\hat{L}^{N,M}$ to complete the proof.

Consider now the contribution of the regular component to the discrete error flux.

**Lemma 2.** Assume (10). For $v, V$, the respective solutions of (5a), (11a), we have that
\[
\varepsilon |D_x^-(V - v)(x_i, t_j)| \leq CN^{-1}, \quad t_j \geq 0.
\]

**Proof.** It is a piecewise-uniform version of the proof of Lemma 1 for the singular component. Consider the discrete function $\tilde{\psi}(x_i, t_j)$ as the solution of the discrete problem
\[
-\varepsilon \delta^2 \tilde{\psi} + (a(x_i, t_j) + \beta t_j) \delta_x \tilde{\psi} = 0, \quad x_i \in (0, 1), \quad \tilde{\psi}(0, t_j) = 1, \quad \tilde{\psi}(1, t_j) = 0.
\]
Define $\tilde{F}_i^j := D_x^- \tilde{\psi}(x_i, t_j) < 0$, which satisfies
\[
H \sum_{i=1}^{N/2} \tilde{F}_i^j + h \sum_{i=N/2+1}^{N} \tilde{F}_i^j = -1; \quad \tilde{F}_i^j = \tilde{F}_N^j \prod_{k=i}^{N-1} (1 + (a(x_k, t_j) + \beta t_j) h_k \varepsilon^{-1})^{-1}, \quad i < N.
\]
Hence,
\[
|D_x^- \tilde{\psi}(1, t_j)| \leq H \sum_{i=N/2+1}^{N-1} (1 + (\|a\|_G + \beta T) h \varepsilon^{-1})^{-(N-i)} + 1) \leq C \varepsilon^{-1}.
\]

Use the barrier function
\[
B_2(x_i, t_j) := C\|L^{N,M}(V - v)\|_{G^{N,M}}(x_i - 1 + \tilde{\psi}(x_i, t_j)),
\]
where $\|L^{N,M}(V - v)\|_{G^{N,M}} \leq CN^{-1}$, to conclude that
\[
|(V - v)(x_i, t_j)| \leq B_2(x_i, t_j), \quad (x_i, t_j) \in G^{N,M}.
\]
Therefore,
\[
\varepsilon |\mathcal{N}^-_{N,j}| \leq \varepsilon |D_x^- B_2(1, t_j)| \leq C \varepsilon \|L^{N,M}(V - v)\|_{G^{N,M}} (1 + |D_x^- \tilde{\psi}(1, t_j)|) \leq CN^{-1}.
\]

**Theorem 4.** Assume (10). Then, for all $t_j \geq 0$,
\[
|D_x^-(V - v)(x_i, t_j)| \leq CN^{-1}, \quad \text{if } x_i \leq 1 - \sigma,
\]
\[
\varepsilon |D_x^- (V - v)(x_i, t_j)| \leq CN^{-1}(\ln N)^2, \quad \text{if } x_i > 1 - \sigma,
\]
where $V$ is the solution of (11a) and $v$ is the solution of (5a).
Proof. We again apply a stability and consistency argument, but now across the domain \(G_{N,M}^H\) to deduce suitable bounds on \(\gamma^-\). At the interior points, using the bounds (14), we get that
\[
|\hat{\gamma}_{i,j}^{\gamma^-}| \leq C N^{-1}, \quad x_i \neq 1 - \sigma, 1 - \sigma + h,
\]
\[
|\hat{\gamma}_{i,j}^{\gamma^-}| \leq C(\varepsilon + N^{-1}), \quad x_i \neq 1 - \sigma,
\]
and if \(x_i = 1 - \sigma + h\), (14a),
\[
|\hat{\gamma}_{i,j}^{\gamma^-}| \leq C h \left( \varepsilon H \|v_{xxx}\|_G + H \|v_{xx}\|_G \right) + C N^{-1} \leq \frac{C}{\varepsilon \ln N}.
\]
Using a suitable barrier function we can establish that
\[
|(V - v)(x_1, t_j)| \leq C x_i N^{-1} \quad \text{and, hence,}
\]
\[
|D_x (V - v)(x_1, t_j)| \leq C \frac{x_1}{H} N^{-1} \leq C N^{-1}.
\]
Summarising, we have established that
\[
|\hat{\gamma}_{i,j}^{\gamma^-}| \leq \left\{
\begin{array}{ll}
CN^{-1}, & \text{if } x_1 \leq x_i < 1 - \sigma, \\
C(\varepsilon + N^{-1}), & \text{if } x_i = 1 - \sigma, \\
C(\varepsilon \ln N)^{-1}, & \text{if } x_i = 1 - \sigma + h, \\
CN^{-1}, & \text{if } 1 - \sigma + h < x_i < 1, \\
C(\varepsilon H)^{-1}, & \text{if } x_i = 1.
\end{array}
\right.
\]
We deduce appropriate bounds for \(\gamma_{i,j}^-\) using a suitable barrier function.
Define the following two mesh functions:
\[
R(x_i) := \left\{
\begin{array}{ll}
\frac{x_i}{1 - \sigma}, & \text{if } x_i \leq 1 - \sigma, \\
1, & \text{if } 1 - \sigma < x_i \leq 1
\end{array}
\right.; \quad S(x_i) := \left\{
\begin{array}{ll}
0, & \text{if } x_i \leq 1 - \sigma, \\
1, & \text{if } 1 - \sigma < x_i \leq 1
\end{array}
\right.
\]
Observe that
\[
\hat{\gamma}_{i,j}^{R(x_i)} \geq \left\{
\begin{array}{ll}
\alpha, & \text{if } x_i < 1 - \sigma, \\
\varepsilon H^{-1} + \alpha, & \text{if } x_i = 1 - \sigma, \\
b + D_x a, & \text{if } x_i > 1 - \sigma.
\end{array}
\right.
\]
and
\[
\hat{\gamma}_{i,j}^{S(x_i)} \geq \left\{
\begin{array}{ll}
0, & \text{if } x_i < 1 - \sigma, \\
-\varepsilon N H^{-1}, & \text{if } x_i = 1 - \sigma, \\
\varepsilon N h^{-1} + \alpha h^{-1}, & \text{if } x_i = 1 - \sigma + h, \\
b + D_x a, & \text{if } x_i > 1 - \sigma + h.
\end{array}
\right.
\]
Define the piecewise linear barrier function
\[
B_3(x_i) = CN^{-1}\varepsilon^{-1}(R(x_i) + N^{-1}S(x_i)) + CN^{-1}R(x_i).
\]
Then, the discrete maximum principle establishes the bound
\[
|\gamma_{i,j}^-| \leq CB_3(x_i) \leq CN^{-1}\varepsilon^{-1}.
\]
By using a sharper barrier function, we will next remove the scaling factor $\varepsilon$ outside the layer. Define the local mesh Peclet numbers as follows:

$$\xi := \frac{\alpha H}{\varepsilon}; \quad \rho := \frac{\alpha h}{\varepsilon},$$

and the following two mesh functions:

$$P(x_i) := \begin{cases} (1 + \xi)^{i-N/2}, & \text{if } x_i \leq 1 - \sigma, \\ 1, & \text{if } x_i = 1 - \sigma + h, \\ (1 + \theta \rho)^{i-N/2-1}, & \text{if } 1 - \sigma + h < x_i \leq 1. \end{cases}$$

$$Q(x_i) := \begin{cases} 0, & \text{if } x_i \leq 1 - \sigma, \\ (1 + \theta \rho)^{i-N/2-1}, & \text{if } x_i > 1 - \sigma. \end{cases}$$

When we take $\theta = 0.5$, we have the following:

$$\varepsilon D^+_x P = \alpha P, \quad \varepsilon (1 + \xi) D^+_x P = \alpha P; \quad \varepsilon D^2_x P = \alpha D^+_x P, \quad x_i < 1 - \sigma,$n

$$\varepsilon D^+_x P = \theta \alpha P, \quad \varepsilon (1 + \theta \rho) D^+_x P = \theta \alpha P; \quad \varepsilon D^2_x P = \theta \alpha D^+_x P, \quad x_i > 1 - \sigma + h,$$

$$N > P(1) = Q(1) \geq 0.25N.$$

Observe that

$$\hat{L}^{N,M} P(x_i) \geq \begin{cases} (a(x_{i-1}, t_j) - \alpha) D^-_x P, & \text{if } x_i < 1 - \sigma, \\ H^{-1} \alpha + \frac{a(x_{i-1}, t_j) \alpha}{\varepsilon(1+\xi)}, & \text{if } x_i = 1 - \sigma, \\ -\alpha \theta h^{-1}, & \text{if } x_i = 1 - \sigma + h, \\ (a(x_{i-1}, t_j) - \theta \alpha) D^-_x P, & \text{if } x_i > 1 - \sigma + h, \end{cases}$$

and

$$\hat{L}^{N,M} Q(x_i) \geq \begin{cases} 0, & \text{if } x_i < 1 - \sigma, \\ -\varepsilon NH^{-1}, & \text{if } x_i = 1 - \sigma, \\ \varepsilon (H h)^{-1} + (a(x_{i-1}, t_j) - \theta \alpha) h^{-1}, & \text{if } x_i = 1 - \sigma + h, \\ (a(x_{i-1}, t_j) - \theta \alpha) D^+_x Q, & \text{if } x_i > 1 - \sigma + h. \end{cases}$$

Hence, using $2\varepsilon \ln N \leq 1$ and $N$ sufficiently large,

$$\hat{L}^{N,M} (P(x_i) + \alpha N^{-1} \varepsilon^{-1} Q(x_i)) \geq \begin{cases} 0, & \text{if } x_i \leq 1 - \sigma, \\ CN\sigma^{-1}, & \text{if } x_i = 1 - \sigma + h, \\ C\varepsilon^{-1}, & \text{if } x_i > 1 - \sigma + h. \end{cases}$$

Form the barrier function

$$B_4(x_i) := CN^{-1} R(x_i) + C(P(1))^{-1} (P(x_i) + \alpha N^{-1} \varepsilon^{-1} Q(x_i)).$$

Then we have established that $|\gamma_{i,j}^- (x_i, t_j)| \leq CB_4(x_i)$. This bound is of little value in the area $|1 - \sigma, 1|$ (as $B_4(x_i) \leq C(1 + N^{-1} \varepsilon^{-1})$ in this area). However, in the coarse mesh region $x_i \in [0, 1 - \sigma]$, we deduce

$$|\gamma_{i,j}^-| \leq CB_4(x_i) \leq CN^{-1}, \quad x_i \leq 1 - \sigma.$$
Using the bounds obtained for the regular and singular components, we obtain the following result.

**Theorem 5.** Assume (10). Then, for all $t_j \geq 0$,

\[
\begin{align*}
|D_x^-(U - u)(x_i, t_j)| &\leq CN^{-1}, & \text{if } 0 < x_i \leq 1 - \sigma, \\
\varepsilon|D_x^-(U - u)(x_i, t_j)| &\leq CN^{-1} (\ln N)^2, & \text{if } 1 - \sigma < x_i \leq 1,
\end{align*}
\]

(20)

where $U$ is the solution generated by the numerical method (3) and $u$ is the solution of the continuous problem (1).

## 5 Nodal approximation of time derivatives

We follow the approach outlined in [9, Appendix A.2] and [11, §8.2] to deduce nodal approximations of the time derivatives. We note that the proof in [11] was given only for the case of constant $a$.

**Lemma 3.** Assume (10). The following bounds hold, for all $t_j > 0$

\[
\begin{align*}
|(D_t^- V - v_t)(x_i, t_j)| &\leq CN^{-1} \ln N, & \text{if } x_i \in [0, 1], \\
|(D_t^- W - w_t)(x_i, t_j)| &\leq CN^{-1}, & \text{if } x_i \in [0, 1 - \sigma], \\
|(D_t^- W - w_t)(x_i, t_j)| &\leq CN^{-1} (\ln N)^3, & \text{if } x_i \in [1 - \sigma, 1],
\end{align*}
\]

where $v, w$ are the solutions of problems (5a), (6a) and $V, W$ are defined in (11).

**Proof.** Using the bounds on the components $v, w$, of the solution $u$ of problem (4) we deduce that for all $(x_i, t_j) \in [0, 1] \times [\tau, T]$

\[
|(D_t^- u - u_t)(x_i, t_j)| \leq C\|u_{tt}\| \leq CN^{-1}.
\]

Hence,

\[
\begin{align*}
|(D_t^- V - v_t)(x_i, t_j)| &\leq |D_t^- (V - v)(x_i, t_j)| + CN^{-1}, \\
|(D_t^- W - w_t)(x_i, t_j)| &\leq |D_t^- (W - w)(x_i, t_j)| + CN^{-1}.
\end{align*}
\]

Note that along the side boundaries

\[
D_t^- (V - v)(x_i, t_j) = 0, \quad (x_i, t_j) \in (\Gamma_L \cup \Gamma_R) \cap \tilde{G}^{N,M}, \quad t_j \geq \tau;
\]

and from the error bound in Theorem 2 we deduce that,

\[
|D_t^- (V - v)(x_i, \tau)| \leq CN^{-1}.
\]

At the interior points $(x_i, t_j) \in G^{N,M} \cap \{t_j \geq 2\tau\}$, we will first estimate

\[
|L^{N,M}(D_t^- (V - v)(x_i, t_j)|.
\]
We recall that for all $t \xi$

Consider the following mesh function

Now we consider the singular component. Outside the layer region, we use the bound to deduce that

Use the following barrier function

It now remains to bound above to deduce the truncation error estimate

We wish to reverse the order of the operators $L^{N,M}$ and $D_t^-$. To this end, we again use the identity (12) to obtain

We recall that for all $t_j \geq 0$,

Consider the following mesh function

with $\xi = \alpha \varepsilon / H$. Note that

Use the following barrier function

to deduce that

Now we consider the singular component. Outside the layer region, we use the bound

max\{\|W(x_i, t_j)\|, |w(x_i, t_j)|\} \leq CN^{-2}, \quad x_i \leq 1 - \sigma,

to deduce that, for all $t_j \geq \tau$,

It now remains to bound $D_t^-(W - w)$ within the layer region. We repeat the argument from above to deduce the truncation error estimate

At the first time level $t_j = \tau$, using the bounds in Theorem 2 we deduce that

and at the right boundary, $D_t^-(W - w)(1, t_j) = 0$. Use the discrete barrier function

$C \left( \frac{x_i - (1 - \sigma)}{\varepsilon} \right)^N \leq \ln N^2 + N^{-1} \ln N$, to complete the proof.
Motivated by the bounds in [14, Corollary 3] we can sharpen the bound given in Theorem 5 for points within the layer region, in the special case where $N^{-2} \leq \varepsilon$.

**Theorem 6.** Assume (10). If $N^{-2} \leq \varepsilon$, then

$$|D_x(U - u)(x_i, t_j)| \leq CN^{-1}(\ln N)^3, \quad x_i < 1 - \frac{\varepsilon}{\alpha} \ln(1/\varepsilon), \quad t_j \geq 0,$$

where $U$ and $u$ are the respective solutions of (9) and (4).

**Proof.** If $N^{-2} \leq \varepsilon$, then $\varepsilon \ln(1/\varepsilon) \leq 2\varepsilon \ln N$. From the previous result, we only need to consider the mesh points in the region $(1 - 2\frac{\varepsilon}{\alpha} \ln N, 1 - \frac{\varepsilon}{\alpha} \ln \frac{1}{\varepsilon})$. Within the fine mesh, the error satisfies

$$-\varepsilon \frac{\varepsilon}{h}(\mathcal{U}_{i+1,j}^r - \mathcal{U}_{i,j}^r) + a(x_i, t_j)\mathcal{U}_{i,j}^r = \hat{T}_{i,j},$$

where

$$\hat{T}_{i,j} = L_{N,M}^r(U - u)(x_i, t_j) - D_x(U - u)(x_i, t_j) - b(x_i, t_j)(U - u)(x_i, t_j).$$

Note that

$$\|\hat{T}\| \leq C(N^{-1} \ln N \varepsilon^{-1} e^{-\alpha(1-x_i)/\varepsilon} + N^{-1}(\ln N)^3).$$

Thus, with $\rho := \frac{\alpha h}{\varepsilon}$, we have

$$|\mathcal{U}_{i,j}^r| = (1 + \frac{h}{\varepsilon} a(x_i, t_j))^{-1} \left|\frac{h}{\varepsilon} \hat{T}_{i,j} + \mathcal{U}_{i+1,j}^r\right| \leq (1 + \rho)^{-1} \left(\frac{h}{\varepsilon} \|\hat{T}\| + |\mathcal{U}_{i+1,j}^r|\right).$$

Thus, we have the following estimate at $x_i$ (within the fine mesh)

$$|\mathcal{U}_{i,j}^r| \leq (1 + \rho)^{-1} \frac{h}{\varepsilon} \|\hat{T}\| + C(1 + \rho)^{-1} \left(\frac{h}{\varepsilon} \|\hat{T}\| + |\mathcal{U}_{i+1,j}^r|\right) + C \frac{N^{-1}(\ln N)^2}{\varepsilon}(1 + \rho)^{-1}.$$ 

Since $x_i < 1 - \frac{\varepsilon}{\alpha} \ln \frac{1}{\varepsilon}$, there exists some $\theta > 1$ such that $x_i \leq 1 - \theta \frac{\varepsilon}{\alpha} \ln \frac{1}{\varepsilon}$. For $N$ sufficiently large, we note that for $\theta > 1$

$$(1 + \rho)^{-1} \leq e^{-\theta} / \rho, \quad \rho \leq \theta \ln \theta.$$

Hence, for $x_i \leq 1 - \theta \frac{\varepsilon}{\alpha} \ln \frac{1}{\varepsilon}$

$$|\mathcal{U}_{i,j}^r| \leq C \left(N^{-1}(\ln N)^2 e^{-\frac{\alpha(1-x_i)}{\varepsilon}} + N^{-1}(\ln N)^3\right) \leq CN^{-1}(\ln N)^3.$$

\qed
6 Global accuracy in weighted $C^1$-norm

In this section, we examine the global accuracy (in the weighted $C^1$-norm $\| \cdot \|_{1,\chi,G}$) of the bilinear interpolant

$$\bar{U}(x,t) := \sum_{i,j=1}^{N-1,M} U(x_i,t_j)\phi_i(x)\psi_j(t), \quad (x,t) \in \bar{G},$$

where $\phi_i(x), \psi_j(t)$ are piecewise linear basis functions in space and time, defined by the nodal values of $\phi_i(x_k) = \delta_{i,k} = \psi_i(t_k)$. Note the following bound on the bilinear interpolant $\bar{g}$ of a function $g$ (see e.g. [19, Lemma 4.1]) in the rectangular cell $R_{ij} := (x_{i-1},x_i) \times (t_{j-1},t_j)$

$$\|g - \bar{g}\|_{R_{ij}} \leq C \min\{h_i^2 \|g_{xx}\|_{R_{ij}}, \max_{t \in [t_{j-1},t_j]} \int_{x_{i-1}}^{x_i} |g_x(s,t)| ds\}$$

$$+ C \min\{\tau^2 \|g_t\|_{R_{ij}}, \max_{x \in [x_{i-1},x_i]} \int_{t_{j-1}}^{t_j} |g_t(x,s)| ds\}. \quad (23)$$

Theorem 7. Assume [10]. Then,

$$\|\bar{U} - u\|_{1,\chi,G} \leq CN^{-1}\ln N. \quad (24)$$

where $U$ is the solution generated by the numerical method [3] and $u$ is the solution of the continuous problem [7].

Proof. Using the decomposition $u = v + w$ and splitting the argument to inside and outside the computational layer region $[1 - \sigma, 1] \times (0,T)$, we have the following interpolation error (see e.g. [19 Theorem 4.2])

$$\|u - \bar{u}\|_G \leq C(N^{-1} \ln N)^2 + C\tau^2. \quad (25)$$

Hence, the following global error estimate follows:

$$\|u - \bar{U}\|_G \leq CN^{-1} \ln N.$$

Note that

$$(\bar{U} - \bar{u})_t(x,t) = \sum_{i=1}^{N-1} D^t_i(U - u)(x_i,t_j)\phi_i(x), \quad t \in (t_{j-1},t_j),$$

$$(\bar{U} - \bar{u})_x(x,t) = \sum_{j=1}^{M} D^x_j(U - u)(x_i,t_j)\psi_j(t), \quad x \in (x_{i-1},x_i).$$

Using the bounds in Lemma [3] for the discrete time derivatives; the bounds in Theorem [5] when $\varepsilon \leq N^{-2}$ and the bound in Theorem [6] when $\varepsilon \geq N^{-2}$ for the discrete space derivatives, we have that

$$\|\bar{U} - \bar{u}\|_{1,\chi,G} \leq CN^{-1}(\ln N)^3. \quad (26)$$
We are left to estimate the interpolation error \( \|u - \bar{u}\|_{1,\chi,G} \). For \( x \in (x_{i-1}, x_i) \), we have

\[
(\bar{u} - u)_x(x, t) = \sum_{j=1}^{M} (D_x^\epsilon u(x_i, t_j) - u_x(x_t, t_j)) \psi_j(t) + \sum_{j=1}^{M} u_x(x_t, t_j) \psi_j(t) - u_x(x, t).
\]

Therefore, in the rectangular cell \( R_{ij} \), we obtain

\[
\|(g - \bar{g})_x\|_{R_{ij}} \leq \min\{h_i \|g_{xx}\|_{R_{ij}}, \|g_x\|_{R_{ij}}\} + \min\{\tau \|g_{xt}\|_{R_{ij}}, \|g_t\|_{R_{ij}}\}. \tag{27}
\]

We employ the decomposition \( u = v + w \). For the regular component it trivially follows that

\[
\|(v - \bar{v})_x\|_{R_{ij}} \leq C(N^{-1} + \tau).
\]

For the layer component, we split the argument to inside and outside the layer region \([1 - 2\frac{x}{\alpha} \ln \frac{1}{\epsilon}, 1] \times (0, T)\) and deal with the two cases of \( \epsilon \leq N^{-2} \) and \( \epsilon \geq N^{-2} \). We observe the following: If \( \epsilon \leq N^{-2} \) then \( 2\epsilon \ln N \leq \epsilon \ln \frac{1}{\epsilon} \) and in this case

\[
\|w_x\|_{R_{ij}} \leq C \epsilon \leq C N^{-2}, \quad x_i \leq 1 - 2(\epsilon/\alpha) \ln(1/\epsilon),
\]

\[
\|w_x\|_{R_{ij}} \leq C \epsilon^{-1} N^{-2}, \quad 1 - 2(\epsilon/\alpha) \ln(1/\epsilon) < x_i \leq 1 - \sigma,
\]

\[
h_i \|w_{xx}\|_{R_{ij}} + \tau \|w_{xt}\|_{R_{ij}} \leq C \epsilon^{-1} N^{-1} \ln N, \quad x_i > 1 - \sigma.
\]

In the second case, where \( \epsilon \geq N^{-2} \), then we distinguish two subcases: If \( \epsilon \geq N^{-1} \), then

\[
h_i \|w_{xx}\|_{R_{ij}} + \tau \|w_{xt}\|_{R_{ij}} \leq C(h_i + \tau), \quad x_i \leq 1 - 2(\epsilon/\alpha) \ln(1/\epsilon),
\]

\[
h_i \|w_{xx}\|_{R_{ij}} + \tau \|w_{xt}\|_{R_{ij}} \leq C \epsilon^{-1} N^{-1} \ln N, \quad x_i > 1 - 2(\epsilon/\alpha) \ln(1/\epsilon).
\]

On the other hand, if \( N^{-1} \geq \epsilon \geq N^{-2} \), then

\[
h_i \|w_{xx}\|_{R_{ij}} + \tau \|w_{xt}\|_{R_{ij}} \leq C(h_i + \tau), \quad x_i \leq 1 - 2(\epsilon/\alpha) \ln(1/\epsilon),
\]

\[
\|w_x\|_{R_{ij}} \leq C \epsilon^{-1} N^{-2}, \quad 1 - 2(\epsilon/\alpha) \ln(1/\epsilon) < x_i \leq 1 - \sigma,
\]

\[
h_i \|w_{xx}\|_{R_{ij}} + \tau \|w_{xt}\|_{R_{ij}} \leq C \epsilon^{-1} N^{-1} \ln N, \quad x_i > 1 - \sigma.
\]

Combining all these bounds, we deduce that

\[
\|\chi(x)(u - \bar{u})_x\|_G \leq CN^{-1} \ln N,
\]

where \( \chi(x) \) is defined in [3]. Similarly, for \( t \in (t_{j-1}, t_j) \),

\[
(\bar{u} - u)_t(x, t) = \sum_{i=1}^{N-1} (D_t^\epsilon u(x_i, t_j) - u_t(x_i, t)) \phi_i(x) + \sum_{i=1}^{N-1} u_t(x_i, t) \phi_i(x) - u_t(x, t).
\]

In the rectangular cell \( R_{ij} \),

\[
\|(g - \bar{g})_t\|_{R_{ij}} \leq C\tau \|g_t\|_{R_{ij}} + \min\{h_i \|g_{xt}\|_{R_{ij}}, \|g_t\|_{R_{ij}}\}. \tag{28}
\]

By again using the decomposition \( u = v + w \) and splitting the argument to inside and outside the layer region \([1 - 2\frac{x}{\alpha} \ln \frac{1}{\epsilon}, 1] \times (0, T)\), we deduce that

\[
\|(u - \bar{u})_t\|_G \leq CN^{-1}.
\]

\[\square\]
Remark 3. In the classical case of $\sigma = 0.5$ we have that $\varepsilon^{-1} \leq \ln N$. Using this inequality, we can easily extend the bound \([24]\) in Theorem \([7]\) to the case of $\sigma = 0.5$. From Theorem 1,

$$\| \frac{\partial^{k+m} u}{\partial x^k \partial t^m} \|_G \leq C \varepsilon^{-k}(1 + \varepsilon^{2-m}).$$

We have the following nodal error estimate,

$$|(U - u)(x_i, t_j)| \leq CN^{-1}(\ln N)^2 \min\{t_j, x_i, \tilde{\psi}(x_i, t_j)\}, \quad (x_i, t_j) \in G_{N,M},$$

where the function $\tilde{\psi}$ is defined in Lemma 2. Then, using the arguments in Lemma 2, one deduces that

$$|D_x^- (U - u)(1, t_j)| \leq CN^{-1}(\ln N)^3.$$

Looking at the proof of Theorem 4, we note that

$$|\hat{L}_N^M (D_x^- (U - u)(x_i, t_j))| \leq CN^{-1}(\ln N)^3, \quad x_1 \leq x_i < 1.$$

Hence

$$|D_x^- (U - u)(x_i, t_j)| \leq CN^{-1}(\ln N)^3, \quad x_1 \leq x_i \leq 1.$$

Also,

$$|D_t^- (U - u)(x_i, t_j)| \leq CN^{-1}(\ln N)^3, \quad \tau \leq t_j.$$

Thus, in the case of $\sigma = 0.5$,

$$\|(\bar{U} - u)_{x}\|_G + \|(\bar{U} - u)_{t}\|_G + \|\bar{U} - u\|_G \leq CN^{-1}(\ln N)^3.$$
The uniform global errors in the norm $\| \cdot \|_{1, \chi, G}$ and their orders of convergence are estimated as follows

$$E^{N,M} := \max_{\varepsilon \in S} E^{N,M}_\varepsilon, \quad Q^{N,M} := \log_2 (E^{N,M} / E^{2N,2M}),$$

with $S = \{2^0, 2^{-1}, 2^{-2}, \ldots, 2^{-30}\}$.

In Table 1 we present the computed global $E^{N,M}_\varepsilon$ and uniform global $E^{N,M}$ errors for $N = M = 2^j, j = 4, 5, 6, 7, 8, 9$ with their corresponding orders of convergence associated with the finite difference scheme (9) on the Shishkin mesh for test problem (29). The numerical results in Table 1 indicate that the method is uniformly convergent in the weighted $C^1$-norm $\| \cdot \|_{1, \chi, G}$. The computed orders of convergence in this example are slightly higher than the theoretical order of convergence established in Theorem 7, but this is a well-known effect when the errors are estimated by considering the computed solution on a fine mesh as the exact solution.

8 Appendix: Reaction-Diffusion problem

In this appendix, we consider the reaction-diffusion problem

$$L_\varepsilon u := -\varepsilon u_{xx} + b(x,t)u + u_t = f(x,t), \text{ in } G := \Omega \times (0, T],$$

$$u = 0, \quad \text{on } \Gamma_B \cup \Gamma_L \cup \Gamma_R, \quad b(x,t) \geq \beta > 0. \quad (30a)$$

Layers of width $O(\sqrt{\varepsilon})$ occur on both sides of the domain and we have the following bounds on the solution and its components.

**Theorem 8.** [16] The solution of (30) can be written in the form $u = v + w_L + w_R$, where the regular component $v \in C^{6+\gamma}(G)$ satisfies

$$L_\varepsilon v = f, \text{ in } G, \quad v = u, \text{ on } \Gamma_B, \quad (31a)$$
and \(v = v^*\) can be specified on the boundary \(\Gamma_R \cup \Gamma_L\), so that

\[
\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\|_G \leq C(1 + \varepsilon^{2-k/2}), \quad 0 \leq k + 2m \leq 6.
\] (31b)

The singular components \(w_L, w_R\) satisfy the problems

\[
L_\varepsilon w_L = 0, \text{ in } G, \quad w_L = 0, \text{ on } \Gamma_B \cup \Gamma_R, \quad w_L = u - v \text{ on } \Gamma_L, \tag{32a}
\]
\[
L_\varepsilon w_R = 0, \text{ in } G, \quad w_R = 0, \text{ on } \Gamma_B \cup \Gamma_L, \quad w_R = u - v \text{ on } \Gamma_R, \tag{32b}
\]

and for all points \((x,t) \in G\) and \(0 \leq k + 2m \leq 6\)

\[
\left| \frac{\partial^{k+m} w_R}{\partial x^k \partial t^m} (x,t) \right| \leq C\varepsilon^{-k/2}e^{-\sqrt{\varepsilon}(1-x)}, \tag{32c}
\]
\[
\left| \frac{\partial^{k+m} w_L}{\partial x^k \partial t^m} (x,t) \right| \leq C\varepsilon^{-k/2}e^{-\sqrt{\varepsilon}x}. \tag{32d}
\]

The Shishkin mesh for the reaction-diffusion problem is fitted to the two boundary layers by splitting the space domain as follows:

\[
[0, \hat{\sigma}] \cup [\hat{\sigma}, 1 - \hat{\sigma}] \cup [1 - \hat{\sigma}, 1]. \tag{33a}
\]

The mesh points are distributed in the ratio \(N/4 : N/2 : N/4\) across these three subintervals. The transition point \(\hat{\sigma}\) is taken to be

\[
\hat{\sigma} := \min \{0.25, 2\sqrt{\varepsilon} \ln N\}. \tag{33b}
\]

The discrete solution of problem \(\langle 9 \rangle\) with \(a(x,t) \equiv 0\) on this new mesh is decomposed \(U = V + W_L + W_R\) in the obvious way \(\langle 16 \rangle\) and we have the following global error estimate \(\langle 16 \rangle\)

\[
\|u - U\|_G \leq C(N^{-1} \ln N)^2 + C\tau.
\]

We now show how the arguments in the earlier sections are applicable and simplify in the case of problem \(\langle 32 \rangle\). We will deduce an error estimate in the following weighted \(C^1\)-norm

\[
\|v\|_{1,\kappa,G} := \|\kappa(x)v_x\|_G + \|v_t\|_G + \|v\|_G,
\] (34)

\[
\kappa(x) := \begin{cases} \sqrt{\varepsilon}, \text{ if } x \in (0, \sqrt{\varepsilon} \ln \frac{1}{\varepsilon}) \cup (1 - \sqrt{\varepsilon} \ln \frac{1}{\varepsilon}, 1), \\ 1, \text{ otherwise.} \end{cases}
\]

Using the nodal error bound \(\|(U - u)(x_i, t_j)\| \leq Ct_j(\tau + N^{-1})\) one deduces that at the first time level

\[
|D_\tau^{-}(U - u)(x_i, \tau)| \leq C(\tau + N^{-1}), \quad 0 \leq x_i \leq 1.
\]

The argument from \(\S 5\) simplifies and one deduces that

\[
\|D_\tau^{-}(U - u)\|_{G^{N,M}} \leq C(\tau + N^{-1}).
\]
Arguments from §4 are now applied to establish bounds on the error in estimating \( u_x \). Using \( u_{tt}(0, t) = 0 \), we have for some \( \eta \in (0, x_i) \),

\[
\left| (u_t - D^-_t u)(x_i, t_j) \right| = \frac{x_i}{\tau} \left| \int_{t=s=t}^{t=t_{j-1}} \int_{s=t}^{t_j} u_{ttx}(\eta, s) ds dt \right|
\leq \frac{x_i}{\tau} \int_{t_{j-1}}^{t_j} \int_{s=t}^{t_j} \| u_{ttx} \| G ds dt \leq C \frac{x_i \tau}{\sqrt{\varepsilon}}.
\]

Hence, one can deduce the nodal error bound,

\[
\left| (U - u)(x_i, t_j) \right| \leq C((N^{-1} \ln N)^2 + \tau \min \{x_i, \frac{1-x_i}{\sqrt{\varepsilon}}\}).
\]

From this bound, the boundary error fluxes are bounded as follows

\[
\max_{p=x_1,x_N} \sqrt{\varepsilon} |D^-_x (U - u)(p, t_j)| \leq CN^{-1} \ln N + C\tau. \tag{35}
\]

Let us again assume that

\[
\hat{\sigma} = 2 \sqrt{\frac{\varepsilon}{\beta}} \ln N, \quad C_1 N \leq M \leq C_2 N. \tag{36}
\]

**Remark 4.** Note that on the piecewise uniform mesh

\[
\delta^2 x_x \equiv \begin{cases} 
\frac{1}{h} (2(1 - 2\hat{\sigma}) D^+_x - D^-_x), & \text{if } x_i = \hat{\sigma}, \\
\frac{1}{H} (D^+_x - 2(1 - 2\hat{\sigma}) D^-_x), & \text{if } x_i = \hat{\sigma} + H, \\
\frac{1}{H} (4\hat{\sigma} D^+_x - D^-_x), & \text{if } x_i = 1 - \hat{\sigma}, \\
\frac{1}{h} (D^+_x - 4\hat{\sigma} D^-_x), & \text{if } x_i = 1 - \hat{\sigma} + h, \\
\delta^2 x_x, & \text{otherwise.}
\end{cases}
\]

Note that

\[
|\delta^2 x_x w(x_i, t_j)| \leq \max_{x \in (x_i-1, x_i+1)} |w_{xx}(x, t_j)|,
\]
and using the truncation error bounds (14) we can then deduce the bound,

\[ |\hat{L}^{N,M}_{i,j}| \leq CN^{-1}(1 + \|u_{tx}\|_{G_1}) \]

Define the following four mesh functions:

\[
R_1(x_i) := \begin{cases} \frac{x_i}{\sigma}, & \text{if } x_i \leq \hat{\sigma}, \\ 1, & \text{if } \hat{\sigma} < x_i \leq 1 \end{cases}; \quad R_2(x_i) := \begin{cases} \frac{x_i}{\sigma + H}, & \text{if } x_i \leq \hat{\sigma} + H, \\ 1, & \text{if } \hat{\sigma} + H < x_i \leq 1 \end{cases}; \\
R_3(x_i) := \begin{cases} 1, & \text{if } x_i \leq 1 - \hat{\sigma}, \\ \frac{1 - x_i}{\sigma}, & \text{if } 1 - \hat{\sigma} < x_i \leq 1 \end{cases}; \quad R_4(x_i) := \begin{cases} 1, & \text{if } x_i \leq 1 - \hat{\sigma} + h, \\ \frac{1 - x_i}{\sigma - h}, & \text{if } 1 - \hat{\sigma} + h < x_i \leq 1 \end{cases}.
\]

Observe that they satisfy

\[
\hat{L}^{N,M}R_1(x_i) \geq \begin{cases} \frac{\varepsilon}{N} \geq \frac{\beta N}{(4\ln N)^2}, & \text{if } x_i = \hat{\sigma} \\ 0, & \text{if } x_i \neq \hat{\sigma} \end{cases}; \quad \hat{L}^{N,M}R_2(x_i) \geq \begin{cases} \frac{2\varepsilon(1-2\hat{\sigma})}{H(H+\sigma)} \geq \frac{\varepsilon N}{2}, & \text{if } x_i = \hat{\sigma} + H \\ 0, & \text{if } x_i < \hat{\sigma} + H \end{cases}; \\
\hat{L}^{N,M}R_3(x_i) \geq \begin{cases} \frac{4\varepsilon}{H} \geq 2\varepsilon N, & \text{if } x_i = 1 - \hat{\sigma} \\ 0, & \text{if } x_i > 1 - \hat{\sigma} \end{cases}; \quad \hat{L}^{N,M}R_4(x_i) \geq \begin{cases} \frac{\varepsilon}{N(\sigma - h)} \geq \frac{\beta N}{(4\ln N)^2}, & \text{if } x_i = 1 - \hat{\sigma} + h \\ 0, & \text{if } x_i \neq 1 - \hat{\sigma} + h \end{cases}.
\]

Form the barrier function

\[ CN^{-1} \ln N \sqrt{\varepsilon}(R_1(x_i) + R_4(x_i)) + CN^{-1}(R_2(x_i) + R_3(x_i)) + C(\sqrt{\varepsilon}N)^{-1} \ln N, \]

to deduce the error bound of

\[ \sqrt{\varepsilon}|\mathcal{U}^{-1}_{i,j}| \leq CN^{-1} \ln N. \]
Define the local mesh Peclet numbers as follows:

\[ \xi_1 := \sqrt{\frac{\beta}{\varepsilon} H}; \quad \rho_1 := \sqrt{\frac{\beta}{\varepsilon} h}, \]

and consider the following two mesh functions

\[
P_1(x_i) = \begin{cases} 
(1 + 0.5\rho_1)^{-i}, & \text{if } x_i \leq \hat{\sigma}, \\
(1 + 0.5\rho_1)^{-N/4}(1 + \xi_1)^{N/4-i}, & \text{if } \hat{\sigma} \leq x_i \leq 0.5, \\
(1 + 0.5\rho_1)^{-N/4}(1 + \xi_1)^{-3N/4}, & \text{if } 0.5 \leq x_i < 1 - \hat{\sigma}, \\
(1 + 0.5\rho_1)^{-N}, & \text{if } x_i \geq 1 - \hat{\sigma}.
\end{cases}
\]

\[
Q_1(x_i) = \begin{cases} 
1, & \text{if } x_i \leq \hat{\sigma}, \\
0, & \text{if } \hat{\sigma} < x_i \leq 1 - \hat{\sigma}, \\
1, & \text{if } x_i < 1 - \hat{\sigma}.
\end{cases}
\]

Let us consider the points where \( x_i < x_{N/2} \). At the transition point \( \hat{\sigma} \),

\[ N^{-1} \leq P_1(\hat{\sigma}) \leq 4N^{-1}, \]

and for the remaining points we have

\[
\sqrt{\varepsilon}(1 + 0.5\rho_1)D_x^+P_1 = -0.5\sqrt{\beta}P_1 \\
\varepsilon D_x^-P_1 = -0.5\sqrt{\beta}P_1 \\
\varepsilon(1 + 0.5\rho_1)\delta_x^2P_1 = 0.25\beta P_1
\]

if \( x_i < \hat{\sigma} \),

\[
\sqrt{\varepsilon}(1 + \xi_1)D_x^+P_1 = -\sqrt{\beta}P_1 \\
\varepsilon D_x^-P_1 = -\sqrt{\beta}P_1 \\
\varepsilon(1 + \xi_1)\delta_x^2P_1 = \beta P_1
\]

if \( \hat{\sigma} < x_i < x_{N/2} \).

These expressions allow us to establish the following lower bounds

\[
\hat{L}^{N,M}P_1(x_i) \geq C \begin{cases} 
1, & \text{if } x_i = x_1, 1, \\
P_1(x_i) \geq e^{-\sqrt{\varepsilon}x_i}, & \text{if } x_i \in (x_1, \hat{\sigma}), \\
P_1(x_i) \geq e^{-\sqrt{\varepsilon}(1-x_i)}, & \text{if } x_i \in (1 - \hat{\sigma} + h, 1), \\
-\frac{1}{\ln N}, & \text{if } x_i = \hat{\sigma}, 1 - \hat{\sigma} + h, \\
-N\sqrt{\varepsilon}P_1 \geq -\sqrt{\varepsilon}, & \text{if } x_i = 1 - \hat{\sigma}, \hat{\sigma} + H, \\
N^{-1}, & \text{if } x_i \in (\hat{\sigma} + H, 1 - \hat{\sigma}) \text{ and } x_i \neq x_{N/2}, \\
-N\sqrt{\varepsilon}(1 + \xi_1)^{-N/4}, & \text{if } x_i = x_{N/2}.
\end{cases}
\]

\[
\hat{L}^{N,M}Q_1(x_i) \geq C \begin{cases} 
0, & \text{if } x_i \neq \hat{\sigma}, \hat{\sigma} + H, 1 - \hat{\sigma}, 1 - \hat{\sigma} + h, \\
\frac{N\varepsilon}{h}, & \text{if } x_i = \hat{\sigma}, 1 - \hat{\sigma} + h, \\
-\frac{N\varepsilon}{H}, & \text{if } x_i = \hat{\sigma} + H, 1 - \hat{\sigma}.
\end{cases}
\]

Hence, we can construct a majoring function for the mesh function \( W^- \)

\[
|W^\pm_{i,j}| \leq CN^{-1}\ln N \sum_{j=1}^{4} R_j(x_i) + C N^{-1}\ln N \left( P_1(x_i) + \frac{CN^{-2}}{\sqrt{\varepsilon}} Q_1(x_i) \right).
\]
Noting that
\[ \frac{N^{-1}}{\sqrt{\varepsilon}} |P_1(\hat{\sigma} + H)| \leq \frac{N^{-1}}{\sqrt{\varepsilon}} \frac{N^{-1}}{1 + \xi} \leq CN^{-1}, \]
we deduce the following estimates for the discrete error flux
\[ \sqrt{\varepsilon}|\mathcal{U}_{i,j}^-| \leq CN^{-1} \ln N, \quad \text{if } x_i \in (0, \hat{\sigma}] \cup [1 - \hat{\sigma}, 1], \]  
(37a)
\[ |\mathcal{U}_{i,j}^-| \leq CN^{-1} \ln N, \quad \text{otherwise.} \]  
(37b)

**Theorem 9.** Assume (36). In the case of problem of (36) and for all \( t_j \geq 0 \),
\[ |D_x (U - u)(x_i, t_j)| \leq CN^{-1} \ln N, \quad x_i \in (\sqrt{\varepsilon} \frac{1}{\beta} \ln \frac{1}{\varepsilon}, 1 - \sqrt{\varepsilon} \frac{1}{\beta} \ln \frac{1}{\varepsilon}). \]  
(38)

**Proof.** From the bounds given in (37), we only need to consider the case where \( N^{-2} \leq \varepsilon \), as then \( \ln \frac{1}{\varepsilon} \leq 2 \ln N \). Moreover, we only need to consider the mesh points in the region
\[ (\sqrt{\varepsilon} \frac{1}{\beta} \ln \frac{1}{\varepsilon}, 2 \sqrt{\varepsilon} \frac{1}{\beta} \ln N) \cup (1 - 2 \sqrt{\varepsilon} \frac{1}{\beta} \ln N, 1 - \sqrt{\varepsilon} \frac{1}{\beta} \ln \frac{1}{\varepsilon}). \]

Let us examine the subregion \( (\sqrt{\varepsilon} \frac{1}{\beta} \ln \frac{1}{\varepsilon}, 2 \sqrt{\varepsilon} \frac{1}{\beta} \ln N) \). Since \( x_i > \sqrt{\varepsilon} \frac{1}{\beta} \ln \frac{1}{\varepsilon} \), there exists some \( \theta > 1 \) such that \( x_i \geq \theta \sqrt{\varepsilon} \frac{1}{\beta} \ln \frac{1}{\varepsilon} \). For \( N \) sufficiently large, we note that
\[ (1 + \rho)^{-1} \leq e^{-\rho/\theta}, \quad \rho \leq \theta \ln \theta, \quad \text{for } \theta > 1. \]

Hence, looking at the barrier function \( P_1 \), it follows in this case that
\[ P_1(x_i) \leq (1 + 0.5 \rho_1)^{-1} \leq e^{-0.5 \rho_1/\theta} \leq \sqrt{\varepsilon}. \]

Using this inequality we establish that, in this particular case, the following estimate for the discrete error flux
\[ |D_x^-(U - u)(x_i, t_j)| \leq CN^{-1} \ln N. \]

\[ \square \]

**Theorem 10.** Assume (36). Then,
\[ \|\bar{U} - u\|_{1,K,G} \leq CN^{-1} \ln N, \]  
(39)
where \( u \) is the solution of (36) and \( \bar{U} \) is the bilinear interpolant of the numerical solution on the piecewise-uniform mesh defined by (33).

**Proof.** Using the decomposition \( u = v + w_L + w_R \) and splitting the argument to inside and outside the computational layer regions, we have
\[ \|u - \bar{u}\|_G \leq C(N^{-1} \ln N)^2. \]

Hence, the following global error estimate follows:
\[ \|u - \bar{U}\|_G \leq CN^{-1} \ln N. \]
Using Theorem \ref{thm:interpolation} we also have
\[
\|\bar{U} - \bar{u}\|_{1,\kappa,G} \leq C N^{-1} \ln N.
\]
The interpolation error \(\|u - \bar{u}\|_{1,\kappa,G}\) is next to be bounded. First, we note that for \(t \in (t_j - 1, t_j]\),
\[
(\bar{u} - u)_t(x, t) = \sum_{i=1}^{N-1} (D^-_i u(x_i, t_j) - u_t(x_i, t)) \phi_i(x) + \sum_{i=1}^{N-1} u_t(x_i, t) \phi_i(x) - u_t(x, t).
\]
Then, in the rectangular cell \(R_{ij}\), it holds that
\[
\|(g - \bar{g})_t\|_{R_{ij}} \leq C \tau \|g_{tt}\|_{R_{ij}} + \min\{h_i \|g_{xt}\|_{R_{ij}}, \|g_t\|_{R_{ij}}\}.
\]
By using the decomposition and splitting the argument to inside and outside the layer region, we obtain
\[
\|(u - \bar{u})_t\|_G \leq C N^{-1}.
\]
Secondly, for \(x \in (x_{i-1}, x_i]\), we have
\[
(\bar{u} - u)_x(x, t) = \sum_{j=1}^{M} (D^-_x u(x_i, t_j) - u_x(x, t_j)) \psi_j(t) + \sum_{j=1}^{M} u_x(x, t_j) \psi_j(t) - u_x(x, t)
\]
Therefore, in the rectangular cell \(R_{ij}\), we obtain
\[
\|(g - \bar{g})_x\|_{R_{ij}} \leq \min\{h_i \|g_{xx}\|_{R_{ij}}, \|g_x\|_{R_{ij}}\} + \min\{\tau \|g_{xt}\|_{R_{ij}}, \|g_x\|_{R_{ij}}\}.
\] (40)
We employ the decomposition \(u = v + w_L + w_R\). The regular component satisfies the following estimate
\[
\|(v - \bar{v})_x\|_{R_{ij}} \leq C N^{-1}.
\]
For the layer component \(w_L\), we split the argument to inside and outside the layer region \((0, \sqrt{\frac{\varepsilon}{\beta}} \ln \frac{1}{\varepsilon}] \times (0, T]\) and deal with the two cases of \(\varepsilon \leq N^{-2}\) and \(\varepsilon \geq N^{-2}\). We observe the following: If \(\varepsilon \leq N^{-2}\) then \(2 \ln N \leq \ln \frac{1}{\varepsilon}\) and in this case we obtain
\[
\|(w_{L,x})_x\|_{R_{ij}} \leq C \sqrt{\varepsilon} \leq C N^{-1}, \quad \text{if } x_i \geq \sqrt{\frac{\varepsilon}{\beta}} \ln \frac{1}{\varepsilon},
\]
\[
\|(w_{L,x})_x\|_{R_{ij}} \leq C \varepsilon^{-1/2} N^{-2}, \quad \text{if } x_i \leq \sqrt{\frac{\varepsilon}{\beta}} \ln \frac{1}{\varepsilon},
\]
\[
h_i \|(w_{L,x})_{xx}\|_{R_{ij}} + \tau \|(w_{L,x})_{xt}\|_{R_{ij}} \leq C \varepsilon^{-1/2} N^{-1} \ln N, \quad \text{if } x_i \leq \hat{\sigma}.
\]
In the second case, where \(\varepsilon \geq N^{-2}\), then
\[
h_i \|(w_{L,x})_{xx}\|_{R_{ij}} + \tau \|(w_{L,x})_{xt}\|_{R_{ij}} \leq C (h_i + \tau \sqrt{\varepsilon}), \quad \text{if } x_i > \sqrt{\frac{\varepsilon}{\beta}} \ln \frac{1}{\varepsilon},
\]
\[
h_i \|(w_{L,x})_{xx}\|_{R_{ij}} + \tau \|(w_{L,x})_{xt}\|_{R_{ij}} \leq C \varepsilon^{-1/2} N^{-1} \ln N, \quad \text{if } x_i \leq \sqrt{\frac{\varepsilon}{\beta}} \ln \frac{1}{\varepsilon}.
\]
Similar estimates are obtained for the right boundary layer component \(w_R\). Combining all these bounds completes the proof. 
\[\square\]
We finish by presenting some numerical results, using the piecewise-uniform Shishkin mesh defined by (33), for the reaction-diffusion test problem:

\[-\varepsilon u_{xx} + (1 + x + t)u + u_t = 4^3 x^3 t^2 (1 - x)^3,\] (41a)

\[u(0, t) = u(1, t) = t^3; \quad u(x, 0) = (4x(1 - x))^3.\] (41b)

The computed solution is displayed in Figure 2 for \(\varepsilon = 2^{-20}\) and \(N = M = 64\). Boundary layers are visible near \(x = 0\) and \(x = 1\).

![Figure 2: Computed solution with the numerical scheme (9) for \(\varepsilon = 2^{-20}\) and \(N = M = 64\)](image)

In Table 2, we present the computed global \(E_{\varepsilon}^{N,M}\) and uniform global \(E^{N,M}\) errors, measured in the global norm \(\| \cdot \|_{1,\kappa,G}\), for \(N = M = 2^j, j = 4, 5, 6, 7, 8, 9\) with their corresponding orders of convergence. The numerical results are in agreement with the bounds given in Theorem 10.

**Acknowledgments**

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**References**


Table 1: Finite difference scheme (9) on the Shishkin mesh: Computed global errors in $\| \cdot \|_{1,\chi,G}$ estimated by $E_{N,M}$ and uniform global errors $E_{N,M}$ with their corresponding computed orders of convergence $Q_{N,M}^{E_{N,M}}$, $Q_{N,M}$ for the test problem [29].

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$E_{N,M}$, $Q_{N,M}$
Table 2: Finite difference scheme [9] on the Shishkin mesh: Computed global errors in \( \| \cdot \|_{1,\kappa,G} \) estimated by \( E_{c}^{N,M} \) and uniform global errors \( E_{\infty}^{N,M} \) with their corresponding computed orders of convergence \( Q_{c}^{N,M} \), \( Q_{\infty}^{N,M} \) for the test problem (41).

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<td>2.475E-001</td>
<td>1.061E-001</td>
<td>1.222</td>
<td>4.785E-002</td>
<td>0.999</td>
<td>3.994E+000</td>
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</tr>
<tr>
<td>( c = 2^{-10} )</td>
<td>3.903E+000</td>
<td>2.297E+000</td>
<td>1.452E+000</td>
<td>7.391E-001</td>
<td>3.473E-001</td>
<td>1.487E-001</td>
<td>1.224</td>
<td>4.785E-002</td>
<td>0.999</td>
<td>3.994E+000</td>
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<td>2.278E+000</td>
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<td>8.593E-001</td>
<td>4.563E-001</td>
<td>2.145E-001</td>
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<tr>
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<td>8.593E-001</td>
<td>4.563E-001</td>
<td>2.145E-001</td>
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<td>4.563E-001</td>
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<td>4.563E-001</td>
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<td>4.785E-002</td>
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