Localized parameter-explicit bounds on the derivatives of the solution to a singularly perturbed two parameter elliptic problem

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Abstract

In this paper, a class of singularly perturbed elliptic partial differential equations posed on a rectangular domain is studied. The differential equation contains two singular perturbation parameters. The solutions of these singularly perturbed problems are decomposed into a sum of regular, boundary layer and corner layer components. Parameter—explicit bounds on the derivatives of each of these components are derived. A numerical algorithm based on an upwind finite difference operator and a tensor product of piecewise-uniform Shishkin meshes is analysed. Parameter—uniform asymptotic error bounds for the numerical approximations are established.

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1 Introduction

Consider the following class of singularly perturbed elliptic problems posed on the unit square $\Omega := (0,1)^2$: find $u(x,y)$ such that

\[
L_{\varepsilon, \mu} u := \varepsilon \Delta u + \mu \vec{a} \cdot \nabla u - bu = f(x,y), \quad (x,y) \in \Omega, \tag{1a}
\]

\[
u = 0, \quad (x,y) \in \partial \Omega, \tag{1b}
\]

\[
\vec{a} := (a_1(x), a_2(y)), \tag{1c}
\]

\[
\mu^2 \geq \frac{\gamma \varepsilon}{\alpha}, \quad \gamma := \min_{\Omega} \left\{ \frac{1}{\frac{1}{\varepsilon^2}, \frac{1}{\mu^2}} \right\}, \quad 0 < \varepsilon \leq 1, \quad 0 < \mu \leq \mu_0 \tag{1g}
\]

where $a_1, a_2, b, f$ are smooth functions and $\mu_0$ is a sufficiently small constant.

For sufficiently compatible and smooth boundary data, there is no loss in generality in assuming zero boundary data. For $u$ to be in $C^{3,\varsigma}(\bar{\Omega})$, $0 \leq \varsigma < 1$ it suffices [4, Theorem 3.2] that $a_1, a_2, b$ are smooth, $f \in C^{1,\varsigma}(\bar{\Omega})$ and $f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0$. Since $a_1 \geq \alpha > 0, a_2 \geq \alpha > 0$, there are no characteristic layers [3] present in the solution. In general, the solution will contain exponential layers [17] with different widths in the vicinity of the sides and the corners of the domain.

Classical numerical methods are inappropriate for singularly perturbed problems [3, 17]. A numerical method is said to be parameter–uniform [3, 17], if an error bound of the form $\|U - u\|_{\Omega_N} \leq CN^{-p}$, $p > 0$, exists, where $U$ is the numerical approximation generated on a mesh $\Omega_N$, $N$ is the number of mesh elements used in each coordinate direction, $\|v\|_{D} := \max_{(x,y) \in D} |v(x,y)|$ is the maximum pointwise norm and, crucially, the error constant $C$ is independent of all perturbation parameters present in the differential equation.

To establish parameter-uniform convergence, we derive pointwise bounds on the derivatives (up to third order) of the solution which explicitly identify how these derivatives depend on the parameters $\varepsilon$ and $\mu$. To obtain this detailed information, we generate a decomposition of the solution into several components. To guarantee that each component is sufficiently smooth for our purposes, we assume the additional compatibility conditions given in (1e), (1f). In the special case of $\mu = 0$, parameter-explicit bounds on a
suitable decomposition of the solution can be obtained assuming no compatibility conditions \( \| \). In the other special case of \( \mu = 1 \), some compatibility conditions (but not as stringent as \( (1c), (1l) \)) are required to obtain suitable bounds on the derivatives of the components in a decomposition of the solution \( 15 \| 10 \). In the general case of arbitrary \( (\varepsilon, \mu) \), the compatibility conditions \( (1c), (1l) \) suffice to construct our solution decomposition. The question of necessary compatibility conditions for a sufficiently smooth decomposition to exist is not addressed here.

We make extensive use of the assumption \( (1c) \) in establishing bounds on derivatives of several components of the solution \( u \), especially in the proof of Lemma 5.1. If one is interested solely in the special case of \( \varepsilon \leq CN^{-1}, \mu^2 \leq CN^{-1} \), then we can replace a significant part (but not all) of the analysis in \( \| 5 \) (and relax the compatibility constraints \( (1c), (1l) \) on the data) without using \( (1c) \) (see Remarks \( 4.1, 5.1 \)). Instead, one could use an asymptotic expansion as an approximation to the solution in the final error analysis in \( 8 \). However, in this paper, we choose not to assume this restriction \( \varepsilon \leq CN^{-1}, \mu^2 \leq CN^{-1} \) on the parameter space. It is worth remarking that the argument establishing the final asymptotic error bound on the discretization errors, given in \( 8 \), relies on the existence of the bounds established in the earlier sections, but not on the assumption \( (1c) \) explicitly.

There are very few papers in the literature dealing with parameter-uniform methods for two parameter singularly perturbed elliptic problems in two dimensions. Li \( 9 \) established a first order parameter-uniform numerical method for a two parameter elliptic problem in two dimensions, where \( a_1 \equiv 0, a_2 \) is a constant and \( \mu^2 \leq C\varepsilon \). Teoferov and Roos \( 20 \) present a decomposition of the solution to a singularly perturbed two parameter elliptic problem where \( a_2 \equiv 0 \) and \( b(x, y) = b(x) \) in \( (1a) \) and in a companion paper Teoferov and Roos \( 21 \) prove that a finite element method on a suitable layer-adapted mesh is uniformly convergent in an energy norm. The fundamental character of the solutions of such singularly perturbed problems are significantly different to the character of the solutions of problems from the class \( (11) \). Shishkin has examined problems involving differential equations of the form \( (1a) \) on bounded \( 15 \) and unbounded \( 19 \) domains, where the convective term is of the form \( (a_1(x, y), \mu a_2(x, y)) \cdot \nabla u \). Note that the fully reduced problem \( (\varepsilon, \mu) = (0, 0) \) in \( 15 \) is different to the fully reduced problem in the present paper. To the best of our knowledge, this paper is the first paper in the literature to study parameter-uniform methods for \( (11) \).

Parameter-uniform numerical methods have been analysed for the one-dimensional two-parameter problem in \( 12, 11, 13 \). In this paper, we extend these ideas to a two parameter problem in two space dimensions. If condition
is replaced by $0 \leq \mu^2 \leq \frac{2\alpha}{\varepsilon}$, then a first order parameter-uniform method was constructed and analysed in [14]. In this paper we require that $\mu_0$ is sufficiently small (so that $4\mu_0(\|a_1\|_2 + \|a_2\|) \leq \beta$, e.g. see proof of Lemma 5.1). This is not a constraint on the data; because if $\mu^2 \geq \frac{2\alpha}{\varepsilon}$ and $\mu \geq \mu_0 > 0$ then the problem is effectively the standard singularly perturbed convection-diffusion elliptic problem (set $\mu = 1$), for which a parameter-uniform method was analysed (for example) in [15]. Hence, this paper in conjunction with [15] [14] completes the study of two-parameter elliptic problems of the form (1) for all values of the parameters in the parameter space $P := \{(\varepsilon, \mu)|0 < \varepsilon \leq 1, 0 \leq \mu \leq 1\}$. In the final section, we present a parameter-uniform numerical method for all values of the parameters within $P$.

In §2, we present some preliminary analysis of problem (1) which yields the bounds

$$\left\| \frac{\partial^k u}{\partial x^i \partial y^j} \right\|_{\infty, \Omega} \leq C \mu^k \varepsilon^{-k}, \quad 0 \leq i + j = k \leq 3.$$  

These bounds crudely identify how the derivatives of the solution depend on the small parameters. However, these global bounds do not identify the localized nature of the large derivatives. In singularly perturbed problems, arbitrary large derivatives only occur in narrow layer regions of the domain. To achieve sharper localized pointwise bounds on the derivatives of the solution, this paper constructs a decomposition of the solution into a sum $u = v + w$ of a regular component $v$ and a layer component $w$. The regular component satisfies $L_{\varepsilon, \mu} v = f$ and the layer component (and any of its subcomponents) satisfy the homogeneous differential equation $L_{\varepsilon, \mu} w = 0$. The boundary values for the regular component $v$ are specified in §4 so that all the partial derivatives up to and including the second order are bounded independently of both perturbation parameters $\varepsilon$ and $\mu$. To identify the appropriate boundary values for the regular component, we first examine in §3 the first order singularly perturbed problem

$$\mu \bar{a} \cdot \nabla z - bz = f, x, y < 1; \quad z(x, 1), z(1, y) \text{ given.}$$

We identify the nature of the boundary layers that can arise in this first order problem near the edges which correspond to the inflow boundaries of the elliptic problem (1).

In §5 - §7, the layer component $w$ is further decomposed into a sum of the form $w = w_L + w_T + w_R + w_B + w_{LT} + w_{BR} + w_{LB}$, where $w_L, w_T, w_R, w_B$ are boundary layer functions associated with one of the edges of the domain and $w_{LT}, w_{BR}, w_{LB}$ are associated with one of the corners. The high order
of compatibility assumed at the inflow corner (1, 1) results in there being no corner layer $w_{RT}$ appearing in the above expansion. Based on this decomposition, a numerical method is constructed in §8 and shown to be essentially a first order parameter-uniform numerical method. In §9, some numerical results are displayed which are consistent with the theoretical error bound established in §8.

**Notation.** We introduce the parameter

$$\tau := \varepsilon (\mu + \sqrt{\varepsilon})^{-1}.$$  

By (1g) we have that $\tau \leq \sqrt{\varepsilon} \leq C\mu$ and $\tau^{-1} \leq C\mu\varepsilon^{-1}$. Here and thoughout $C$ denotes a generic constant that is independent of both singular perturbation parameters $\varepsilon$ and $\mu$. Define the zero, first and second order differential operators $L_0, L_{\mu}$ and $L_{\varepsilon, \mu}$ as follows:

$$L_{\varepsilon, \mu} z := \varepsilon \Delta z + \mu \vec{a} \cdot \nabla z + L_0 z := \varepsilon \Delta z + \mu \vec{a} \cdot \nabla z - b z.$$  

For nonnegative integers $k$, we define the following semi-norms $|\cdot|_{k,D}$ and norms $\|\cdot\|_{k,D}$ on $\mathcal{C}_k(D)$, $D \subset \mathbb{R}^2$ by

$$|v|_{k,D} := \sum_{i+j=k} \sup_{(x,y) \in D} \left| \frac{\partial^k v}{\partial x^i \partial y^j} \right|, \quad \|v\|_{k,D} := \sum_{0 \leq j \leq k} |v|_{j,D}.$$  

If $D = \bar{\Omega}$ we omit the subscript $D$ and if $k = 0$ we omit the subscript $k$. The space $\mathcal{C}_\zeta(D)$ is the set of all functions that are Hölder continuous of degree $\zeta < 1$ with respect to the Euclidean norm $\|\cdot\|_e$. That is if $f \in \mathcal{C}_\zeta(D)$ if

$$[f]_{0,\zeta,D} := \sup_{u \neq v, u, v \in D} \frac{|f(u) - f(v)|}{\|u - v\|_e}$$

is finite. The space $\mathcal{C}^{k,\zeta}(D)$ is the set of all functions in $\mathcal{C}^k(D)$ whose derivatives of order $k$ are Hölder continuous of degree $\zeta$. Also $\|\cdot\|_{k,\zeta,D}$, $[\cdot]_{k,\zeta,D}$ are the associated norms and semi-norms defined by

$$[v]_{k,\zeta,D} := \sum_{i+j=k} \left[ \frac{\partial^k v}{\partial x^i \partial y^j} \right]_{0,\zeta,D}, \quad \|v\|_{k,\zeta,D} := \sum_{0 \leq \zeta \leq k} |v|_{\zeta,D} + [v]_{k,\zeta,D}.$$  

On the boundary $\partial \Omega$, define the following parameter weighted semi-norms

$$|u|_{\star,k,\partial \Omega} := \varepsilon |u|_{k+2,\partial \Omega} + \mu |u|_{k+1,\partial \Omega} + |u|_{k,\partial \Omega},$$

$$[u]_{\star,k,\zeta,\partial \Omega} := \varepsilon [u]_{k+2,\zeta,\partial \Omega} + \mu [u]_{k+1,\zeta,\partial \Omega} + [u]_{k,\zeta,\partial \Omega}$$

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where, for example,
\[ |u|_{\partial \Omega} := \max_{x \in [0,1]} \left( |(u(x,0))'| + |(u(x,1))'| + \max_{y \in [0,1]} |(u(y,0))'| + |(u(y,1))'| \right). \]

If \( D \) is the domain of some function \( f \) and \( D \subset D^* \), then throughout we denote the extension of the function to the larger domain by \( f^* \). Certain functions \( f \) will be extended in different ways in the paper, but for notational convenience we will simply denote all such extensions by \( f^* \).

2 Preliminary a priori bounds on the solution

In this section we establish preliminary a priori parameter explicit bounds on the solution of (1) and its derivatives. We begin by stating a continuous minimum principle for the differential operator in (1), whose proof is standard. Throughout this section, we only require the following non-negativity constraints on the coefficients.

\[ a_1 \geq \alpha_1 \geq 0, \quad a_2 \geq \alpha_2 \geq 0, \quad b \geq \beta > 0. \]  \hspace{1cm} (2)

Lemma 2.1. \([10]\) Assume (2). If \( w \in C^2(\Omega) \cap C^0(\bar{\Omega}) \) such that \( L_{\varepsilon, \mu} w |_{\Omega} \leq 0 \) and \( w |_{\partial \Omega} \geq 0 \), then \( w |_{\bar{\Omega}} \geq 0 \).

Consider the following singularly perturbed problems: find \( z(x,y) \) s.t.

\[ L_{\varepsilon, \mu} z = p(x,y; \mu), \quad (x,y) \in (0,1)^2, \]  \hspace{1cm} (3a)
\[ z(x,0) = s_1(x), \quad z(x,1) = s_2(x), \quad 0 \leq x \leq 1, \]  \hspace{1cm} (3b)
\[ z(0,y) = q_1(y), \quad z(1,y) = q_2(y), \quad 0 \leq y \leq 1, \]  \hspace{1cm} (3c)

where the data is sufficiently smooth and compatible so that \( z \in C^3(\bar{\Omega}) \).

Lemma 2.2. Assume (2), then the solution \( z \) of (3) satisfies

\[ ||z|| \leq ||z||_{\partial \Omega} + C||p||; \]
\[ ||z|| \leq ||z||_{\partial \Omega} + C||p||; \]
\[ |z|_1 + \tau^k |z|_1,\kappa \leq C \tau^{-1} ||z|| + C \tau \varepsilon^{-1} \left( ||p||_0 + ||z||_{0,\kappa,\bar{\Omega}} + \tau^{l}(||p||_{0,\kappa} + ||z||_{0,\kappa,\bar{\Omega}}) \right), \]

and for \( l = 0, 1 \)

\[ |z|_{l+2} + \tau^k |z|_{l+2,\kappa} \leq C \tau^{-2} ||z|| + C \tau^{-1} \varepsilon^{-1} \sum_{k=0}^{l} \tau^k \left( ||p||_k + ||z||_{k,\kappa,\partial \Omega} \right) + C \varepsilon^{-1} \tau^k \left( ||p||_{l,\kappa} + ||z||_{l,\kappa,\partial \Omega} \right). \]
Proof. Since \( b \geq \beta > 0 \), the first bound on the solution follows directly from the comparison principle. Note that the zero-order compatibility conditions for problem (3) are \( s_1(0) = q_1(0); q_1(1) = s_2(0); s_2(1) = q_2(1) \) and \( s_1(1) = q_2(0) \). Hence if \( h(x,y) = (s_1(x) - s_1(0)(1-x))(1-y) + (s_2(x) - s_2(1)x)y + (q_1(y) - q_1(1)y)(1-x) + (q_2(y) - q_2(0)(1-y))x \) and \( \omega = z - h \), then \( \omega \) is the solution of

\[
L_{\varepsilon,c}\omega = p - L_{\varepsilon,c}h =: \tilde{p}, \text{ on } \Omega, \quad \omega = 0, \text{ on } \partial\Omega.
\]  

Consider the transformation \( \xi = x/\tau \) and \( \eta = y/\tau \). The stretched domain \( \tilde{\Omega} \) is given by \( \tilde{\Omega} = (0, 1/\tau)^2 \). Applying this transformation, (3) now becomes

\[
\bar{\omega}_{\xi\xi} + \bar{\omega}_{\eta\eta} + \tau \mu \bar{\omega}^{-1} (\hat{a}_1 \bar{\omega}_{\xi} + \hat{a}_2 \bar{\omega}_{\eta}) - \tau \omega^{-1} \tilde{b} \bar{\omega} = \tilde{p}, \quad \text{on } \tilde{\Omega},
\]

where \( \bar{\omega}(\xi, \eta) := \omega(x,y) \), the transformed coefficients \( \hat{a}_1, \hat{a}_2, \tilde{b} \) are defined similarly and \( \tilde{p}(\xi, \eta) := \tau^2 \omega^{-1} \tilde{p}(x, y) \). For each \((\zeta_1, \zeta_2) \in \tilde{\Omega} \) and each \( \delta > 0 \), we denote the rectangle \((\zeta_1 - \delta, \zeta_2 + \delta)^2 \cap \tilde{\Omega} \) by \( R_\delta(\zeta_1, \zeta_2) \). Using [8, page 110] we see that for all \((\xi, \eta) \in \tilde{\Omega} \) and any \( R_\delta \) we have that

\[
||\bar{\omega}||_{l+2,\kappa,R_\delta} \leq C(||\tilde{p}\|_{l,\kappa,R_\delta} + ||\bar{\omega}||_{R_{2\delta}}), \quad \text{for } l = 0, 1.
\]

Transforming back to the original variables and following the arguments in [10], we deduce that for all \((x, y) \in \Omega \) and \( R_\delta = R_\delta(x, y) \)

\[
|\omega|_{1,R_\delta} + \tau^\kappa [\omega]_{1,\kappa,R_\delta} \leq C \tau^{-1} ||\omega||_{R_{2\delta}} + C \tau \varepsilon^{-1} \left( ||\tilde{p}|_{0,R_{2\delta}} + \tau^\kappa [\tilde{p}]_{0,\kappa,R_{2\delta}} \right),
\]

and for \( l = 0, 1 \)

\[
|\omega|_{l+2,R_\delta} + \tau^\kappa [\omega]_{l+2,\kappa,R_\delta} \leq C \tau^{-(l+2)} ||\omega||_{R_{2\delta}} + C \varepsilon^{-1} \left( \sum_{k=0}^{l} \tau^k ||\tilde{p}|_{k,R_{2\delta}} + \tau^\kappa [\tilde{p}]_{l,\kappa,R_{2\delta}} \right).
\]

Replacing \( \tilde{p} \) by \( p - L_{\varepsilon,c}h \) and using the definition of \( h \) gives us

\[
||\tilde{p}|_{k,R_{2\delta}} = |p|_{k,R_{2\delta}} + |z|_{*,k,R_{2\delta} \cap \partial\Omega}.
\]

Since \( \Omega \) can be covered by the neighbourhoods \( R_\delta \) of a finite number of points and noting that \( z = \omega + h \), the result follows.

\[ \square \]

**Corollary 2.1.** Assume (2), \( p \in C^{2,\xi}(\bar{\Omega}) \), \( z = 0, (x, y) \in \partial\Omega \) and \( |p|_{n} \leq C \mu^{-n}, n = 0, 1, 2 \). Then the solution \( z \) of (3) satisfies the following bounds

\[
|z|_{n} + \tau^\kappa |z|_{n,\kappa} \leq C \tau^{-n}, \quad n = 0, 1, 2, 3.
\]
Proof. From the previous lemma \(|z| \leq C\). Moreover, using the inequality 
\(|\tilde{p}|_0, \varsigma \leq C|\tilde{p}|_1\) in the stretched variables, we have that
\[
\tau(|z|, R, \delta + \tau \varsigma |z| \leq C \left( \|z\|_2^2 + \varepsilon^{-1} \sum_{\nu=0}^{l} \tau \nu |p|_{\nu, R} \right)
\]
and for \(l = 0, 1\)
\[
\tau^{l+2}(|z|, R, \delta + \tau \varsigma |z| \leq C \left( \|z\|_2^2 + \varepsilon^{-1} \sum_{\nu=0}^{l+1} \tau \nu |p|_{\nu, R} \right).
\]
Use the bounds on \(p\) to deduce that 
\(\tau v |p|_{v, R} \leq C\).

Corollary 2.2. The solution \(u\) of (1) satisfies the following bounds
\(|u| + \tau \varsigma |u| \leq C \tau^{-n}, \ n = 0, 1, 2, 3\).

3 Singularly perturbed first order problem

In this section we discuss the singularly perturbed problem associated with formally setting \(\varepsilon = 0\) in (1a). We establish the following comparison principle for the first order differential operator \(L_\mu\). Note \(b \geq 0\) is not used in the proof of Lemma 3.1; but we do use \(a_1 + a_2 > 0\).

Lemma 3.1. Assume \(a_1, a_2 \geq 0, a_1 + a_2 > 0\). and let \(\Omega_1 = [0, 1]^2\). If
\[z \in C^1(\Omega_1) \cap C^0(\Omega_1), L_\mu z \geq 0 \text{ and } z \geq 0, \text{ then } z \geq 0.\]

Proof. Let \(z = e^{-\varepsilon^2/2}w\), where \(\kappa\) is chosen such that \((a_1 + a_2)\kappa + b > 0, \forall (x, y) \in \Omega_1\). Assume that \(\min_{\Omega_1} z < 0\) and let \((x_0, y_0)\) be the point where 
\(w(x_0, y_0) = \min_{\Omega_1} w < 0\). At this point, \(w_x(x_0, y_0) \geq 0\) and \(w_y(x_0, y_0) \geq 0\). Hence \(L_\mu z(x_0, y_0) > 0\), which is a contradiction.

Consider the first order problem: find \(r(x, y)\) such that
\[
\begin{align*}
a_1 r_x + a_2 r_y - br &= f_0, \quad (x, y) \in D = (0, L) \times (0, L), \quad (5a) \\
r(x, 0) &= g_x(x), \quad 0 \leq x \leq 1; \quad r(0, y) = g_w(y), \quad 0 < y \leq 1; \quad (5b) \\
a_1, a_2, f_0 &\in C^{m, \gamma}(D), \quad a_1 > 0, \ a_2 > 0, \ (x, y) \in D. \quad (5c)
\end{align*}
\]
Bobisud [2] gives explicit compatibility and regularity conditions so that 
\(r \in C^2(D)\) and indicates how to derive necessary and sufficient conditions so
Lemma 3.2. Assume (6) in order to simplify the analysis in later sections. Less stringent necessary and sufficient regularity conditions are given in \[10\], but we assume (6) in order to simplify the analysis in later sections.

**Proof.** Use (5) to establish the regularity of \(z\) and use Lemma 3.1 to obtain the required bound on \(z\). Differentiate (7a) with respect to \(x\) to obtain

\[
L_\mu^1 z_x := \mu a_1(x)z_x + \mu a_2(z_x) - b_1(z_x) = p_1,
\]

(8)

where \(z_x(1, 1) = 0\) and using (7a) we have \(|z_x(1, y)| \leq (a_1(1))^{-1}|p(1, y)|\). To bound \(\|z_x\|\), use the barrier function

\[
\psi(x, y) = \frac{|p(1, y)|}{a_1(1)\mu} + \frac{1}{\beta} (\|p_x\| + \|b_x\| |z||) e^{A(1-x)} e^{A(1-y)},
\]

where \(A\) is chosen sufficiently large so that

\[L_\mu^1 \psi \pm z_x = \psi (-\mu ((a_1 + a_2)A - (a_1)x) - b) \pm p_1 \leq 0.\]
Repeating the above argument, but this time, differentiate \(7a\) with respect to \(y\) to obtain the bound
\[
\|z_y\| \leq \frac{|p(x,1)|}{a_2(1)\mu} + \frac{1}{\beta} \left(\|p_y\| + \|b_y\|\|z\|\right).
\]

Differentiate \(7a\) \(l\) times with respect to \(x\) to obtain
\[
L_z^{[l]} v_l := \mu a_1 (v_l)_x + \mu a_2 (v_l)_y - b_l v_l = p_l(x,y), \quad v_l := \frac{\partial^l z}{\partial x^l},
\]
\[
b_l := b - l\mu (a_1)_x, \quad l \geq 0; \quad p_l(x,y) := \frac{\partial^l p}{\partial x^l} + \sum_{k=0}^{l-1} \frac{\partial^k p}{\partial x^k} \frac{\partial^{l-k} z}{\partial x^{l-k}} \left(\sum_{i=0}^{k} c_{k,i} b_i\right), \quad l \geq 1,
\]
for some specified constants \(c_{k,i}\). Note that \(p_l(x,y)\) involves \(p\) and its derivatives with respect to \(x\) up to order \(l\), \(z\) and its derivatives with respect to \(x\) up to order \(l - 1\) and the coefficients and their derivatives. We see that \(v_l(x,1) = 0\) and \(v_l(1,y) = \phi_l(y)\). Using the differential equation \(7a\), we can show that
\[
|\phi_l(y)| \leq C \mu^{-l} \sum_{r+s=0}^{l-1} \mu^{r+s} \left|\frac{\partial^{r+s} p}{\partial x^r \partial y^s}(1,y)\right|.
\]

As above we can then deduce that
\[
\|v_l\| \leq C \left(\|z\| + \left\|\frac{\partial^l p}{\partial x^l}\right\| + \frac{1}{\mu^l} \sum_{r+s=0}^{l-1} \mu^{r+s} \left|\frac{\partial^{r+s} p}{\partial x^r \partial y^s}\right|\right).
\]

The bounds on all the \(y\)-derivatives can be established in an analogous fashion. Note that by using \(8\) we have the following bounds on the boundary
\[
\mu \|z_{xy}(x,1)\| \leq C \|p_x\| \quad \text{and} \quad \mu \|z_{xy}(1,y)\| \leq C \|p_y\|.
\]

Differentiate \(8\) with respect to \(y\). Repeating the maximum principle argument, one gets
\[
\|z_{xy}\| \leq C ((\|z\| + \|p_{xy}\| + \mu^{-1} \|p_x\| + \mu^{-1} \|p_y\| + \mu^{-1} \|p\|).
\]

Finish the proof for the bounds on the remaining mixed derivatives using an inductive argument. \(\square\)
Corollary 3.1. Assume (1c) and that $z$ satisfies the first order problem

\begin{align}
L_\mu z &= 0, \quad (x, y) \in [0, 1)^2, \tag{9a} \\
z(x, 1) &= 0; \quad z(1, y) = s(y), \quad 0 \leq x, y < 1; \tag{9b}
\end{align}

$s \in C^{n+1}(0,1]$ and $\frac{d^j s}{dy^j}(1) = 0, \; j \leq n$, \tag{9c}

then $z \in C^{n,\varsigma}(\overline{\Omega})$ and for $0 \leq k + m \leq n$

$$
\left| \frac{\partial^{k+m} z}{\partial x^k \partial y^m}(x, y) \right| \leq C \mu^{-k}(1-y)^{n+1-k-m}e^{-2\gamma(1-x)/\mu},
$$

Proof. Use (3) to establish the regularity of $z$ and use Lemma 3.1 to bound $|z(x,y)|$. Follow the argument from the proof of Lemma 3.2 with $p \equiv 0$. To obtain the desired bound on $z_y$ (and then $z_x$), differentiate (9a) with respect to $y$, note that

$$
|z_y(1,y)| = |s'(y)| \leq C(1-y)^n, \; z_y(x,1) = 0
$$

and use the barrier function

$$C(1-y)^n e^{A(1-y)} e^{-2\gamma(1-x)/\mu},$$

where $A$ is chosen to be sufficiently large. Differentiate (9a) $l$ times with respect to $y$, note that $|\hat{v}_l(1,y)| \leq C(1-y)^{n+1-l}$, $\hat{v}_l(x,1) = 0$, $\hat{v}_l := \frac{\partial^l z}{\partial y^l}$, and use induction to obtain the bounds on $|\hat{v}_l(1,y)|$. Complete the proof on the mixed derivatives by noting that $\mu|z_{xy}(1,y)| \leq C\mu|s''(y)| + C|s'(y)| + C|s(y)|$

and by differentiating (9a) $m$ times with respect to $y$ and $k$ times with respect to $x$. \hfill \Box

Corollary 3.2. Assume (1c) and that $z$ satisfies

\begin{align}
L_\mu z &= p(x, y; \mu), \quad (x, y) \in [0, 1)^2; \quad z(x, 1) = z(1, y) = 0, \quad 0 \leq x, y \leq 1; \\
p \in C^{n,\varsigma}(\overline{D}), \quad \frac{\partial^{k+m} p}{\partial x^k \partial y^m}(1,1) = 0, \; k + m \leq n - 1, \quad \text{and}
\end{align}

$$
\left| \frac{\partial^{k+m} p}{\partial x^k \partial y^m}(x, y) \right| \leq C \mu^{-k}(1-y)^{n+1-k-m}e^{-\gamma(1-x)/\mu}, \; k + m \leq n,
$$

then $z \in C^{n,\varsigma}(\overline{\Omega})$ and for $0 \leq k + m \leq n$

$$
\left| \frac{\partial^{k+m} z}{\partial x^k \partial y^m}(x, y) \right| \leq C \mu^{-k}(1-y)^{n+1-k-m}e^{-\gamma(1-x)/\mu}.
$$
Proof. Combine the arguments from Lemma 3.2 and Corollary 3.1.

Remark 3.1. In the proofs of Lemma 3.2, Corollary 3.1 and Corollary 3.2 we make use of assumption (1c). However, by using a parameter explicit integral representation of the exact solution (akin to the closed form representation of the exact solution to a first order problem given in [2] pg.396) it may be possible to establish the results in this section for the more general case of \((a_1(x,y), a_2(x,y))\). As the proofs in later sections do require assumption (1c), we have chosen to use a proof in this section which is close in spirit to the proofs of the later sections of the paper.

4 Regular component

The regular component of the solution \(u\) to (1a), (1b) will be constructed so that all the derivatives up to second order of the regular component are bounded independently of both small parameters. In order to minimize the imposition of artificial compatibility conditions, we consider the extended region \(\Omega_1^* = [-d, 1] \times [-d, 1],\ d > 0.\) The differential operators \(L_{\varepsilon,\mu}^*\) and \(L_0^*\) coincide with the operators \(L_{\varepsilon,\mu}\) and \(L_0\) respectively in \(\Omega\). We can also construct smooth extensions \(a_1^*, a_2^*, b^*\) and \(f^*\) of the functions \(a_1, a_2, b\) and \(f\) to \(\Omega_1^*\) so that \(a_1^*, a_2^* \geq \alpha/2 > 0,\ b^* \geq \beta/2 > 0,\ (x, y) \in \Omega_1^*\).

Consider the differential equation

\[
L_{\varepsilon,\mu}^* v^* = f^*, \quad (x, y) \in (-d, 1) \times (-d, 1). \tag{10}
\]

Decompose \(v^*\) as follows,

\[
v^*(x, y; \varepsilon, \mu) = (v_0^* + \varepsilon v_1^*)(x, y; \mu) + \varepsilon^2 v_2^*(x, y; \varepsilon, \mu) \tag{11a}
\]

where \(v_0^*(x, y, \mu)\) is further decomposed via the expansion

\[
v_0^*(x, y; \mu) = (s_0^* + \mu s_1^*)(x, y) + \mu^2 s_2^*(x, y; \mu) \tag{11b}
\]

and

\[
L_0^* s_0^* = f^*, \quad \mu L_0^* s_1^* = (L_0^* - L_{\varepsilon,\mu}^*) s_0^*, \text{ on } \Omega_1^*, \tag{11c}
\]

\[
\mu^2 L_{\varepsilon,\mu}^* s_2^* = \mu (L_0^* - L_{\varepsilon,\mu}^*) s_1^*, \text{ on } \Omega_1^*, \ s_2^*|_{\partial \Omega_1^*} = 0. \tag{11d}
\]

The appropriate boundary conditions for \(v^*\) are determined from

\[
L_{\varepsilon,\mu}^* v_0^* = f^*, \text{ on } \Omega_1^*, \ v_0^*|_{\partial \Omega_1^*} = s_0^* + \mu s_1^*, \tag{11e}
\]

\[
\varepsilon L_{\varepsilon,\mu}^* v_1^* = (L_{\varepsilon,\mu}^* - L_{\varepsilon,\mu}^*) v_0^*, \text{ on } \Omega_1^*, \ v_1^*|_{\partial \Omega_1^*} = 0, \tag{11f}
\]

\[
\varepsilon^2 L_{\varepsilon,\mu}^* v_2^* = \varepsilon (L_{\varepsilon,\mu}^* - L_{\varepsilon,\mu}^*) v_1^*, \text{ on } (-d, 1) \times (-d, 1) \tag{11g}
\]

\[
v_2^*|_{\bar{\Omega}_1^\text{\textbackslash}(-d, 1) \times (-d, 1)} = 0. \tag{11h}
\]
Observe that $v^*_0$ and $v^*_1$ are solutions of singularly perturbed first order differential equations, but $v^*_2$ satisfies an elliptic problem. The bounds in §3 will be used to bound $v^*_0, v^*_1$ and the bounds in §2 will be used to bound $v^*_2$.

**Lemma 4.1.** Define the regular component $v$ to be the solution of

$$L_{\varepsilon, \mu} v = f, \quad (x, y) \in \Omega, \quad v = v^*, \quad (x, y) \in \partial \Omega$$

(12)

where $v^*$ is defined in the decomposition (11). Then the regular component of (1) satisfies the following bounds

$$|v|_n + \tau^\kappa |v|_{n, \kappa} \leq C \left( 1 + \varepsilon^{2-n} \mu^{n-2} \right), \quad n = 0, 1, 2, 3.$$  

(13)

**Proof.** The compatibility assumptions (11) ensure that $s^*_2 \in C^{6,\kappa}(\bar{\Omega}^*_1)$ and the function $s^*_2$ satisfies a similar equation to $z$ in (14). Note that $s^*_0, s^*_1$ satisfy algebraic equations and thus their derivatives are bounded independently of $\mu$. We can apply Lemma 3.2 to obtain

$$\|s^*_2\|_{\Omega^*_1} \leq C|s^*_1|_{1, \Omega^*_1} \leq C,$$

and

$$\frac{\partial^n s^*_2}{\partial x^n \partial y^m}(1, 1) = 0; \quad \|s^*_2\|_{n, \Omega^*_1} \leq C\mu^{-n} \quad 0 \leq n \leq 6.$$  

(14)

It follows by (11b) that $v^*_0 \in C^{6,\kappa}(\bar{\Omega}^*_1)$ and

$$\frac{\partial^n v^*_0}{\partial x^n \partial y^m}(1, 1) = 0; \quad \|v^*_0\|_{n, \Omega^*_1} \leq C(1 + \mu^{2-n}), \quad 0 \leq n \leq 6.$$  

Since $v^*_1$ satisfies a similar equation to $z$ in (14), we can again apply Lemma 3.2 and the bounds above to obtain $v^*_1 \in C^{4,\kappa}(\bar{\Omega}^*_1)$

$$\|v^*_1\|_{\Omega^*_1} \leq C\|\Delta v^*_0\|_{\Omega^*_1} \leq C,$$

and for $1 \leq k + m = n \leq 4$

$$\frac{\partial^n v^*_1}{\partial x^n \partial y^m}(1, 1) = 0, \quad \|v^*_1\|_{n, \Omega^*_1} \leq C \left( \|v^*_1\|_{\Omega^*_1} + |\Delta v^*_0|_{n, \Omega^*_1} + \sum_{p=0}^{n-1} \mu^{p-n} |\Delta v^*_0|_{p, \Omega^*_1} \right).$$  

It follows that

$$|v^*_1|_{n, \Omega^*_1} \leq C\mu^{-n}, \quad 0 \leq n \leq 4.$$  

Since $v^*_2$ satisfies a similar elliptic equation to $u$, we obtain from (14) that $v^*_2 \in C^{4,\kappa}(\bar{\Omega}^*_1)$. By Lemma 2.2

$$\|v^*_2\|_{\Omega^*_1} \leq C\|\Delta v^*_0\|_{\Omega^*_1} \leq C\mu^{-2}.$$
Using the stretched variables $\eta = x/\tau$ and $\zeta = y/\tau$, we deduce from the proof of Lemma 3.2 that

$$|\tilde{v}_0^* + \varepsilon v_1^*|_{n,\tilde{\Omega}^*_1} \leq C\tau^n(1 + \mu^{2-n}), \quad n = 0, 1, 2, 3, 4;$$

where $\tilde{v}_0^*(\eta, \zeta) = v_0^*(x, y)$. Since $\tau \leq C\mu$, this implies that $[\tilde{v}_0^* + \varepsilon v_1^*]_{n,\tilde{\Omega}^*_1} \leq C\tau^n(1 + \mu^{2-n})$ for $n = 1, 2, 3$. Then returning to the original variables, we get that

$$\tau^n (|v_0^* + \varepsilon v_1^*|_n + \tau^\varepsilon [v_0^* + \varepsilon v_1^*]_{n,\Omega^*_1}) \leq C\tau^n(1 + \mu^{2-n}), \quad n = 0, 1, 2, 3.$$ 

Finally we use Corollary 2.1 to obtain,

$$|v_2^*|_{n,\Omega^*_1} + \tau^\varepsilon [v_2^*]_{n,\Omega^*_1} \leq C\mu^{-2}\tau^{-n}, \quad n = 0, 1, 2, 3.$$ 

Combining all these bounds with (15) completes the proof.

Remark 4.1. An asymptotic expansion $v_{asp}$ for the regular component $v$ is given by $v_{asp} = s_0^* + \mu s_1^* + \varepsilon R_1^* + \mu^2 R_2^*$, where $s_0^*, s_1^*$ are defined in (11c). The remainder terms satisfy $R_1^* = R_2^* = 0, (x, y) \in \partial\Omega^*_1$ and

$$L_{\varepsilon, \mu} R_1^* = -\Delta (s_0^* + \mu s_1^*), \quad L_{\varepsilon, \mu} R_2^* = \vec{a} \cdot \nabla s_1^*, \quad (x, y) \in \Omega^*_1.$$

Without using (1c), one can show that $\|R_1^*\| \leq C, \|R_2^*\| \leq C$.

5 Inflow boundary layer components

In this section, we define the boundary layer function $w_R$ (and $w_T$) associated with the edge $x = 1$ (and $y = 1$). To specify appropriate boundary conditions at the outflow edges, we decompose the boundary layer function $w_R$ as the regular component $v$ was decomposed. That is, define

$$w_R^*(x, y; \varepsilon, \mu) = (w_0^* + \varepsilon w_1^*) (x, y; \mu) + \varepsilon^2 w_2^*(x, y; \varepsilon, \mu), \quad (15)$$

where $v(1, y) = \left( \frac{\ell}{\ell} - \left( \frac{\ell}{\ell} \right) a \cdot \nabla(\frac{\ell}{\ell}) \right)(1, y)$ is given in (11b) and

$$L_{\mu}^* w_0^* = 0, \quad \text{on } \Omega^*_1, \quad w_0^*(x, 1) = 0, \quad w_0^*(1, y) = -v^*(1, y), \quad \varepsilon L_{\mu}^* w_1^* = \left( L_{\mu}^* - L_{\varepsilon, \mu}^* \right) w_0^*, \quad \text{on } \Omega^*_1, \quad w_1^*(x, 1) = w_1^*(1, y) = 0.$$

Note that we are free to choose the extensions in §4 so that $v^*(1, y) \equiv 0, \forall y \leq -d/4$. From the compatibility conditions (11) we recall that

$$\frac{\partial^j v}{\partial y^j}(1, 1) = 0, \quad 0 \leq j \leq 6,$$
which, by \([13]\), suffices for \(w_0^* \in C^{6,\varsigma}(\tilde{\Omega}_1^*)\) and \(w_1^* \in C^{4,\varsigma}(\tilde{\Omega}_1^*)\). To obtain bounds on \(w_R^*\), the extensions of \(b^*, a_1^*, a_2^*\) are constructed so that for all points in \(\tilde{\Omega}_1^*\), there exists a sufficiently large constant \(A\) such that

\[
Aa_1^* - 6(a_1^*)_x \geq 0, \quad Aa_2^* - 6(a_2^*)_y \geq 0, \quad (16a)
\]

\[
3a_1^* \gamma \leq 2b^*, \quad 3a_2^* \gamma \leq 2b^*, \quad (16b)
\]

\[
a_1^* \geq \alpha / 2 > 0, \quad a_2^*(y) \geq \delta(d + y), \quad \delta > 0, \quad b^* \geq \beta / 2 > 0, \quad (16c)
\]

and also

\[
\frac{a_2^*}{a_1^*} \equiv m > 0, \quad (x, y) \in [-d, 1] \times [-d/2, -d/4], \quad (16d)
\]

\[
v^*(1, y) \equiv 0, \quad b^* \equiv \beta / 2, \quad (x, y) \in [-d, 1] \times [-d, -d/4]. \quad (16f)
\]

Compare these conditions on the extended region to the corresponding conditions \([14]\) on the original domain \(\Omega\). By a suitable, sufficiently small choice for the constant slope \(m\), we can also have that \(w_0^*, w_1^*\), which are solutions of first order problems, satisfy

\[
w_0^*(x, y) \equiv 0, \quad w_1^*(x, y) \equiv 0, \quad (x, y) \in \bar{R}_d = [-d, 1] \times [-d, -d/2] \subset \tilde{\Omega}_1^*.
\]

Note that by this choice of \([15c-e]\) the results stated in Corollaries 3.1 and 3.2 remain valid on the extended region \(\tilde{\Omega}_1^*\). Hence, we can deduce the following bounds on the first two components in the expansion of \(w_R^*\). For all \((x, y) \in \tilde{\Omega}_1^*\),

\[
\left| \frac{\partial^n w_0^*}{\partial x^k \partial y^m}(x, y) \right| \leq C \mu^{-k}(1 - y)^{7-n} e^{-\gamma(1-x)\mu}, \quad 0 \leq n \leq 6, \quad (17a)
\]

\[
\left| \frac{\partial^n w_1^*}{\partial x^k \partial y^m}(x, y) \right| \leq C \mu^{-k-2}(1 - y)^{5-n} e^{-\gamma(1-x)\mu}, \quad 0 \leq n \leq 4. \quad (17b)
\]

Define the function \(g^* := -\Delta w_1^* \in C^{2,\varsigma}(\tilde{\Omega}_1^*)\). Then

\[
\left| \frac{\partial^n g^*}{\partial x^k \partial y^m}(x, y) \right| \leq C \mu^{-k-4}(1 - y)^{3-n} e^{-\gamma(1-x)\mu}, \quad 0 \leq n \leq 2 \quad (18a)
\]

and from the extensions in \([16]\), we get that

\[
\frac{\partial^n g^*}{\partial x^k \partial y^m}(x, y) = 0, \quad 0 \leq n \leq 3, \quad -d \leq y \leq -3d/4. \quad (18b)
\]

Define the final correction \(w_2^* \in C^{3,\varsigma}(\tilde{\Omega}_1^*)\) to \(w_R^*\) as the solution of

\[
L_{x,y}^* w_2^* = g^*, \quad (x, y) \in (-d, 1) \times (-d, 1), \quad w_2^*|_{\Omega_1^* \setminus (-d, 1) \times (-d, 1)} = 0. \quad (19)
\]
Lemma 5.1. Assume (1c). When $w_2^*$ is defined as in (19), then

$$|w_2^*(x,y)| \leq C \mu^{-4}(1-y)(y+d)e^{-\frac{\gamma}{\mu}(1-\tau)},$$

$$|w_2^*|_n + \tau^\epsilon [w_2^*]_{n,\varsigma} \leq C \mu^{-4}\epsilon^{-n}, \quad n = 1, 2, 3.$$

Moreover, in the direction orthogonal to the layer, for $\mu$ sufficiently small,

$$\|(w_2^*)_y\| \leq C \mu^{-4}, \quad \|(w_2^*)_yy\| \leq C \mu^{-3}\epsilon^{-1}, \quad \|(w_2^*)_yyy\|_{\Omega_1^*_{(-d,1)^2}} \leq C \mu^{-2}\epsilon^{-2}.$$

Proof. See appendix.

The function $w_R^*$ constructed above is the solution of

$$L_{\epsilon,\mu}^* w_R^*(x,y) = 0, \quad (x,y) \in (-d,1)^2, \quad (20a)$$

$$w_R^*(1,y) = -v^*(1,y), \quad w_R^*(-d,y) = (w_0^* + \epsilon w_1^*)(-d,y), \quad y \in [-d,1], \quad (20b)$$

$$w_R^*(x,1) = 0, \quad w_R^*(x,-d) = (w_0^* + \epsilon w_1^*)(x,-d), \quad x \in [-d,1]. \quad (20c)$$

From the compatibility conditions (11) we have that

$$\frac{\partial^k v^*}{\partial x^j \partial y^j}(1,1) = 0, \quad 0 \leq k \leq 6,$$

which is more than sufficient for $w_R^* \in C^3(\Omega_1^*)$.

Lemma 5.2. The right layer function $w_R^*$ defined in (20) satisfies

$$|w_R^*(x,y)| \leq C (1-y)(y+d)e^{-\frac{\gamma}{\mu}(1-\tau)},$$

and its derivatives satisfy the following bounds

$$|w_R^*|_n + \tau^\epsilon [w_R^*]_{n,\varsigma} \leq C \mu^{-n}, \quad n = 1, 2, \quad |w_R^*|_3 + \tau^\epsilon [w_R^*]_{3,\varsigma} \leq C \mu^{-1}\epsilon^{-1};$$

$$\left|\frac{\partial^j w_R^*}{\partial y^j}\right| \leq C(1 + \mu^{1-j}), \quad j = 1, 2, 3.$$

Proof. Using the stretched variables $\xi = x/\tau$ and $\eta = y/\tau$, we deduce from (17) that

$$|\tilde{w}_0^* + \epsilon \tilde{w}_1^*|_n \leq C \tau^n \mu^{-n}, \quad n = 0, 1, 2, 3, 4;$$

which implies that $|\tilde{w}_0^* + \epsilon \tilde{w}_1^*|_{n,\varsigma} \leq C \tau^n \mu^{-n}$ for $n = 1, 2, 3$, since $\tau \leq C \mu$.

Returning to the original variables, we get that

$$\tau^n (|w_0^* + \epsilon w_1^*|_n + \tau^\epsilon [w_0^* + \epsilon w_1^*]_{n,\varsigma}) \leq C \tau^n \mu^{-n}.$$
Lemma 5.3. The function $w^*_T$ defined in (21), satisfies

$|w^*_T(x,y)| \leq C(1-x)(x+d)e^{-\frac{\gamma}{d}(1-y)}$,

$|w^*_T|_n + \tau^i |w^*_T|_{n,i} \leq C\mu^{-n}$, $n = 1, 2$, $|w^*_T|_3 + \tau^i |w^*_T|_{3,i} \leq C\varepsilon^{-1}\mu^{-1}$,

$|\frac{\partial^i w^*_T}{\partial x^i}| \leq C(1 + \mu^{1-i})$, $i = 1, 2, 3$. 

Based on the bounds on the components in the expansion of $w^*_R$ given in (19), all the bounds follow immediately, except for the sharper bounds on $(w^*_R)_{yy}$ and $(w^*_R)_y(0,y)$. On the boundary,

$$\left|\frac{\partial^3 w^*_R}{\partial y^3}(x,y)\right|_{\Omega^*_1 \setminus (-d,1)^2} \leq C(1 + \mu^{-2}).$$

Differentiate (20b) three times with respect to $y$ and use the maximum principle to obtain the bound on $(w^*_R)_{yyy}$.

Define $w^*_T$, the boundary layer function associated with the top edge $y = 1$, as the solution of

$$L^*_{\varepsilon,\mu} w^*_T = 0, \quad (x,y) \in (-d,1)^2, \quad (21a)$$

$$w^*_T(x,y;\varepsilon,\mu) = (\tilde{w}^*_0 + \varepsilon\tilde{w}^*_1)(x,y;\mu) + \varepsilon^2 \tilde{w}^*_2(x,y;\varepsilon,\mu), \quad (21b)$$

where $v(x,1) = v_0(x,1) = \left(\frac{1}{T} - \left(\frac{y}{T}\right) a \cdot \nabla(\frac{1}{T})\right)(x,1)$ is given in (11b) and

$$L^*_{\varepsilon,\mu} \tilde{w}^*_0 = 0 \text{ on } \Omega^*_1, \quad \tilde{w}^*_0(1,1) = 0, \quad \tilde{w}^*_0(x,1) = -v^*(x,1), \quad (21c)$$

$$\varepsilon L^*_{\varepsilon,\mu} \tilde{w}^*_1 = (L^*_{\varepsilon,\mu} - L^*_{\varepsilon,\mu})\tilde{w}^*_0 \text{ on } \Omega^*_1, \quad \tilde{w}^*_1(1,1) = \tilde{w}^*_1(x,1) = 0, \quad (21d)$$

$$\varepsilon^2 L^*_{\varepsilon,\mu} \tilde{w}^*_2 = (\varepsilon(L^*_{\varepsilon,\mu} - L^*_{\varepsilon,\mu})\tilde{w}^*_1), \quad (x,y) \in (-d,1)^2, \quad \tilde{w}^*_2 \big|_{\Omega^*_1 \setminus (-d,1)^2} = 0. \quad (21e)$$

To obtain bounds on $w^*_T$, the extensions of $b^*, a^*_1, a^*_2$ are constructed so that at all points in $\Omega^*_1$ we have for sufficiently large $A$

$$A a^*_2 - 6(a^*_2)_y \geq 0, \quad A a^*_1 - 6(a^*_1)_x \geq 0, \quad (22a)$$

$$3a^*_2 \gamma \leq 2b^*, \quad 3a^*_1 \gamma \leq 2b^*, \quad a^*_1 \geq \delta(x+d), \quad a^*_2 \geq \alpha/2, \quad b^* \geq \beta/2 > 0, \quad (22b)$$

and also $

\frac{a^*_2}{a^*_1} \equiv m > 0, \quad (x,y) \in [-d/2,-d/4] \times [-d,1] \quad (22c)$

$$a^*_1(x) \geq \alpha/2, \quad x \geq -d/2, \quad a^*_1(-d) = 0 \quad (22d)$$

$$v^*(x,1) \equiv 0, \quad b^* \equiv \beta/2, \quad (x,y) \in [-d,-d/4] \times [-d,1]. \quad (22e)$$

Note that these extensions are not as in (10). Using the extensions (22), we deduce the following result.

Lemma 5.3. The function $w^*_T$ defined in (21), satisfies
Define the boundary layer function \( w_R \) (and \( w_T \)) associated with the right edge \( x = 1 \) (top edge \( y = 1 \)) by

\[
L_{\epsilon,\mu} w_R = 0, \quad (x, y) \in \Omega, \quad w_R = w^*_R, \quad (x, y) \in \partial \Omega. \tag{23a}
\]
\[
L_{\epsilon,\mu} w_T = 0, \quad (x, y) \in \Omega, \quad w_T = w^*_T, \quad (x, y) \in \partial \Omega. \tag{23b}
\]

\textbf{Remark 5.1.} An asymptotic expansion \( w^*_{R,asp} \) for the boundary layer component \( w^*_R \) is given by

\[
w^*_{R,asp} = w^*_{0,asp} + \epsilon w^*_{1,asp} + \epsilon^2 w^*_{2,asp} + \mu^2 R^*_3 + \epsilon R^*_4,
\]

where all subcomponents in this expansion are set to zero on relevant boundaries except that

\[
w^*_{0,asp}(1, y) = -v^*(1, y). \quad \text{Also}
\]

\[
w^*_{0,asp} := w^*_{0,0} + \mu w^*_{0,1}, \quad w^*_{1,asp} := w^*_{1,0} + \mu w^*_{1,1}
\]
\[
M_{\mu} w^*_{0,0} := (\mu a_1 \frac{\partial}{\partial x} - b)w^*_{0,0} = 0, \quad w^*_{0,0}(1, y) = -v^*(1, y)
\]
\[
M_{\mu} w^*_{0,1} = -a_2 \frac{\partial w^*_{0,0}}{\partial y}, \quad M_{\mu} w^*_{1,0} = -\frac{\partial^2 w^*_{0,asp}}{\partial x^2},
\]
\[
M_{\mu} w^*_{1,1} = -a_2 \frac{\partial w^*_{1,0}}{\partial y}, \quad L_{\epsilon,\mu} w^*_{2,asp} = -\frac{\partial^2 w^*_{1,asp}}{\partial x^2}.
\]

\[
L_{\epsilon,\mu} R^*_3 = -a_2 \frac{\partial}{\partial y}(w^*_{0,1} + \epsilon w^*_{1,1}), \quad L_{\epsilon,\mu} R^*_4 = -\frac{\partial^2}{\partial y^2}(w^*_{0,asp} + \epsilon w^*_{1,asp}).
\]

Without using (1c) one can show that \( \|R^*_3\| \leq C, \|R^*_4\| \leq C \) for \( \epsilon \leq C \mu^2 \); but (1c) is still needed in a proof of the bounds on \( w^*_{2,asp} \) in Lemma 5.1.

\section{6 Corner layers within the flow}

As \( a_1, a_2 > 0 \), we view the flow as entering through either of the edges \( x = 1 \) or \( y = 1 \) and exiting via \( x = 0 \) or \( y = 0 \). By assuming sufficient compatibility at the inflow corner (1,0), we ensure that there is no corner layer associated with this corner. In this section, we consider \( w_{BR} \) (and \( w_{LT} \)), the boundary layer function associated with the corner (1,0) (and (0,1)). Define \( w^*_BR \) to be the solution of

\[
L^*_{\epsilon,\mu} w^*_{BR} = 0, \quad (x, y) \in (-d, 1) \times (0, 1), \tag{24a}
\]
\[
w^*_{BR}(x, 0) = (-w_R)^*(x, 0), \quad w^*_{BR}(x, 1) = 0, \quad x \in [-d, 1], \tag{24b}
\]
\[
w^*_{BR}(-d, y) = w^*_{BR}(1, y) = 0, \quad y \in [0, 1]. \tag{24c}
\]
The boundary value \( w^*_R(x, 0) \) is constructed so that \( w^*_R(x, 0) = w^*_R(x, 0), \) \( 0 \leq x \leq 1 \) and \( w^*_R(x, 0) \) is a smooth extension of \( w^*_R(x, 0) \) to the interval \( -d \leq x \leq 1 \) so that \( w^*_R(x, 0) \equiv 0, \) \( -d \leq x \leq -d/2. \) Using the compatibility conditions (1c), one can also deduce that

\[
\frac{\partial^i w^*_R}{\partial x^i} (1, 0) = 0, \quad i \leq 2
\]

which suffices for \( w^*_R \in C^{3, \kappa}([-d, 1] \times [0, 1]). \)

**Lemma 6.1.** The corner layer function \( w^*_{BR} \) defined in (24), satisfy

\[
|w^*_{BR}(x, y)| \leq C e^{-\frac{2\alpha}{\nu} (1-x)} e^{-\frac{\alpha y}{2}} e^{-\frac{\alpha y}{2}}
\]

\[
|w^*_{BR}| + \tau^i [w^*_{BR}]_{i, \kappa} \leq C \tau^{-i}, \quad i = 1, 2, 3,
\]

\[
\left| \frac{\partial w^*_{BR}}{\partial x} \right| \leq C \mu^{-1}, \quad \left| \frac{\partial^2 w^*_{BR}}{\partial x^2} \right| \leq C \varepsilon^{-1}, \quad \left| \frac{\partial^3 w^*_{BR}}{\partial x^3} \right| \leq C \varepsilon^{-2} \mu.
\]

**Proof.** Use the extensions specified in (10) and from (10b, c) in particular we have that

\[
L^*_{i, \mu} e^{-\frac{2\alpha}{\nu} (1-x)} e^{-\frac{\alpha y}{2}} e^{-\frac{\alpha y}{2}} \leq 0.
\]

Using the bounds in Lemma 6.2 we deduce that the weighted semi-norms on the edge \( x = 0 \) are bounded as follows

\[
|w^*_{BR}(x, 0)|_{s, l, [-d, 1]} + \tau^i [w^*_{BR}(x, 0)]_{s, l, [-d, 1]} \leq C \mu^{-l}, \quad l = 0, 1.
\]

Using Lemma 2.2 one can deduce that

\[
|w^*_{BR}| + \tau^i [w^*_{BR}]_{i, \kappa} \leq C \tau^{-1} + C \tau \varepsilon^{-1} \leq C \tau^{-1},
\]

\[
|w^*_{BR}|^2 + \tau^i [w^*_{BR}]_{2, \kappa} \leq C \tau^{-2} + C \varepsilon^{-1} \leq C \tau^{-2},
\]

\[
|w^*_{BR}|^3 + \tau^i [w^*_{BR}]_{3, \kappa} \leq C \tau^{-3} + C \tau^{-1} \varepsilon^{-1} (1 + \tau \mu^{-1}) \leq C \tau^{-3}.
\]

Since \( w^*_{BR}(1, y) = w^*_{BR}(-d, y) = 0 \) and \( |(w^*_{BR}(x, 0))_2 \| \leq C \mu^{-1}, \) we get that \( |w^*_{BR}| \leq C \mu^{-1} (1 - x)(d + x). \) Combining this with the differential equation (24), one can further deduce that

\[
\left| \frac{\partial w^*_{BR}(1, y)}{\partial x} \right| \leq C \mu^{-1}, \quad \left| \frac{\partial w^*_{BR}(-d, y)}{\partial x} \right| \leq C \mu^{-1},
\]

\[
\left| \frac{\partial^2 w^*_{BR}(1, y)}{\partial x^2} \right| \leq C \varepsilon^{-1}, \quad \left| \frac{\partial^2 w^*_{BR}(-d, y)}{\partial x^2} \right| \leq C \varepsilon^{-1}.
\]
In this section, we complete the decomposition of 7 Outflow layer components with three components.

Assuming that $\mu$ is sufficiently small, the bounds on the first two $x$-derivatives follow using the same argument as in the proof of Lemma 5.1. Introduce the auxiliary function

$$\zeta_1 = \varepsilon (w_{BR}^*)_{xx} + a_1^\mu(x) \mu (w_{BR}^*)_x.$$  

Using (24), it follows that $\zeta_1(1, y) = \zeta_1(-d, y) = \zeta_1(x, 1) = 0$ and

$$|\zeta_1(x, 0)| \leq C(1 - x)(\varepsilon \|(w_{BR}^*)_{xxx}\| + \mu \|(w_{BR}^*)_{xx}\|) \leq C(1 - x) \mu^{-1}.$$  

We also have that

$$L_{\varepsilon, \mu}^* \zeta_1 = (\pm b_{xx}^\varepsilon + \pm a_1^\mu b_{xx}^\varepsilon) w_{BR}^* + 2\varepsilon \mu b_{xx}^\varepsilon (w_{BR}^*)_x =: g_1.$$  

Then $|L_{\varepsilon, \mu}^* \zeta_1| \leq C \mu^{-1}(1 - x)$ and so $|\zeta_1| \leq C \mu^{-1}(1 - x)$. This, in turn, implies that $|(|\zeta_1|)_{x}(1, y)| \leq C \mu^{-1}$, which will yield $|(w_{BR}^*)_{xxx}(1, y)| \leq C \mu^{-2}$. Noting the extensions specified in (16) and from (16a) in particular we have that $g_1(-d, y) = 0$. Using a similar argument to above we can also deduce that $|(w_{BR}^*)_{xxx}(-d, y)| \leq C \mu^{-1} \varepsilon^{-1}$. Note further that $(w_{BR}^*)_{xxx}(x, 1) = 0$ and $|(w_{BR}^*)_{xxx}(x, 0)| \leq C \mu^{-1} \varepsilon^{-1}$. Complete the argument by differentiating the differential equation (24) three times with respect to the $x$-variable and use the comparison principle.

Analogous bounds hold for the other corner layer function $w_{LT}^*$. The boundary layer function $w_{BR}$ ($w_{LT}$) associated with the corner $(1, 0)$ (and $(0, 1)$) is defined by

$$L_{\varepsilon, \mu} w_{BR} = 0, \quad (x, y) \in \Omega; \quad w_{BR} = w_{BR}^*, \quad (x, y) \in \partial \Omega. \quad (25a)$$

$$L_{\varepsilon, \mu} w_{LT} = 0, \quad (x, y) \in \Omega; \quad w_{LT} = w_{LT}^*, \quad (x, y) \in \partial \Omega. \quad (25b)$$

### 7 Outflow layer components

In this section, we complete the decomposition of $u$ by defining the remaining three components $w_L$, $w_B$, and $w_{LB}$ associated respectively with the edge $x = 0$, the edge $y = 0$ and the corner $(0, 0)$. We define $w_L^*$ as the solution of

$$L_{\varepsilon, \mu}^* w_L^* = 0, \quad (x, y) \in (0, 1) \times (-d, 1), \quad (26a)$$

$$w_L^*(0, y) = -(v + w_R)^*(0, y), \quad w_L^*(1, y) = 0, \quad y \in [-d, 1], \quad (26b)$$

$$w_L^*(x, -d) = w_L^*(x, 1) = 0, \quad x \in [0, 1]. \quad (26c)$$
The extensions are constructed so that \( w^*_L(0, -d) = 0 \) and from the compatibility conditions (1e) we deduce that

\[
\frac{\partial^k v^*}{\partial x^i \partial y^j}(0, 1) = 0, \quad 0 \leq k \leq 4,
\]

which is sufficient for \( w^*_L \in C^3([0, 1] \times [-d, 1]) \). From the bounds on the derivatives of \( w^*_R \) given in Lemma 5.2, we have that for \( 1 \leq j \leq 3 \)

\[
\left| \frac{\partial^j w^*_R}{\partial y^j}(0, y) \right| \leq C(1 + \mu^{1-j}).
\]

**Lemma 7.1.** When \( w^*_L \) is defined as in (26) then

\[
|w^*_L(x, y)| \leq Ce^{-\frac{\mu}{2\varepsilon}x}, \quad |w^*_L|_n + \tau^i[w^*_L|_{n, \varepsilon}] \leq C\tau^{-n}, \quad n = 1, 2, 3,
\]

\[
\left| \frac{\partial^i w^*_L}{\partial x^i} \right| \leq C(1 + \left(\frac{\mu}{\varepsilon}\right)^{i-1}), \quad i = 1, 2, 3.
\]

**Proof.** Note that, using the extensions in (16) we have that

\[
L^*_\varepsilon,\mu = \left(\frac{\mu^2}{2\varepsilon} - a_1^* - b^*\right)te^{-\frac{\mu}{2\varepsilon}x}.
\]

Note that

\[
|w^*_L(0, y)|_{w, \ell, [-d, 1]} + \tau^i[w^*_L(0, y)]_{n, \varepsilon} \leq C\mu^{-l}, \quad l = 0, 1
\]

and as in the proof of Lemma 6.1 we have \( |w^*_L|_n + \tau^i[w^*_L|_{n, \varepsilon}] \leq C\mu^n\varepsilon^{-n} \) for \( n = 1, 2, 3 \). In the direction orthogonal to the layer we sharpen these bounds by noting that \( |w^*_L(x, y)| \leq C(1 - y)(y + d) \) and then follow the argument in the proof of Lemma 6.1. 

The layer component \( w^*_B \) is defined analogously, as the solution of

\[
L^*_\varepsilon,\mu w^*_B = 0, \quad (x, y) \in (-d, 1) \times (0, 1), \quad (27a)
\]

\[
w^*_B(x, 0) = -(v + w_T)^*(x, 0), \quad w^*_B(x, 1) = 0, \quad x \in [-d, 1], \quad (27b)
\]

\[
w^*_B(-d, y) = w^*_B(1, y) = 0, \quad y \in [0, 1]; \quad (27c)
\]

where \(-(v + w_T)(x, 0)\) is extended to \( x \in [-d, 1] \) so that sufficient compatibility conditions are satisfied.

**Lemma 7.2.** When \( w^*_B \) is defined as in (27) we have the bounds

\[
|w^*_B(x, y)| \leq Ce^{-\frac{\mu}{2\varepsilon}y}, \quad |w^*_B|_n + \tau^i[w^*_B|_{n, \varepsilon}] \leq C\tau^{-n}, \quad n = 1, 2, 3,
\]

\[
\left| \frac{\partial^i w^*_B}{\partial x^i} \right| \leq C(1 + \left(\frac{\mu}{\varepsilon}\right)^{i-1}), \quad i = 1, 2, 3.
\]

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The boundary layer function \( w_L (w_B) \) associated with the left edge \( x = 0 \) (bottom edge \( y = 0 \)) is the solution of

\[
L_{\varepsilon, \mu} w_L = 0, \quad (x, y) \in \Omega; \quad w_L = w_L^*, \quad (x, y) \in \partial \Omega.
\] (28a)

\[
L_{\varepsilon, \mu} w_B = 0, \quad (x, y) \in \Omega; \quad w_B = w_B^*, \quad (x, y) \in \partial \Omega.
\] (28b)

Finally we consider the outflow corner layer function \( w_{LB} \) associated with the corner \((0, 0)\). We define \( w_{LB} \) to be the solution of

\[
L_{\varepsilon, \mu} w_{LB} = 0, \quad (x, y) \in \Omega; \quad w_{LB}(0, y) = -(w_B + w_{BR})(0, y), \quad w_{LB}(1, y) = 0, \quad 0 \leq y \leq 1,
\] (29a)

\[
w_{LB}(x, 0) = -(w_L + w_{LT})(x, 0), \quad w_{LB}(x, 1) = 0, \quad 0 \leq x \leq 1.
\] (29b)

**Lemma 7.3.** We have the following bounds on the solution of (29).

\[
|w_{LB}(x, y)| \leq C e^{-\frac{\mu}{2\varepsilon}(x+y)}, \quad |w_{LB}|_n \leq C(1 + \tau^{-n}) \quad \text{for } n = 1, 2, 3.
\]

**Proof.** Note that

\[
L_{\varepsilon, \mu} \left( e^{-\frac{\mu}{2\varepsilon}(x+y)} \right) = e^{-\frac{\mu}{2\varepsilon}(x+y)} \left( \frac{\alpha^2}{4\varepsilon} (\alpha - 2a_1 + \alpha - 2a_2) - b \right) \leq 0.
\]

Complete the proof using arguments from Lemma 7.1. \( \square \)

### 8 Discrete problem

Consider the following discrete problem: find a mesh function \( U \) such that

\[
L_{N,M}^U = f, \quad (x_i, y_j) \in \Omega_{N,M}, \quad U = u, \quad (x_i, y_j) \in \partial \Omega_{N,M},
\] (30a)

\[
L_{N,M}^U := \varepsilon \delta_x^2 U + \varepsilon \delta_y^2 U + \mu a_1 D_x^+ U + \mu a_2 D_y^+ U - b U,
\] (30b)

where the mesh \( \Omega_{N,M} = \Omega_N \times \Omega_M \) is defined to be the tensor product of two piecewise–uniform Shishkin meshes \( \Omega_N \) and \( \Omega_M \). Here \( D^+, \delta^2 \) denote the standard forward finite difference operator and the central difference operator, respectively. The mesh points \( x_i \in \Omega_N \) are given by

\[
x_i := \begin{cases} 
\frac{4\sigma_1^N i}{N}, & \frac{N}{4} \leq i \leq \frac{N}{2} \\
\sigma_1^N + (i - \frac{N}{4}) H, & \frac{N}{2} \leq i \leq \frac{3N}{4} \\
1 - \sigma_2^N + (i - \frac{3N}{4}) \frac{4\sigma_2^N}{N}, & \frac{3N}{4} \leq i \leq N
\end{cases}
\] (30c)
where the two transition points are defined by

\[
\sigma^N_1 := \min \left\{ \frac{1}{4}, \frac{2\varepsilon}{\mu \alpha} \ln N \right\} \quad \text{and} \quad \sigma^N_2 := \min \left\{ \frac{1}{4}, \frac{2\mu}{\gamma} \ln N \right\}.
\] (30d)

Here \(NH = 2(1 - \sigma^N_1 - \sigma^N_2)\) and \(\Omega^M\) is defined analogously with transition points \(\sigma^M_1\) and \(\sigma^M_2\).

A discrete minimum principle holds for the upwind finite difference operator \(L^{N,M}\). We have the following decomposition of the discrete solution

\[
U = V + W_L + W_R + W_B + W_T + W_{LT} + W_{BT} + W_{LB},
\] (31a)

where

\[
L^{N,M}V = f, \quad (x_i, y_j) \in \Omega^{N,M}, \quad V|_{\partial \Omega^{N,M}} = v|_{\partial \Omega^{N,M}},
\] (31b)

\[
L^{N,M}W_L = 0, \quad (x_i, y_j) \in \Omega^{N,M}, \quad W_L|_{\partial \Omega^{N,M}} = w_L|_{\partial \Omega^{N,M}},
\] (31c)

\[
L^{N,M}W_{LB} = 0, \quad (x_i, y_j) \in \Omega^{N,M}, \quad W_{LB}|_{\partial \Omega^{N,M}} = w_{LB}|_{\partial \Omega^{N,M}},
\] (31d)

with the other layer functions defined similarly.

**Theorem 8.1.** We have the following bounds on the discrete boundary layer functions,

\[
|W_L(x_i, y_j)| \leq C \prod_{s=1}^{i} \left( 1 + \frac{\mu \alpha}{2\varepsilon} h_s \right)^{-1} =: \Psi_L(x_i), \quad \Psi_L(0) = C,
\]

\[
|W_R(x_i, y_j)| \leq C \prod_{s=i+1}^{N} \left( 1 + \frac{\gamma}{2\mu} h_s \right)^{-1} =: \Psi_R(x_i), \quad \Psi_R(1) = C,
\]

\[
|W_B(x_i, y_j)| \leq C \prod_{r=1}^{j} \left( 1 + \frac{\mu \alpha}{2\varepsilon} k_r \right)^{-1} =: \Psi_B(y_j), \quad \Psi_B(0) = C,
\]

\[
|W_T(x_i, y_j)| \leq C \prod_{r=j+1}^{M} \left( 1 + \frac{\gamma}{2\mu} k_r \right)^{-1} =: \Psi_T(y_j), \quad \Psi_T(1) = C,
\]

where \(h_s = x_s - x_{s-1}\) and \(k_r = y_r - y_{r-1}\).

**Proof.** We begin by bounding \(W_L\). By Lemma \[\text{2.1}\] \(|W_L(x_i, 0)| \leq \Psi_L(x_i)\). Note that \(L^{N,M} \Psi_L(x_i) \leq 0\) as

\[
L^{N,M} \Psi_L(x_i) = \left( \frac{\mu^2 \alpha^2}{2\varepsilon} \left( \frac{h_{i+1}}{2h_i} - 1 \right) + \left( \frac{\mu^2 \alpha(\alpha - a_1)}{2\varepsilon} - b \right) - \frac{\mu \alpha}{2\varepsilon} b h_{i+1} b \right) \Psi_L(x_{i+1}).
\]

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Use the discrete minimum principle to obtain the required bound. The proof to bound \( W_B \) is analogous. Using the exponential bounds in Lemma 5.2 we see that \(|W_R(0, y)| = |w_R(0, y)| \leq C e^{-\frac{\gamma}{\mu}} \leq \Psi_R(0)\). To obtain the bound on \( W_R \), combine this with the fact that, since \( \gamma \varepsilon \leq \alpha \mu^2 \),

\[
L^{N,M}\Psi_R(x_i) = \Psi_R(x_{i-1}) \left( 2\varepsilon \left( \frac{\gamma}{2\mu} \right)^2 \left( \frac{h_i}{2h_{i-1}} - 1 \right) + \left( 2\varepsilon \left( \frac{\gamma}{2\mu} \right)^2 h_i \right) \right) \leq 0.
\]

The proof to derive the bound on \( W_T \) is analogous.

We have the following bounds on the discrete corner layer functions

\[
|W_{LB}(x_i, y_j)| \leq C\Psi_L(x_i)\Psi_B(y_j), \quad |W_{LT}(x_i, y_j)| \leq C\Psi_L(x_i)\Psi_T(y_j), \quad |W_{BR}(x_i, y_j)| \leq C\Psi_R(x_i)\Psi_B(y_j).
\]

**Theorem 8.2.** We have the following pointwise error bound

\[
\|U - u\|_{\Omega^{N,M}} \leq C(N^{-1} + M^{-1})(\ln N + \ln M)^2,
\]

where \( C \) is a constant independent of \( \varepsilon, \mu, N \) and \( M \).

**Proof.** Using the usual truncation error argument at each mesh point \((x_i, y_j)\), we have that

\[
|L^{N,M}(U - u)| \leq CN^{-1}(\varepsilon\|u_{xxx}\| + \mu\|u_{xx}\|) + CM^{-1}(\varepsilon\|u_{yyy}\| + \mu\|u_{yy}\|).
\]

(32)

Note that if \( \sigma^N_1 = 1/4 \) then \( \mu/\varepsilon \leq C \ln N \). From the truncation error bound (32) and the bounds in Corollary 2.1 it follows that

\[
\|U - u\|_{\Omega^{N,M}} \leq C\mu(N^{-1} + M^{-1})(\ln N)^2, \quad \text{if} \quad \sigma^N_1 = 1/4. \quad (33a)
\]

Hence, from now on we focus on the case of \( \sigma^N_1 < 1/4 \) and \( \sigma^M_1 < 1/4 \). Also, note that when \( \sigma^N_2 = 1/4 \) then \( 1/\mu \leq C \ln N \). In the case of the regular component, by (32) and (13) it follows that

\[
\|V - v\|_{\Omega^{N,M}} \leq C\mu(N^{-1} + M^{-1}), \quad \text{if} \quad \sigma^N_1 < 1/4 \quad (33b)
\]

where \( v \) is the solution of (12), and \( V \) is the solution of (31b). Let us now examine the errors in each of the layer components.
Use (32) and the bounds in Lemma 7.2 to obtain
\[ |L^{N,M}(W_R - w_R)(x_i, y_j)| \leq C_1\mu^{-1}(h_{i+1} + h_i) + C_2(k_{j+1} + k_j). \]
Consider first the case of \( \sigma_2^N < 1/4 \). By the bounds in Theorem 8.4 and Lemma 5.2, we have the following bound in the region \((0, 1 - \sigma_2^N) \times (0, 1)\)
\[ |(W_R - w_R)(x_i, y_j)| \leq |W_R(x_i, y_j)| + |w_R(x_i, y_j)| \leq CN^{-1}, \quad \text{when} \quad \sigma_2^N < 1/4. \]
In the fine mesh region \((1 - \sigma_2^N, 1) \times (0, 1)\), use (32) to get
\[ |L^{N,M}(W_R - w_R)| \leq C_1N^{-1}\ln N + C_2M^{-1}, \quad \sigma_2^N < 1/4. \]
This bound holds throughout \( \Omega^{N,M} \) when \( \sigma_2^N = \frac{1}{4} \). Using the discrete comparison principle, we conclude that
\[ \|W_R - w_R\|_{\Omega^{N,M}} \leq C(N^{-1}\ln N + M^{-1}). \quad (33c) \]
From Lemma 5.1 and the bound (32) we obtain
\[ |L^{N,M}(W_{LT} - w_{LT})(x_i, y_j)| \leq C\mu^3\varepsilon^{-2}(h_{i+1} + h_i) + C\mu\varepsilon^{-1}(k_{j+1} + k_j). \]
From the bounds on both the discrete and continuous corner layer function,
\[ |(W_{LT} - w_{LT})(x_i, y_j)| \leq CN^{-1}, \quad (x_i, y_j) \in \Omega^{N,M}\setminus(0, \sigma_1^N) \times (1 - \sigma_2^M, 1). \]
In the corner region \((0, \sigma_1^N) \times (1 - \sigma_2^M, 1)\) we obtain
\[ |L^{N,M}(W_{LT} - w_{LT})(x_i, y_j)| \leq C\mu^2\varepsilon^{-1}(N^{-1}\ln N + M^{-1}\ln M). \]
Using the barrier function \(\mu\varepsilon^{-1}(\sigma_1^N - x_i)\), it follows that
\[ \|W_{LT} - w_{LT}\|_{\Omega^{N,M}} \leq C(N^{-1}\ln N + M^{-1}\ln M)\ln N. \quad (33d) \]
Let us now consider the error in one of the outflow boundary layer components \(W_L\). Use Lemma 7.1 and (32) to deduce that
\[ |L^{N,M}(W_L - w_L)(x_i, y_j)| \leq C\mu^3\varepsilon^{-2}(h_{i+1} + h_i) + C\mu^2\varepsilon^{-1}M^{-1}. \]
In the outer region \([\sigma_1^N, 1) \times (0, 1)\),
\[ |(W_L - w_L)(x_i, y_j)| \leq CN^{-1}, \quad x_i \geq \sigma_1^N, \quad \text{when} \quad \sigma_1^N < 1/4. \]
In the fine mesh region \((0, \sigma_1^N) \times (0, 1)\), use the fine mesh step to obtain
\[ |L^{N,M}(W_L - w_L)(x_i, y_j)| \leq C\mu^2\varepsilon^{-1}(N^{-1}\ln N + M^{-1}). \]

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Use the barrier function $\mu \epsilon^{-1}(\sigma^N - x_i)$ to deduce that
$$\|W_L - w_L\|_{\Omega^{N,M}} \leq C(N^{-1} \ln N + M^{-1}) \ln N. \quad (33e)$$

In analogous fashion we can also conclude that
$$\|W_B - w_B\|_{\Omega^{N,M}} \leq C(N^{-1} + M^{-1} \ln M) \ln M, \quad (33f)$$
$$\|W_T - w_T\|_{\Omega^{N,M}} \leq C(N^{-1} + M^{-1} \ln M) \ln M \quad (33g)$$
$$\|W_{BR} - w_{BR}\|_{\Omega^{N,M}} \leq C(N^{-1} + M^{-1} \ln M) \ln M + N^{-1} \ln N. \quad (33h)$$

Finally we examine the component associated with the outflow corner. From the bounds on both the discrete and continuous corner layer function,
$$|(W_{LB} - w_{LB})(x_i, y_j)| \leq CN^{-1}M^{-1}, \quad (x_i, y_j) \in \Omega^{N,M} \setminus (0, \sigma^N) \times (0, \sigma^M),$$

Consider the corner region $(0, \sigma^N) \times (0, \sigma^M)$, where
$$|L^{N,M}(W_{LB} - w_{LB})(x_i, y_j)| \leq C\mu^2 \epsilon^{-1}(N^{-1} \ln N + M^{-1} \ln M).$$

Using a suitable barrier function, it follows that
$$\|W_{LB} - w_{LB}\|_{\Omega^{N,M}} \leq C \left( N^{-1} \ln N + M^{-1} \ln M \right) (\ln N + \ln M). \quad (33i)$$

\[ \square \]

**Remark 8.3.** Consider the singularly perturbed first order problem
$$\mu \alpha \cdot \nabla u - bu = f(x, y), \quad (x, y) \in G := (0, 1)^2,$$
$$u = 0, \quad (x, y) \in \bar{G} \setminus G, \quad \frac{\partial^j f}{\partial x^i \partial y^j}(0, 0) = 0, \quad i + j \leq 1,$$
and (1c), (1d). The solution can be decomposed via
$$u = v + w_L + w_B + w_{LB}, \quad \text{where}$$
$$L_\mu f^i (x, y) \in G, \quad v = v^i (x, y) \in \bar{G} \setminus G,$$
$$L_\mu w_L = 0, \quad (x, y) \in G, \quad w_L(0, y) = -v, \quad |w_L(x, 0)| \leq C e^{-\alpha x/\mu}$$
$$L_\mu w_B = 0, \quad (x, y) \in G, \quad w_B(x, 0) = -v, \quad |w_B(0, y)| \leq C e^{-a y/\mu}$$
$$L_\mu w_{LB} = 0, \quad (x, y) \in G, \quad w_{LB}(x, 0) = -w_L(x, 0), \quad |w_{LB}(0, y)| = w_B(0, y).$$

Appropriate $\mu$-explicit bounds on the first two derivatives of these components can be deduced using the techniques in earlier sections. A fitted mesh for this problem can be a tensor product of two piecewise-uniform meshes on the domain $[0, \tau], [\tau, 1]$ where $\tau := \min\left\{ \frac{1}{2}, \frac{2}{3} \mu \ln N \right\}$. The difference scheme is $\mu a_D x U + \mu a_2 D_y U - bU = f$. Using the same techniques as above, one can easily deduce that $\|U - u\| \leq CN^{-1} \ln N.$
9 Numerical results

Combining the theoretical result in this paper with the result in [14], one now has a numerical method that is parameter-uniform for all values of \((\varepsilon, \mu) \in (0, 1) \times (0, 1]\). The upwind scheme (30) is employed on a tensor-product Shishkin mesh (30c) with \(N = M\) (for convenience). The transition points in both directions are defined to be

\[
\sigma_1^N := \sigma_2^N := \min \left\{ \frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma}} \ln N \right\}, \quad \text{if} \quad \alpha \mu^2 \leq \gamma \varepsilon,
\]

\[
\sigma_1^N := \frac{1}{4}, \quad \sigma_2^N := \min \left\{ \frac{1}{4}, \frac{2\varepsilon}{\mu} \ln N \right\}, \quad \text{if} \quad \alpha \mu^2 \geq \gamma \varepsilon.
\]

Consider the following particular test problem:

\[
\varepsilon \Delta u + \mu (1 + x, 1 + y) \cdot \nabla u - u = 2x(x - 1) + 2y(y - 1), \quad u = 0, \quad \text{on } \partial \Omega. \quad (34)
\]

Figures 1 and 2 display surface and contour plots of the numerical solution with \(N = 32\), for different values of \(\varepsilon\) and \(\mu\).

![Figure 1: Numerical solutions of example (34) when \(\alpha \mu^2 \leq \gamma \varepsilon\).](image)

The uniform order of convergence is estimated using the double mesh principle [3]. Define the double mesh differences to be

\[
D_N^\mu := \max_{\mu \in R_\mu} D_N^\mu, \quad D^\varepsilon := \max_{\varepsilon \in R_\varepsilon} D_N^\varepsilon, \quad D_{\varepsilon, \mu}^N := \max_{\Omega^N} \left| U^N - \hat{U}^{2N} \right|,
\]

where \(\hat{U}^{2N}\) is the numerical solution on a mesh containing the mesh points of the original Shishkin mesh \(\Omega^N\) and its mid-points \(((x_{i+1} + x_i)/2, y_{j+1} + \)
Figure 2: Numerical solutions of example (34) when $\alpha \mu^2 \geq \gamma \varepsilon$.

$y_j/2)$. Here the range of the singular perturbation parameters over which the numerical performance of the schemes was tested was taken to be

$$R_\varepsilon := \{\varepsilon = 2^0, 2^{-1}, \ldots, 2^{-30}\}, \quad R_\mu := \{\mu = 2^0, 2^{-1}, \ldots, 2^{-30}\}.$$  

From these quantities the orders of convergence $p^N, p^N_\mu$ are computed from

$$p^N := \log_2 \left( \frac{D^N}{D^{2N}} \right) \quad \text{and} \quad p^N_\mu := \log_2 \left( \frac{D^N_\mu}{D^{2N}_\mu} \right).$$

In Table 1 we display these orders for the above test problem. These computed orders are in line with the theoretical order of convergence $N^{-1} (\ln N)^2$. In this example, the constraint $[1R]$ is satisfied for $\mu \geq 2^{-16}$. As $\mu \to 0$, the orders of convergence are tending towards $(N^{-1} \ln N)^2$, which is the rate expected $[1]$ when dealing with a reaction-diffusion problem (where $\mu = 0$).

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**References**


10 Appendix: Proof of Lemma 5.1

Proof. Use Lemma 2.1 to obtain the bound on $|w_2^*(x,y)|$. Using (16e),

$$(-2b^*(d + y) + 4a_2^*\mu)(1 - y) \leq -b^*(d + y)(1 - y), \quad -d \leq y \leq 1,$$

for $\mu_0$ sufficiently small. Combine this fact with the obvious choice of barrier function and Lemma 2.2 to obtain that

$$|w_2^*(x,y)| \leq C\mu^{-4}(1 - y)(d + y)e^{-\theta(1-x)}/2\mu, \quad \theta < \gamma. \quad (35)$$
Using Corollary 2.1 the derivatives of $w^*_2$ satisfy
\[ |w^*_2|_n + \tau^\xi[w^*_2]_{n,\xi} \leq C\mu^{-4}\varepsilon^{-n}, \quad n = 1, 2, 3. \]

Below, we sharpen these bounds in the direction orthogonal to the layer. Using (35) we have that
\[
\left| \frac{\partial w^*_2}{\partial y}(x, 1) \right| \leq C\mu^{-4}e^{-\theta(1-x)/2\mu}, \quad \left| \frac{\partial w^*_2}{\partial y}(x, -d) \right| \leq C\mu^{-4}e^{-\theta(1-x)/2\mu}
\]
and we also have
\[
\frac{\partial^j w^*_2}{\partial y^j}(-d, y) = \frac{\partial^j w^*_2}{\partial y^j}(1, y) = 0, \quad j = 0, 1, 2, 3.
\]

Differentiate (19) with respect to $y$ to obtain
\[
L^*_{1;\varepsilon,\mu} \left( \frac{\partial w^*_2}{\partial y} \right) := L^*_{\varepsilon,\mu} \left( \frac{\partial w^*_2}{\partial y} \right) + \mu \left( \frac{\partial a^*_2}{\partial y} \right) \frac{\partial w^*_2}{\partial y} = \left( \frac{\partial b^*_2}{\partial y} \right) w^*_2 + \left( \frac{\partial g^*_2}{\partial y} \right).
\]

Observe that
\[
L^*_{1;\varepsilon,\mu}e^{-\theta(1-x)/(2\mu)} < ((a^*_1\gamma - 2b^*/3) + (\mu(a^*_2)_y - b^*/3)) e^{-\theta(1-x)/(2\mu)}.
\]

Hence using (16a,b) and for $\mu_0$ sufficiently small we have by (18a) that
\[
\left| \frac{\partial w^*_2}{\partial y}(x, y) \right| \leq C\mu^{-3}\varepsilon^{-1}e^{-\theta(1-x)/(2\mu)}.
\]

Using (19) along the boundary $y = 1$, we have that
\[
\left| \frac{\partial^2 w^*_2}{\partial y^2}(x, 1) \right| \leq C\mu^{-3}\varepsilon^{-1}e^{-\theta(1-x)/2\mu}.
\]

Note further that, by (1be), $(\varepsilon(w^*_2)_{yy})(x, -d) = g^*(x, -d) = 0$. Differentiate (19) twice with respect to $y$ to obtain
\[
L^*_{2;\varepsilon,\mu} \frac{\partial^2 w^*_2}{\partial y^2} := L^*_{\varepsilon,\mu} \frac{\partial^2 w^*_2}{\partial y^2} + 2\mu \left( \frac{\partial a^*_2}{\partial y} \right) \frac{\partial^2 w^*_2}{\partial y^2} = \left( 2 \left( \frac{\partial b^*_2}{\partial y} \right) - \mu \left( \frac{\partial^2 a^*_2}{\partial y^2} \right) \right) \frac{\partial w^*_2}{\partial y} + \left( \frac{\partial^2 g^*_2}{\partial y^2} \right) w^*_2 + \frac{\partial^2 g^*_2}{\partial y^2}.
\]

Using (1g), we deduce that
\[
\left| \frac{\partial^2 w^*_2}{\partial y^2}(x, y) \right| \leq C\mu^{-3}\varepsilon^{-1}e^{-\theta(1-x)/(2\mu)}.
\]
We introduce the auxiliary function
\[ \zeta = \varepsilon (w_2^*)_{yy} + a_2^*(y)\mu (w_2^*)_y. \]

Note that, by using (19) on the top edge with \( g^*(x, 1) = 0 \), it follows that \( \zeta(x, 1) = 0 \) and we have that
\[ L^*_\varepsilon,\mu \zeta \equiv \varepsilon g_{yy}^* + \mu a_2^*g_y^* + (\varepsilon b_{yy}^* + \mu a_2^*b_y^*)w_2^* + 2\varepsilon \mu b_{yy}^* (w_2^*)_y. \]

Then by (18a) \(|L^*_\varepsilon,\mu \zeta| \leq C\mu^{-3}(d + y)\) and, using (16c), \(|\zeta| \leq C(1 - y)\mu^{-4}\). This, in turn, implies that \(|\zeta_y(x, 1)| \leq C\mu^{-4}\), which will yield a bound on
\[
|\left(w_2^*ight)_{yyy}(x, 1)| \leq C\mu^{-4}\varepsilon^{-1} + C\mu \varepsilon^{-1} \left( \left| \frac{\partial^2 w_2^*}{\partial y^2} (x, 1) \right| + \left| \frac{\partial w_2^*}{\partial y} (x, 1) \right| \right)
\]
\[
\leq C\mu^{-4}(\varepsilon^{-1} + \mu^2 \varepsilon^{-2}) \leq C\mu^{-2}\varepsilon^{-2}.
\]

Using a similar argument we can also deduce that \(|(w_2^*)_y(x, -d)| \leq C\mu^{-3}\varepsilon^{-1}\). Therefore we have
\[
\left| \frac{\partial^3 w_2^*}{\partial y^3} (x, y) \right|_{\Omega_1^*(\varepsilon^{-1})^2} \leq C\mu^{-2}\varepsilon^{-2}.
\]

\[\square\]
Table 1: Computed differences $D^N_\mu$, orders of convergence $p^N_\mu$, parameter-uniform differences $D^N$ and orders of convergence $p^N$ for the test problem.

<table>
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<th>$\mu$</th>
<th>$N=16$</th>
<th>$N=32$</th>
<th>$N=64$</th>
<th>$N=128$</th>
<th>$N=256$</th>
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<td>0.220E-1</td>
<td>0.138E-1</td>
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<tr>
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<tr>
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<tr>
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<td>0.422</td>
<td>0.557</td>
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<td>0.639</td>
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</table>

$D^N_\mu$, $p^N_\mu$, $D^N$, $p^N$