Designing tasks to aid understanding of mathematical functions

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Threshold concepts have been defined as concepts which are troublesome, but transformative, integrative and irreversible in nature. While such concepts may present difficulties for a student initially, once understood, a new and previously inaccessible way of thinking about the concept and the wider subject may be opened up for the student and is unlikely to be forgotten. Some advise that curriculum design and teaching should focus on threshold concepts in a particular subject rather than aiming to transmit vast amounts of 'knowledge'.

In Mathematics, various concepts (such as 'function', 'limit') have been characterised as threshold concepts, with many students having difficulty reaching a comprehensive understanding of these concepts. Recent studies have also shown that many sets of mathematical tasks produced for students emphasize lower level skills (such as memorization and the routine application of algorithms or procedures), rather than endeavouring to develop students' understanding of the underlying concepts involved. We report on a study which aimed to use a variety of task types to aid the development of undergraduate students' understanding of the function concept. The task types were drawn from a framework designed by the authors for use in undergraduate calculus courses in order to give students the opportunity to engage meaningfully with concepts and to allow them to transform their understanding appropriately.

INTRODUCTION TO THRESHOLD CONCEPTS

The idea of a 'threshold concept' first emerged from a UK national research project (2001-2005) entitled 'Enhancing Teaching-Learning Environments in Undergraduate Courses' (Cousin, 2006). During the course of this project, which was designed to support departments in undergraduate teaching in thinking about new ways of encouraging high quality learning, Erik Meyer and Ray Land found that certain concepts were held by economists to be essential to the mastery of their subject. Such concepts were named 'threshold concepts' to distinguish them from 'core concepts': the latter may be conceptual building blocks that must be understood by students, but they do not necessarily lead to students' forming a qualitatively different view of their subject, which the former do (Meyer and Land, 2003). As such, threshold concepts have been described as portals, opening up a new and previously inaccessible view of a topic, a view without which students cannot fully progress intellectually. Meyer and Land (2003) identify five characteristics of a threshold concept: transformative, irreversible, integrative, bounded, and troublesome.

Once a threshold concept is understood, it has the potential to trigger a significant shift or *transformation* in the perception of a subject. Furthermore, this shift may be ontological as well as conceptual as on mastering such a concept a student starts to think and act like a professional in the discipline. The change in perception is *irreversible* in the sense that it is unlikely to be forgotten and can be 'unlearned' only with considerable effort. For this reason, it can be difficult for lecturers or experienced practitioners to appreciate the difficulties of their students as this requires them to look back over thresholds they have long since crossed. Threshold concepts can also be described as *integrative* as they often expose the inter-relatedness of a topic and allow previously hidden connections to be displayed. From this perspective, they may act like an anchor for a subject by bringing different aspects together and giving structure to a topic. Often, but not necessarily, threshold concepts may lie on the border between conceptual spaces or may constitute the demarcation line between disciplinary areas, and for this reason, have been described as *bounded*. Finally, threshold concepts are *troublesome* by nature; in part due to the characteristics described above, but also because they are often inherently conceptually difficult, counter-intuitive, apparently paradoxical, or require subtle distinctions to be made between ideas.

A focus on threshold concepts can enable teachers to make refined decisions about what is fundamental to the study and mastery of their subject (Cousin, 2006). Land et al. (2005) discuss the implications such an approach has for curriculum design and teaching. Because of the potentially powerful transformative effects of threshold concepts on a student's learning experience, they advocate treating threshold concepts as 'jewels in the curriculum' around which courses should be organised. In order to enable students to develop an understanding of troublesome concepts, they must actively engage with the conceptual material and so lecturers or teachers should construct a framework of engagement to facilitate this understanding and to allow students to experience the 'ways of thinking and practicing' that are expected of practitioners in their discipline or community of practice.

MATHEMATICAL FUNCTION AS A THRESHOLD CONCEPT

The concept of 'function' is central to any Calculus course and indeed underpins many other areas of Mathematics. Much research has focused on the development of understanding of the function concept. For instance, Vinner (1983) considered the difference between *concept definition* and *concept image* in relation to the concept of function and found that students construct a variety of concept images that are not consistent with the definition (for instance, that a function should be given by one rule).

The concept of 'function' can be viewed as a threshold concept in mathematics (Pettersson et al., 2013) –as it can be characterized as transformative, irreversible, integrative and troublesome (Meyer and Land, 2003). To properly understand functions and to work with them in diverse areas of mathematics, students should be able to conceive of a function as an action, as a process and as an object in its own right (Dubinsky and McDonald, 2001). Sfard (1991) discusses the complementary approaches of dealing with abstract notions such as functions: operationally as processes and structurally as objects. She introduced the term 'reification' to represent the transition of thought involved when a learner progresses to viewing processes as objects. She warns that reification is "an ontological shift, a sudden ability to see something familiar in a new light" (p.19) and a "rather complex phenomenon" (p.30), causing obstacles and frustration for learners — illustrating the transformative but troublesome properties of the concept. Gray and Tall (1994) maintain that the ability to think flexibly in this manner (operationally and structurally) is at the root of successful mathematical thinking. Thus, it can serve as a marker of students' progress in learning mathematics. However, in mathematics teaching 'reification' often remains an implicit learning outcome, a form of tacit knowledge that is not explicitly articulated to learners.

Once such ‘reification’ has taken place, a previously inaccessible means of thinking about the mathematical concept is opened up and is unlikely to be forgotten or reversed. There is a permanent repositioning of the learner in relation to the concept and it is unlikely that there will be any ‘conceptual decay’ over time. As mentioned earlier, Meyer and Land (2003) discuss how expert practitioners in a field can have difficulties looking back over a threshold they have personally long since crossed. Gray and Tall (1994) suggest the flexibility in thought achieved by those who have experienced ‘reification’ (e.g. with ‘function’) can explain why a mathematics expert may find it difficult to appreciate the difficulties of a novice. The integrative nature of the understanding associated with a threshold concept also highlights a distinction between the thinking of a novice and the community of practice within a discipline (Meyer and Land, 2003). This aspect of integration is also a feature of comprehensive understanding of mathematical functions. Dubinsky and McDonald (2001) describe a further stage (beyond action, processes and objects) in the understanding of mathematical concepts. They use the term ‘schema’ to describe the collection of actions, processes and objects an individual associates with a particular concept (e.g. function) and links by general principles, to each other and to other concepts in the subject area, to form a coherent framework in the individual’s mind. A schema outlines previously hidden relations between concepts. Dubinsky and McDonald (2001) suggest that a student who has reached the stage of constructing a coherent schema, integrating aspects and features of the concept in question (e.g. function), is more likely to be successful in using the concept and solving problems involving it. For instance, reaching a comprehensive understanding of the concept of ‘function’ can lend shape and structure to a student’s concept image of Calculus.

Thus, there is ample evidence in the research literature that the concept of function can be considered a threshold concept.

TRADITIONAL APPROACHES TO UNDERGRADUATE MATHEMATICS CURRICULUM DESIGN

Cousin (2006) claims that

> “a tendency among academic teachers is to stuff their curriculum with content, burdening themselves with the task of transmitting vast amounts of knowledge bulk and their students of absorbing and reproducing this bulk (p.4)”.

In particular, mathematics lecturers have been accused of such a practice. Hillel (2001) claims that, generally speaking, undergraduate mathematics courses have traditionally been defined in terms of mathematical content and the techniques students are expected to master or theorems they should be able to prove. Although the main goal of a mathematics lecturer is to foster mathematical understanding in their students, such an understanding is seldom specifically fostered by the mathematical tasks and assessments students are required to complete (Sangwin, 2003), with many authors expressing the view that mathematics at third level suffers from an over-emphasis on procedures and memorisation. For instance, Dreyfus (1991) asserts that many students learn a large number of standardised procedures in their university mathematics courses and, although they end up with a considerable amount of mathematical knowledge, they lack the working methodology of a mathematician and therefore cannot use their knowledge in a flexible manner. This is very much in contrast with the type of approach advocated by Land et al. (2005) and described above.

TASK TYPES & SAMPLES OF TASKS

In an attempt to follow the advice of Land et al. (2005) to provide students with opportunities to actively engage with threshold concepts, we chose to focus on the *tasks* that we would assign to students in the first year undergraduate Calculus courses we were teaching. In fact, Mason (2002) contends that

> "in a sense, all teaching comes down to constructing tasks for students…This puts a considerable burden on the lecturer to construct tasks from which students actually learn (p.105)".

However, Groves and Doig's (2002) assert that

> "insufficient attention is being paid to the critical role of the development of conceptually focussed, robust tasks which can be used to support the development of sophisticated mathematical thinking (p.31)".

Boesen, Lithner and Palm (2010) found that the types of tasks assigned to students affect their learning: when faced with familiar tasks students employed imitative reasoning (that is, reproduced from memory or used well-rehearsed procedures) and, in contrast, used creative mathematically founded reasoning (that is, formulated mathematically well-founded arguments which were new to the students) to tackle unfamiliar tasks. They claim that the solutions to familiar tasks required little or no conceptual understanding and they conjecture that exposure to these types of tasks alone limits the students' ability to reason and gain conceptual understanding. Likewise, Selden, Selden, Hauk and Mason (2000) recommend that lecturers should regularly assign non-routine problems to students in order to develop their mathematical thinking skills.

Many authors (e.g. Dreyfus 1991) agree that the mathematical practices and thinking to be encouraged in learners of mathematics should mirror the practices of professional mathematicians, and Bass (2005) describes these ways of thinking and practicing as including experimentation, reasoning, generalization, and the use of definitions and mathematical language. Cuoco et al (1996) further propose that students need to conjecture, visualise, describe and invent. They claim the inclusion of such 'mathematical habits of mind' will 'give students the tools they will need in order to use, understand and even make the mathematics that does not yet exist' (p.376). Swan (2008) selected five task types which he believed would promote conceptual understanding and encourage the development of mathematical skills amongst secondary school students: those were classifying mathematical objects; interpreting multiple representations; evaluating mathematical statements; creating problems; analysing reasoning and solutions. Sangwin (2003) promote the use of exercises in which students are required to generate or construct their own examples, as mathematicians would.

Drawing on and synthesizing the advice from the literature described above, we identified the following types of tasks as being appropriate for Irish first year undergraduate Calculus students: tasks requiring students to generate examples, evaluate statements, analyse reasoning, conjecture, generalise, visualise, and/or use definitions. Our aim was to move away from a content-driven curriculum, and to provide students with an opportunity to actively engage with mathematical concepts, in particular Calculus threshold concepts, and to enable them to experience and gain an understanding of the ways of thinking and practicing of mathematicians. We designed a number of tasks and include samples of these types of tasks, which deal with functions, below.

Example Generation: Give an example of a function with natural domain R\{2,4}.

Conjecturing/Generalising:

(i) Sketch the graphs of $f_1(x)=x^3$ and $f_2(x)=x^3+4$ (using the natural domains).

(ii) Sketch the graphs of $g_1(x)=1/x^2$ and $g_2(x)=1/x^2+4$ (using the natural domains).

(iii) Sketch the graphs of $h_1(x)=3^x$ and $h_2(x)=3^x+4$ (using the natural domains).

(iv) What is the relationship between the functions in the pairs f_1 and f_2; g_1 and g_2; h_1 and h_2? Can you make a general conjecture regarding the graphs of functions from your observation of the graphs of these pairs?

Visualisation: Sketch a graph of a function, f, which satisfies all of the given conditions:

$f(0)=0$, $f(2)=6$, *f is even.*

Evaluating Mathematical Statements: Suppose $f(x)$ is a function with natural domain R. Decide if each of the following statements is sometimes, always or never true:

(i) There are two different real numbers a and b such that $f(a)=f(b)$.
(ii) There are three different real numbers a, b, c such that $f(a)=b$ and $f(a)=c$.

The latter three tasks, as well as being of the type indicated in order to promote conceptual understanding, also provide the students with an opportunity to encounter a function as an 'object' rather than a 'process' or 'action'. (Note, for instance, that in the third and fourth tasks the students are not given a formulaic representation of a function to work with and so must focus more on *structure* than *operation* to answer the question posed.)

CONCLUDING REMARKS

The concept of 'function' can be considered as a threshold concept in mathematics. As such it is a key concept that students must master and could be viewed as a 'jewel in the curriculum' (Cousin, 2006) of Calculus courses and act as a focal point for teaching. Constructing tasks and activities for students that will engage them with the concept, and effectively develop and transform their understanding of functions then becomes a challenge for teachers and lecturers. Some suggestions have been made here as to the types of tasks suitable for first year undergraduate Calculus students in this regard. In designing the tasks we were mindful of the need to create a rich variety of tasks to avoid a particular type of task becoming over-familiar and to provide students with sufficient opportunities to develop their mathematical thinking skills.

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