Optimization: From Discovery to Assessment

Mairead Greene, *Ph.D.*

Rockhurst University, Kansas City, Missouri, USA

* Calculus students are generally prospective science, business and math majors.
* Focus is on concepts and applications – understanding not memorizing.
* 32 students in a classroom
* Computer lab type classrooms with group tables where students spend much of their time actively working in small groups during class time.

Calculus at Rockhurst

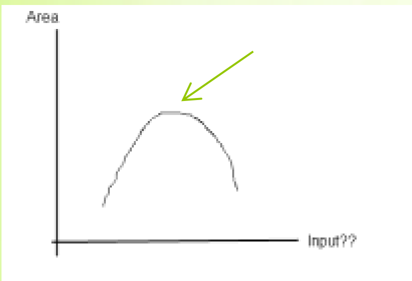
Philosophy on Optimization

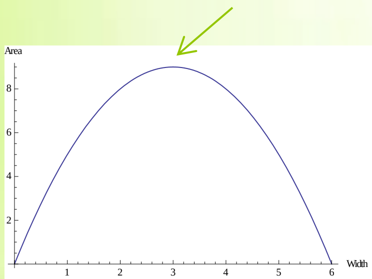
* Introduce as early as possible so that students have time to work with these big ideas.
* Use multiple representations to help students focus on understanding not method and to allow us to ask optimization questions before we have completed all derivative rules. They also helps us assess student understanding.
* Use reading activities to help students develop the abilities they will need to tackle the typical optimization problems.

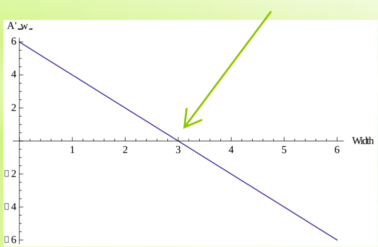
A Typical Optimization Problem

A rectangular window is being built with 12 meters of framing materials. What must the dimensions of the window be to let in the most light?

Graphical Approach







Numerical Approach

Width	Height	Area
2	4	8
1	5	5
3	3	9
3.5	2.5	8.75
3.2	2.8	8.96

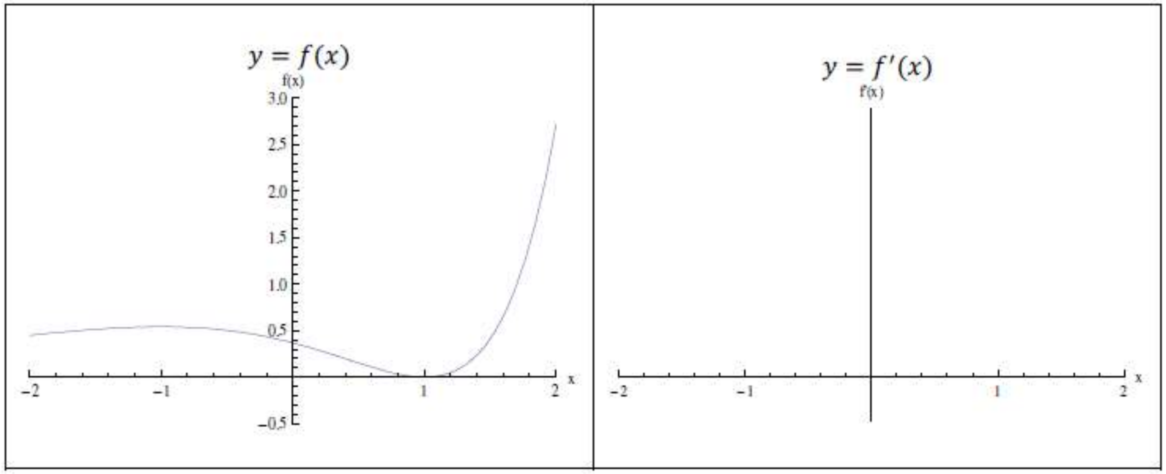
Symbolic Approach

$$Area = w \cdot h$$
$$Per = 12 = 2w + 2h$$
$$A(w) = w(6 - w)$$

$$A(w) = 6w - w^2$$
$$A'(w) = 6 - 2w$$
$$6 - 2w = 0$$
$$w = 3$$
$$h = 6 - 3 = 3$$

First Introduction

4. Consider the graph of $f(x)$ provided below. On the axes on the right, sketch the graph of the derivative of $f(x)$ as a function of x. (You should sketch this graph using only the shape and behavior of the graph of $f(x)$ – you do not need an analytical form of $f(x)$ to do this.)



5. Class Discussion: How did you determine how to sketch $f'(x)$ above? What is a local maxima and local minima of a function?

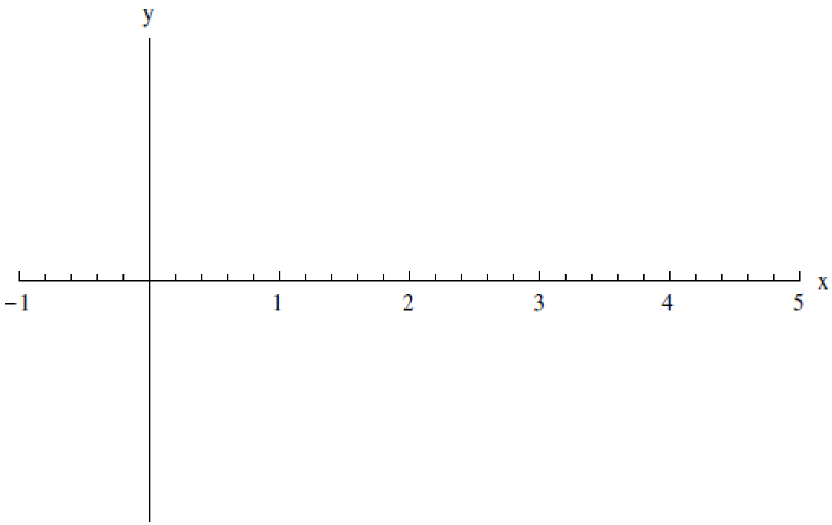
6. At what x-values does the function $f(x)$ have ***local maxima*** or ***local minima***? What do you notice about $f'(x)$ at these values of x?

- Approx 4 weeks into a 15 week semester.
- Graphical introduction and noticing that $f'(x) = 0$.
- Dependence on meaning of f'

Further Development

1. Sketch the graph of a single function $f(x)$ that has the following characteristics:

x	$f(x)$	$f'(x)$	$f''(x)$
0	−	+	−
1	−	**0**	−
2	−	−	**0**
3	−	**0**	+
4	**0**	+	+
5	+	+	+



- Following numerical and graphical work on first and second derivative as well as understanding the connections to increasing/decreasing and concave up/down.
- Observing max/mins as a consequence of interpreting f' and f'' information.

- Results ($n = 24$)

  Average Number of Mistakes = 1.96

2. A function $f(x)$ has the following characteristics:

- Concave up in the interval $(-2,5)$
- Concave down in the interval $(5,10)$
- Second derivative is 0 at $x = 5$
- Has an instantaneous rate of change of 0 at $x = 2$, and $x = 7$
- The instantaneous rate of change is positive between $x = 2$ and $x = 7$

Circle all of the following statements that you definitely know to be true using only the above information:

a. $f(x)$ has a local minimum at $x = 7$

b. $f(x)$ is increasing from $x = 2$ to $x = 7$

c. $f(x)$ has a local maximum at $x = 5$

d. $f(x)$ has a local minimum at $x = 2$

e. $f(x)$ has an inflection point at $x = 5$

f. $f(x)$ has a local maximum at $x = 3$

* Results $(n = 24)$

Average 5.125 out of 6 correct

* More direct relationship between concavity and instantaneous rate of change and max/min

Find all of the local maxima and minima for the function $f(x) = x^3 - 3.6x^2 - 12.96x + 2$ *exactly*. You should solve this problem exactly *by hand*. Justify why the maxima are maxima and why the minima are minima. Show all of your work and explain your reasoning. You may check the reasonableness of your answers by graphing the function but this cannot be the basis for your justification.

* Students work on this question outside of class as a synthesis of their understanding of first and second derivative.
* During their work on this problem outside of class we work on the following quiz which helps them make more general conclusions about max and mins.
* Goal: Reasoning from understanding of f' and f'' instead of following/memorizing a method at this stage.

* Results ($n = 21$)

Critical points correct but no justification attempted – 9 students (43%)

Critical points correct incorrect justification attempted – 4 students (19%)

Fully correct max/min identified with justification – 8 students (38%)

More complete optimization without context

Circle all of the following statements that are true.

a. If $f'(a) = 0$ and $f''(a) = -5$ then $f(x)$ has a maximum at $x = a$.

b. If $f'(a) = 0$ and $f''(a) = 5$ then $f(x)$ has a maximum at $x = a$.

c. If $f'(a) = 0$ and $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > a$ then $f(x)$ has a maximum at $x = a$.

d. If $f'(a) = 0$ and $f'(x) > 0$ for $x < a$ and $f'(x) < 0$ for $x > a$ then $f(x)$ has a maximum at $x = a$.

e. If $f'(a) = 0$ and $f'(x) > 0$ for $x < a$ and $f'(x) > 0$ for $x > a$ then $f(x)$ has a maximum at $x = a$.

* Students work on this quiz in groups of 2-3 in class.
* I recorded their answers after this work without any whole class discussion. (See right.)
* Followed immediately with whole class discussion on problem options after posting answers.
* Goal: To draw some more general conclusions about the justification of max/mins.

* Results ($n = 26$)
  a) True (38%)
  b) False (92%)
  c) False (73%)
  d) True (27%)
  e) False (77%)

Quiz on Optimization

Assessment

* Now there is almost three weeks until the midterm exam which will test them over this material.
* A variety of levels of understanding of optimization are assessed on the midterm exam.
* Use the Conceptual Understanding Weighting System (developed with Dr. Paula Shorter) to help write questions to distinguish these levels.

Conceptual Understanding Weighting System (CUWS)

Check 1 (Skills): Does this problem involve only computational/algebraic skills or memorized facts with no understanding of the concept required? Yes - assign a weight of 0 to the problem and skip the remaining checks; No - continue to the next check.

Check 2 (Method): Could this problem be answered completely by using or adapting a method that the student or teacher might have developed in prior course work? Yes - assign a weight of 1 to the problem and skip the remaining checks; No - continue to the next check.

Check 3 (Conceptual Reasoning Characteristics): Does the problem involve either of the following? No - assign a weight of 2 to the problem; Yes - assign a weight of 3 to the problem.

- interpreting the meaning of a mathematical characteristic in a novel applied setting
- making connections between different representations (numerical, graphical, symbolic or narrative) of a mathematical characteristic

*CUWS

1. (3 points) Consider the following polynomial:

$$f(x) = x^3 - 3x^2 - 18x + 25$$

   a. Calculate the derivative of $f(x)$.

   b. Calculate the second derivative of $f(x)$.

2. (5 points) Consider the function $f(x) = x^2 - x - 6$.

   a. Find the x values where $f(x) = 0$.

   b. Calculate $f(5)$.

Average Score

- 1 - 92%
- 2 - 82%

Skills

Consider the following function and its derivatives, already calculated and factored for you:

$$f(x) = 2x^3 + 9x^2 - 24x$$

$$\begin{aligned} f'(x) &= 6x^2 + 18x - 24 \\ &= 6(x^2 + 3x - 4) \\ &= 6(x-1)(x+4) \end{aligned}$$

$$\begin{aligned} f''(x) &= 12x + 18 \\ &= 6(2x + 3) \end{aligned}$$

Circle all of the choices below that are true for this function, $f(x)$.

a. $f(x)$ has a local maximum at $x = 1$.

b. $f(x)$ has a local minimum at $x = -\frac{3}{2}$.

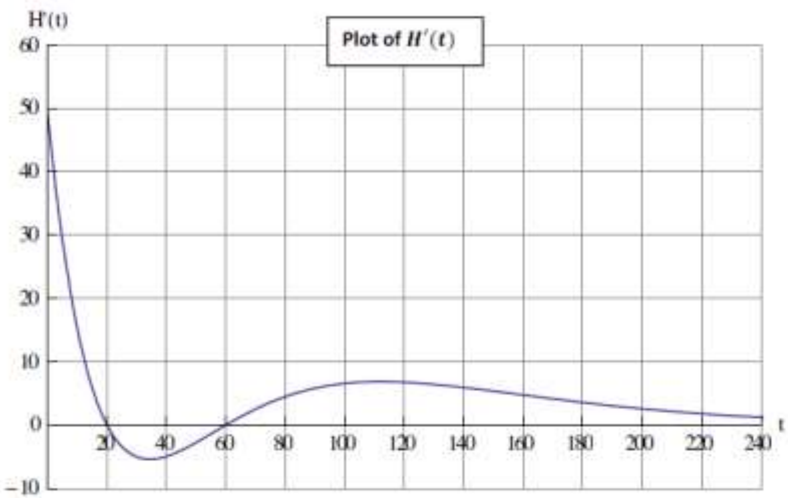
c. $f(x)$ has a local maximum at $x = -4$.

Results

* a – False(73%)
* b - False(88%)
* c - True (69%)

Method

7. (10 points) A patient was given an injection and then the hormone level in their blood was monitored for four hours after the injection. The hormone level, H (measured in International Units per liter of blood), was monitored over time, t (measured in minutes after the injection was administered). Use the graph of the derivative of $H(t)$ ($H'(t)$) below to answer the questions that follow.



a. What was the instantaneous rate of change of this patient's hormone level in the bloodstream 180 minutes after the injection was administered? (Estimate from the graph.) Include units in your answer.

b. This patient's hormone level is at a lowest level *twice* in the first two hours. When does this happen? Include units in your answer.

c. Over what interval of time was this patient's hormone level decreasing?

d. Over what interval of time was this patient's hormone level increasing at an increasing rate?

Average Score

* Overall – 59%

* Concept

* To introduce the concept earlier than might otherwise be possible
* To build a student's understanding of a concept by asking a question in another representation (new setting) where a memorized method will not apply.
* To examine a concept on an exam where we may have worked on a similar question in class but in a different representation.
* To develop students ability to reach for multiple representations as tools to help them answer any question – in particular more typical optimization questions later.

Utilizing multiple representations

* Why are students significantly better at correctly identifying false statements but not true statements on our quiz? Is this a problem with the quiz itself or with the development of the material?
* Does this early introduction to optimization help students feel more comfortable when they come to the more typical optimization problems?
* Revise the activities to make the connections between max/min and the derivatives more explicit.
* Observe the students work on the remainder of our optimization work
* Expand sample size and classroom environment to see impact of working on optimization earlier in different settings.

Interesting Questions/Follow Up

mairead.greene@rockhurst.edu

www.iblcalculus.com (jointly with Dr. Paula Shorter)

Thank you!