Designing Tasks to Aid Understanding of Mathematical Functions

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Overview

- Introduction to Threshold Concepts
- Function as a threshold concept
- Implications for teaching
- Ways of thinking and practising of mathematicians
- Types of tasks
- Samples of tasks
- Concluding remarks

Threshold Concepts

- Emerged from UK National Research Council project (2001-2005) 'Enhancing Teaching-Learning Environments in Undergraduate Courses' — economists Erik Meyer & Ray Land
- Portal: opening new, previously inaccessible way of thinking
- E.g. heat transfer in cooking

Threshold Concepts - Characteristics

- Transformative
  Irreversible
  Integrative
  Bounded
  Troublesome (Meyer & Land 2003)
- Disciplinary examples: opportunity cost in economics, laws of motion in physics, equal temperament in music

Mathematics Perspective

- Mathematics threshold concepts: e.g. fractions, limits, proof
- Function definition: A function (mapping) from a set A to a set B is a rule that assigns each element of A to exactly one element of B.
- Examples from the set of real numbers, R, to R: $f(x)=x+4$, $f(x)=3x$, $f(x)=x^2$

Function as a Threshold Concept -transformative but troublesome

- Understanding 'function' requires conceiving of a function as an action, process, object (Dubinsky and McDonald 2001)
- Reification (from process to object view) is "an ontological shift, a sudden ability to see something familiar in a new light" and a "rather complex phenomenon" (Sfard 1991)

Function as a Threshold Concept -irreversible and integrative

- Difficult for expert practitioners to look back over threshold to understand difficulties of students (Meyer and Land 2003) – flexibility in thought following 'reification' means experts fail to appreciate difficulties of novices (Gray and Tall 1994)
- Schema – coherent framework linking reified concept to other associated concepts (Dubinsky and McDonald 2001)

Recommendations for Teaching

(Land et al. 2005, Cousin 2006)

- Jewels in curriculum – foci of module
- Adopt recursive approach
- Construct framework for engagement
- Ways of thinking and practising

Recommendations for Teaching

(Land et al. 2005, Cousin 2006)

- Listen for understanding
- Tolerate uncertainty
- Be aware of unintended consequences of 'good pedagogy'

Traditional Approaches to Mathematics Curriculum Design

- Cousin (2006)

  a tendency…to stuff their curriculum with content, burdening themselves with the task of transmitting vast amounts of knowledge bulk

- Defining undergraduate mathematics courses in terms of mathematical content (techniques, theorems) (Hillel 2001)
- Students learn standardised procedures but lack working methodology of mathematician (Dreyfus 1991)
- Reliance on superficial, shallow, or rote learning

Ways of Thinking & Practising

- Tasks should mirror practices of mathematicians (Dreyfus 1991; Cuoco et al 1996; Bass 2005)
- Practices of mathematicians/mathematical skills to develop in students:

    Experimenting, inventing, creating, visualising, reasoning, conjecturing, generalising, using definitions and mathematical language, classifying objects, comparing, interpreting, evaluating statements, analysing reasoning.

Task Types

- Tasks identified as appropriate for Irish first year undergraduate Calculus students:
  - generating examples
  - evaluating statements
  - analysing reasoning
  - conjecturing
  - generalising
  - visualising
  - using definitions

Sample Tasks

- *Example Generation:* Give an example of a function with natural domain $R\setminus\{2,4\}$.

- *Visualisation:* Sketch a graph of a function, f, which satisfies all of the given conditions:

$f(0)=0$, $f(2)=6$, f is even.

Sample Tasks

- *Conjecturing/Generalising:*

i. Sketch the graphs of $f_1(x)=x^3$ and $f_2(x)=x^3+4$.

ii. Sketch the graphs of $g_1(x)=1/x^2$ and $g_2(x)=1/x^2+4$.

iii. Sketch the graphs of $h_1(x)=3^x$ and $h_2(x)=3^x+4$.

iv. What is the relationship between the functions in the pairs f_1 and f_2; g_1 and g_2; h_1 and h_2? Can you make a general conjecture regarding the graphs of functions from your observation of the graphs of these pairs?

Sample Tasks

- *Evaluating Mathematical Statements*: Suppose $f(x)$ is a function with natural domain R.

  Decide if each of the following statements is sometimes, always or never true:

  (i) There are two different real numbers a and b such that $f(a)=f(b)$.

  (ii) There are three different real numbers a, b, c such that $f(a)=b$ and $f(a)=c$.

Concluding Remarks

- Importance of identifying Threshold Concepts
- Challenge of constructing effective tasks and activities for students to engage with concepts
- Suggestions made as to the types of tasks suitable for first year undergraduate Calculus students
- Key component of successful understanding of functions involves the ability to move flexibly between different characterisations of functions

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